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Stability in parametric resonance of axially moving viscoelastic beams with time-dependent speed

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Abstract

Stability in transverse parametric vibration of axially accelerating viscoelastic beams is investigated. The governing equation is derived from Newton's second law, the Kelvin constitution relation, and the geometrical relation. When the axial speed is a constant mean speed with small harmonic variations, the governing equation can be regarded as a continuous gyroscopic system under small periodically parametric excitations and a damping term. The method of multiple scales is applied directly to the governing equation without discretization. The stability conditions are obtained for combination and principal parametric resonance. Numerical examples are presented for beams with simple supports and fixed supports, respectively, to demonstrate the effect of viscoelasticity on the stability boundaries in both cases.

1. Introduction

Many engineering devices can be modeled as axially moving beams. One major problem is the occurrence of large transverse vibrations due to tension or axial speed variation. Transverse vibration of axially accelerating beams has been extensively analyzed. Although Pasin [1] first studied the problem as early as in 1972, much progress was achieved recently. Öz et al. [2] applied the method of multiple scales to study dynamic stability of an axially accelerating beam with small

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bending stiffness. Özkaya and Pakdemirli [3] applied the method of multiple scales and the method of matched asymptotic expansions to construct non-resonant boundary layer solutions for an axially accelerating beam with small bending stiffness. Öz and Pakdemirli [4] and Öz [5] used the method of multiple scales to calculate analytically the stability boundaries of an axially accelerating tensioned beam under simply supported conditions and fixed-fixed conditions, respectively. Parker and Lin [6] adopted a 1-term Galerkin discretization and the perturbation method to study dynamic stability of an axially accelerating beam subjected to a tension fluctuation. Özkaya and Öz [7] applied artificial neural network algorithm to determine stability of an axially accelerating beam.

All above-mentioned researchers considered elastic beams, and did not account for any damping. The modeling of dissipative mechanisms is an important research topic of axially moving material vibrations [8,9]. Viscoelasticity is an effective approach to model the damping mechanism because some beam-like engineering devices are composed of some viscoelastic metallic or ceramic reinforcement materials like glass-cord and viscoelastic polymeric materials such as rubber. The literature that is specially related to axially accelerating viscoelastic beams is relatively limited. Based on 3-term Galerkin truncation, Marynowski [10] and Marynowski and Kapitaniak [11] compared the Kelvin model with the Maxwell model and the Bügers model, respectively, through numerical simulation of nonlinear vibration responses of an axially moving beam at a constant speed, and found that all models yield similar results in the case of small damping. Marynowski [12] further studied numerically nonlinear dynamical behavior of an axially moving viscoelastic beam with time-dependent tension based on 4-term Galerkin truncation. Based on 2-term Galerkin truncation, Yang and Chen [13] and Chen et al. [14] applied the averaging method to analyze the stability of axially accelerating linear beams with pinned or clamped ends, and Yang and Chen [15] studied numerically bifurcation and chaos of an axially accelerating nonlinear beam.

In this paper, the stability is investigated for parametric vibration of axially accelerating viscoelastic beams. The governing equation is derived from Newton's second law, the constitution relation, and the strain–displacement relation. The method of multiple scales is applied directly to the governing equation. The stability boundaries for combination and principal resonance are presented for beams with simple supports and fixed supports. The effects of viscoelasticity on the boundaries are numerically demonstrated.

2. The governing equation

A uniform axially moving viscoelastic beam, with density ρ, cross-sectional area A, moment of inertial I and initial tension P_0, travels at the time-dependent axial transport speed $v(T)$ between two prismatic ends separated by distance L. Consider only the bending vibration described by the transverse displacement $V(X, T)$, where T is the time and X is the axial coordinate. The Newton second law of motion yields

$$\rho A\left(\frac{\partial^2 U}{\partial T^2} + 2v\frac{\partial^2 U}{\partial X \partial T} + \frac{\mathrm{d}v}{\mathrm{d}T}\frac{\partial U}{\partial X} + v^2\frac{\partial^2 U}{\partial X^2}\right) = P_0\frac{\partial^2 U(X,T)}{\partial X^2} - \frac{\partial^2 M(X,T)}{\partial X^2}, \tag{1}$$

where $M(X,T)$ is the bending moment given by

$$M(X,T) = -\int_A Z\sigma(X,Z,T)\,\mathrm{d}A, \tag{2}$$

where Z,X-plane is the principal plane of bending, and $\sigma(X,Z,T)$ is the disturbed normal stress. The viscoelastic material of the beam obeys the Kelvin model, with the constitution relation

$$\sigma(X,Z,T) = Ee(X,Z,T) + \eta\,\frac{\partial e(X,Z,T)}{\partial T}, \tag{3}$$

where $e(X,Z,T)$ is the axial strain, E is the stiffness constant, and η is the viscosity coefficient. For small deflections, the strain–displacement relation is

$$e(X,Z,T) = -Z\,\frac{\partial^2 U(X,T)}{\partial X^2}. \tag{4}$$

Substitution of Eqs. (3) and (4) into Eq. (2) and then substitution the resulting equation into Eq. (1) lead to

$$\rho A\left(\frac{\partial^2 U}{\partial T^2} + 2v\,\frac{\partial^2 U}{\partial X\partial T} + \frac{\mathrm{d}v}{\mathrm{d}T}\frac{\partial U}{\partial X} + v^2\,\frac{\partial^2 U}{\partial X^2}\right) - P_0\,\frac{\partial^2 U}{\partial X^2} + EI\,\frac{\partial^4 U}{\partial X^4} + \eta I\,\frac{\partial^5 U}{\partial T\partial X^4} = 0. \tag{5}$$

Introduce the dimensionless variables and parameters:

$$u = \frac{U}{L}, \quad x = \frac{X}{L}, \quad t = T\sqrt{\frac{P_0}{\rho A L^2}}, \quad \gamma = v\sqrt{\frac{\rho A}{P_0}},$$
$$v_f^2 = \frac{EI}{P_0 L^2}, \quad \varepsilon\alpha = \frac{I\eta}{L^3\sqrt{\rho A P_0}}, \tag{6}$$

where bookkeeping device ε is a small dimensionless parameter accounting for the fact that the viscosity coefficient is very small. Eq. (5) can be cast into the dimensionless form

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma\,\frac{\partial^2 u}{\partial x\partial t} + \frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\partial u}{\partial x} + (\gamma^2 - 1)\,\frac{\partial^2 u}{\partial x^2} + v_f^2\,\frac{\partial^4 u}{\partial x^4} + \varepsilon\alpha\,\frac{\partial^5 u}{\partial x^4\partial t} = 0. \tag{7}$$

3. Stability condition via the method of multiple scales

In the present investigation, the axial speed is assumed to be a small simple harmonic variation, with the amplitude $\varepsilon\gamma_1$ and the frequency ω, about the constant mean speed γ_0,

$$\gamma(t) = \gamma_0 + \varepsilon\gamma_1 \sin\omega t. \tag{8}$$

Here the bookkeeping device ε is used to indicate the fact that the fluctuation amplitude is small, with the some order as the dimensionless viscosity coefficient. In spite of the apparent connection between the dimensionless viscosity coefficient and the amplitude of the variation through the bookkeeping device ε, they are actually independent because each of them includes, respectively, an arbitrary parameter α or γ_1 of order one. Substitution of Eq. (8) into Eq. (7) and neglecting higher order ε terms in the resulting equation yield

$$M\frac{\partial^2 u}{\partial t^2}+G\frac{\partial u}{\partial t}+Ku = -2\varepsilon\gamma_1\sin\omega t\frac{\partial^2 u}{\partial x\partial t}-2\varepsilon\gamma_0\gamma_1\sin\omega t\frac{\partial^2 u}{\partial x^2}-\varepsilon\omega\gamma_1\cos\omega t\frac{\partial u}{\partial x}-\varepsilon\alpha\frac{\partial^5 u}{\partial x^4\partial t}, \tag{9}$$

where the mass, gyroscopic, and linear stiffness operators are, respectively, defined as

$$M=I,\quad G=2\gamma_0\frac{\partial}{\partial x},\quad K=(\gamma_0^2-1)\frac{\partial^2}{\partial x^2}+v_f^2\frac{\partial^4}{\partial x^4}. \tag{10}$$

The method of multiple scales will be employed to solve Eq. (9) directly. A first-order uniform approximation is sought in the form

$$u(x,t;\varepsilon)=u_0(x,T_0,T_1)+\varepsilon u_1(x,T_0,T_1)+\cdots, \tag{11}$$

where $T_0=\tau$ is a fast scale characterizing motions occurring at ω_k (one of the natural frequencies of the corresponding unperturbed linear system), and $T_1=\varepsilon\tau$ is a slow scale characterizing the modulation of the amplitudes and phases due to viscoelasticity and possible resonance. Substitution of Eq. (11) and the following relationship

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\cdots,\quad \frac{\partial^2}{\partial t^2}=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial^2}{\partial T_0\partial T_1}+\cdots \tag{12}$$

into Eq. (9) and then equalization of coefficients of ε^0 and ε in the resulting equation lead to

$$M\frac{\partial^2 u_0}{\partial T_0^2}+G\frac{\partial u_0}{\partial T_0}+Ku_0=0 \tag{13}$$

and

$$M\frac{\partial^2 u_1}{\partial T_0^2}+G\frac{\partial u_1}{\partial T_0}+Ku_1 = -2\frac{\partial^2 u_0}{\partial T_0\partial T_1}-2\gamma_0\frac{\partial^2 u_0}{\partial x\partial T_1}-2\gamma_1\sin\omega t\left(\frac{\partial^2 u_0}{\partial x\partial T_0}+\gamma_0\frac{\partial^2 u_0}{\partial x^2}\right)-\gamma_1\omega\cos\omega t\frac{\partial u_0}{\partial x} -\alpha\frac{\partial^5 u_0}{\partial x^4\partial T_0}. \tag{14}$$

Wickert and Mote [16] have obtained the solution to Eq. (13)

$$u_0(x,T_0,T_1)=\sum_{k=0,1,\ldots}\lfloor\phi_k(x)A_k(T_1)e^{i\omega_k T_0}+\bar{\phi}_k(x)\bar{A}_k(T_1)e^{-i\omega_k T_0}\rfloor, \tag{15}$$

where the over bar denotes complex conjugation, and the kth natural frequency and the kth complex eigenfunction can be determined by the boundary conditions.

If the variation frequency ω approaches the sum of any two natural frequencies of system (13), summation parametric resonance may occur. A detuning parameter σ is introduced to quantify the deviation of ω from $\omega_n + \omega_m$, and ω is described by

$$\omega = \omega_n + \omega_m + \varepsilon\sigma, \tag{16}$$

where ω_n and ω_m are, respectively, the nth and mth natural frequencies of system (13).

To investigate the summation parametric response, Eq. (15) can be expressed as

$$u_0(x, T_0, T_1) = \phi_n(x)A_n(T_1)\mathrm{e}^{\mathrm{i}\omega_n T_0} + \phi_m(x)A_m(T_1)\mathrm{e}^{\mathrm{i}\omega_m T_0} + cc, \tag{17}$$

where cc stands for the complex conjugate of all preceding terms on the right hand of an equation. Substitution of Eqs. (16) and (17) into Eq. (14) and expression of the trigonometric functions in exponential form yield

$$\begin{aligned} &M\frac{\partial^2 u_1}{\partial T_0^2} + G\frac{\partial u_1}{\partial T_0} + Ku_1 \\ &= \left\{-2\dot{A}_n(\mathrm{i}\omega_n\phi_n + \gamma_0\phi_n') + \gamma_1\left[\frac{1}{2}(\omega_m - \omega_n)\bar{\phi}_m' + \mathrm{i}\gamma_0\bar{\phi}_m''\right]\bar{\phi}_m\mathrm{e}^{\mathrm{i}\sigma T_1} - \mathrm{i}\alpha\omega_n A_n\phi_n''''\right\}\mathrm{e}^{\mathrm{i}\omega_n T_0} \\ &\quad \left\{-2\dot{A}_m(\mathrm{i}\omega_m\phi_m + \gamma_0\phi_m') + \gamma_1\left[\frac{1}{2}(\omega_n - \omega_m)\bar{\phi}_n' + \mathrm{i}\gamma_0\bar{\phi}_n''\right]\bar{\phi}_n\mathrm{e}^{\mathrm{i}\sigma T_1} - \mathrm{i}\alpha\omega_m A_m\phi_m''''\right\}\mathrm{e}^{\mathrm{i}\omega_m T_0} \\ &\quad + cc + \mathrm{NST}, \end{aligned} \tag{18}$$

where the dot and the prime denote derivation with respect to the slow time variable T_1 and the dimensionless spatial variable x, respectively, and NST stands for the terms that will not bring secular terms into the solution. Eq. (18) has a bounded solution only if a solvability condition holds. The solvability condition demands the orthogonal relationships

$$\begin{aligned} &\left\langle -2\dot{A}_n(\mathrm{i}\omega_n\phi_n + \gamma_0\phi_n') + \gamma_1\left[\frac{1}{2}(\omega_m - \omega_n)\bar{\phi}_m' + \mathrm{i}\gamma_0\bar{\phi}_m''\right]\bar{\phi}_m\mathrm{e}^{\mathrm{i}\sigma T_1} - \mathrm{i}\alpha\omega_n A_n\phi_n'''', \phi_n\right\rangle = 0, \\ &\left\langle -2\dot{A}_m(\mathrm{i}\omega_m\phi_m + \gamma_0\phi_m') + \gamma_1\left[\frac{1}{2}(\omega_n - \omega_m)\bar{\phi}_n' + \mathrm{i}\gamma_0\bar{\phi}_n''\right]\bar{\phi}_n\mathrm{e}^{\mathrm{i}\sigma T_1} - \mathrm{i}\alpha\omega_m A_m\phi_m'''', \phi_m\right\rangle = 0, \end{aligned} \tag{19}$$

where the inner product is defined for complex functions on [0,1] as

$$\langle f, g\rangle = \int_0^1 f\bar{g}\,\mathrm{d}x. \tag{20}$$

Application of the distributive law of the inner product to Eq. (19) leads to

$$\begin{aligned} &\dot{A}_n + \alpha c_{nn}A_n + \gamma_1 d_{nm}\bar{A}_m\mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \\ &\dot{A}_m + \alpha c_{mm}A_m + \gamma_1 d_{mn}\bar{A}_n\mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \end{aligned} \tag{21}$$

where

$$\begin{aligned} c_{kk} &= \frac{\mathrm{i}\omega_k\int_0^1 \phi_k''''\bar{\phi}_k\,\mathrm{d}x}{2(\mathrm{i}\omega_k\int_0^1 \phi_k\bar{\phi}_k\,\mathrm{d}x + \gamma_0\int_0^1 \phi_k'\bar{\phi}_k\,\mathrm{d}x)} \quad (k = n, m), \\ d_{kj} &= -\frac{(\omega_j - \omega_k)\int_0^1 \bar{\phi}_j'\bar{\phi}_k\,\mathrm{d}x + 2\mathrm{i}\gamma_0\int_0^1 \bar{\phi}_j''\bar{\phi}_k\,\mathrm{d}x}{4(\mathrm{i}\omega_k\int_0^1 \phi_k\bar{\phi}_k\,\mathrm{d}x + \gamma_0\int_0^1 \phi_k'\bar{\phi}_k\,\mathrm{d}x)} \quad (k = n, m;\ j = m, n). \end{aligned} \tag{22}$$

These coefficients can be determined by the modal parameters calculated from Eq. (13), and are independent of parametric excitation due to the variation of axial speed.

The transformation

$$A_n(T_1) = B_n(T_1)e^{i\sigma T_1/2}, \quad A_m(T_1) = B_m(T_1)e^{i\sigma T_1/2} \tag{23}$$

changes Eq. (21) into an autonomous system

$$\begin{aligned} &\dot{B}_n + i\frac{\sigma}{2}B_n + \alpha c_{nn}B_n + \gamma_1 d_{nm}\bar{B}_m = 0, \\ &\dot{B}_m + i\frac{\sigma}{2}B_m + \alpha c_{mm}B_m + \gamma_1 d_{mn}\bar{B}_n = 0. \end{aligned} \tag{24}$$

Obviously, Eq. (24) (and thus Eq. (21) has a zero solution. Suppose that the non-zero solutions of Eq. (24) take the form

$$B_n = b_n e^{\lambda T_1}, \quad B_m = b_m e^{\bar{\lambda} T_1}, \tag{25}$$

where b_n and b_m are real coefficients, and λ is a complex to be determined. Substituting Eq. (25) into Eq. (24) and taking the complex conjugate of the second resulting equation yield

$$\begin{aligned} &\left(-\lambda - \frac{\sigma}{2}i - \alpha c_{nn}\right)b_n + \gamma_1 d_{nm}b_m = 0, \\ &\gamma_1 \bar{d}_{mn}b_n + \left(-\lambda + \frac{\sigma}{2}i - \alpha\bar{c}_{mm}\right)b_m = 0. \end{aligned} \tag{26}$$

Eq. (26), a set of homogeneous linear algebraic equations of b_n and b_m, has non-zero solutions if and only if its determinant of coefficient vanishes. Therefore,

$$\lambda^2 + \alpha(c_{nn} + c_{mm})\lambda + \left(\frac{\sigma}{2}i + \alpha c_{nn}\right)\left(-\frac{\sigma}{2}i + \alpha\bar{c}_{mm}\right) - \gamma_1^2 d_{nm}\bar{d}_{mn} = 0. \tag{27}$$

When λ has positive real part, the system is unstable.

Separate λ, c_{nn}, and c_{mm} into real and imaginary parts,

$$\lambda = \lambda^R + i\lambda^I, \quad c_{nn} = c_{nn}^R + ic_{nn}^I, \quad c_{mm} = c_{mm}^R + ic_{mm}^I. \tag{28}$$

Substituting Eq. (28) into Eq. (27) and separating the resulting equation into real and imaginary parts lead to

$$\begin{aligned} &\lambda^{R^2} - \lambda^{I^2} + \alpha(c_{nn}^R + c_{mm}^R)\lambda^R - \alpha(c_{nn}^I + c_{mm}^I)\lambda^I + \alpha^2 c_{nn}^R c_{mm}^R \\ &\quad + \left(\frac{\sigma}{2} + \alpha c_{nn}^I\right)\left(\frac{\sigma}{2} + \alpha c_{mm}^I\right) - \gamma_1^2\,\mathrm{Re}(d_{nm}\bar{d}_{mn}) = 0, \\ &2\lambda^R\lambda^I + \alpha(c_{nn}^I + c_{mm}^I)\lambda^R + \alpha(c_{nn}^R + c_{mm}^R)\lambda^I \\ &\quad + \alpha\left[c_{mm}^R\left(\frac{\sigma}{2} + \alpha c_{nn}^I\right) - c_{nn}^R\left(\frac{\sigma}{2} + \alpha c_{mm}^I\right)\right] - \gamma_1^2\,\mathrm{Im}(d_{nm}\bar{d}_{mn}) = 0. \end{aligned} \tag{29}$$

For $\alpha \neq 0$, Eq. (29) has the solution $\lambda^R = 0$ on the condition

$$\mathrm{Im}(d_{nm}\bar{d}_{mn}) = 0, \quad \mathrm{Re}(d_{nm}\bar{d}_{mn}) > 0, \quad \sigma = \pm\gamma_1\sqrt{\mathrm{Re}(d_{nm}\bar{d}_{mn})}. \tag{30}$$

For $\alpha \neq 0$, substituting $\lambda^{\mathrm{R}} = 0$ into Eq. (29) and eliminating λ^{I} in the resulting equation give

$$\left[\frac{\sigma}{2}(c_{nn}^{\mathrm{R}} - c_{mm}^{\mathrm{R}}) + \alpha(c_{nn}^{\mathrm{R}} c_{mm}^{\mathrm{I}} - c_{mm}^{\mathrm{R}} c_{nn}^{\mathrm{I}}) + \frac{\gamma_1^2}{\alpha}\,\mathrm{Im}(d_{nm}\bar{d}_{mn})\right]^2$$
$$+ (c_{nn}^{\mathrm{R}} + c_{mm}^{\mathrm{R}})(c_{nn}^{\mathrm{I}} + c_{mm}^{\mathrm{I}})\left[\frac{\sigma}{2}(c_{nn}^{\mathrm{R}} - c_{mm}^{\mathrm{R}}) + \alpha(c_{nn}^{\mathrm{R}} c_{mm}^{\mathrm{I}} - c_{mm}^{\mathrm{R}} c_{nn}^{\mathrm{I}}) + \frac{\gamma_1^2}{\alpha}\,\mathrm{Im}(d_{nm}\bar{d}_{mn})\right]$$
$$+ (c_{nn}^{\mathrm{R}} + c_{mm}^{\mathrm{R}})^2\left[\frac{\sigma^2}{4} + \frac{\sigma\alpha}{2}(c_{nn}^{\mathrm{I}} + c_{mm}^{\mathrm{I}}) + \alpha^2(c_{nn}^{\mathrm{R}} c_{mm}^{\mathrm{R}} + c_{nn}^{\mathrm{I}} c_{mm}^{\mathrm{I}}) + \gamma_1^2\,\mathrm{Re}(d_{nm}\bar{d}_{mn})\right] = 0. \quad (31)$$

Eq. (31) is the analytical expression of the stability boundary in summation parametric resonance.

If the variation frequency ω approaches two times of a natural frequency of system (13), principal parametric resonance may occur. Denote

$$\omega = 2\omega_n + \varepsilon\sigma. \quad (32)$$

Let $m = n$ in Eq. (31), then the resulting equation gives the stability boundary in nth principal parametric resonance. For $\alpha = 0$, the stability boundary is expressed by

$$\sigma = \pm\gamma_1 |d_{nn}|. \quad (33)$$

For $\alpha \neq 0$, the stability boundary is expressed by

$$\frac{\gamma_1^4}{\alpha^2}|d_{nn}|^4 + 4\,\frac{\gamma_1^2}{\alpha}\,c_{nn}^{\mathrm{R}} c_{nn}^{\mathrm{I}} |d_{nn}|^2 + 4c_{nn}^{\mathrm{R}^2}\left[\frac{\sigma^2}{4} + \sigma\alpha c_{nn}^{\mathrm{I}} + \alpha^2(c_{nn}^{\mathrm{R}^2} + c_{nn}^{\mathrm{I}^2}) + \gamma_1^2 |d_{nn}|^2\right] = 0, \quad (34)$$

where

$$c_{nn} = \frac{\mathrm{i}\omega_n \int_0^1 \phi_n'''' \bar{\phi}_n \,\mathrm{d}x}{2(\mathrm{i}\omega_n \int_0^1 \phi_n \bar{\phi}_n \,\mathrm{d}x + \gamma_0 \int_0^1 \phi_n' \bar{\phi}_n \,\mathrm{d}x)}, \quad d_{nn} = -\frac{2\mathrm{i}\gamma_0 \int_0^1 \bar{\phi}_n'' \bar{\phi}_n \,\mathrm{d}x}{4(\mathrm{i}\omega_n \int_0^1 \phi_n \bar{\phi}_n \,\mathrm{d}x + \gamma_0 \int_0^1 \phi_n' \bar{\phi}_n \,\mathrm{d}x)}. \quad (35)$$

If the variation frequency ω approaches the difference of any two natural frequencies of system (13), difference parametric resonance may occur. The stability in difference parametric resonance can be treated similarly. Denote

$$\omega = \omega_n - \omega_m + \varepsilon\sigma. \quad (36)$$

The stability boundaries are expressed by Eqs. (30) and (31), respectively, for $\alpha = 0$ and $\alpha \neq 0$, while the coefficients in them are given by

$$c_{nn} = \frac{\mathrm{i}\omega_n \int_0^1 \phi_n'''' \bar{\phi}_n \,\mathrm{d}x}{2(\mathrm{i}\omega_n \int_0^1 \phi_n \bar{\phi}_n \,\mathrm{d}x + \gamma_0 \int_0^1 \phi_n' \bar{\phi}_n \,\mathrm{d}x)}, \quad c_{mm} = \frac{\mathrm{i}\omega_m \int_0^1 \phi_m'''' \bar{\phi}_m \,\mathrm{d}x}{2(\mathrm{i}\omega_m \int_0^1 \phi_m \bar{\phi}_m \,\mathrm{d}x - \gamma_0 \int_0^1 \phi_m' \bar{\phi}_m \,\mathrm{d}x)},$$
$$d_{nm} = \frac{(\omega_m + \omega_n)\int_0^1 \bar{\phi}_m' \bar{\phi}_n \,\mathrm{d}x - 2\mathrm{i}\gamma_0 \int_0^1 \bar{\phi}_m'' \bar{\phi}_n \,\mathrm{d}x}{4(\mathrm{i}\omega_n \int_0^1 \phi_n \bar{\phi}_n \,\mathrm{d}x + \gamma_0 \int_0^1 \phi_n' \bar{\phi}_n \,\mathrm{d}x)},$$
$$d_{mn} = \frac{(\omega_n + \omega_m)\int_0^1 \bar{\phi}_n' \bar{\phi}_m \,\mathrm{d}x + 2\mathrm{i}\gamma_0 \int_0^1 \bar{\phi}_n'' \bar{\phi}_m \,\mathrm{d}x}{4(\mathrm{i}\omega_m \int_0^1 \phi_m \bar{\phi}_m \,\mathrm{d}x - \gamma_0 \int_0^1 \phi_m' \bar{\phi}_m \,\mathrm{d}x)}. \quad (37)$$

4. Stability boundaries of beams with simple supports

For an axially moving beam with simple supports, the boundary conditions in dimensionless form are

$$u(0,t) = u(1,t) = 0, \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=1} = 0. \tag{38}$$

Under the boundary conditions (38), the eigenfunction corresponding the kth natural frequency ω_k is [4]

$$\begin{aligned}\phi_k(x) = {} & e^{i\beta_{1k}x} - \frac{(\beta_{4k}^2 - \beta_{1k}^2)(e^{i\beta_{3k}} - e^{i\beta_{1k}})}{(\beta_{4k}^2 - \beta_{2k}^2)(e^{i\beta_{3k}} - e^{i\beta_{2k}})} e^{i\beta_{2k}x} - \frac{(\beta_{4k}^2 - \beta_{1k}^2)(e^{i\beta_{3k}} - e^{i\beta_{1k}})}{(\beta_{4n}^2 - \beta_{3n}^2)(e^{i\beta_{3k}} - e^{i\beta_{3k}})} e^{i\beta_{2k}x} \\ & - \left[1 - \frac{(\beta_{4k}^2 - \beta_{1k}^2)(e^{i\beta_{3n}} - e^{i\beta_{1k}})}{(\beta_{4k}^2 - \beta_{2k}^2)(e^{i\beta_{3k}} - e^{i\beta_{2k}})} - \frac{(\beta_{4k}^2 - \beta_{1k}^2)(e^{i\beta_{3k}} - e^{i\beta_{1k}})}{(\beta_{4k}^2 - \beta_{3k}^2)(e^{i\beta_{3k}} - e^{i\beta_{3k}})}\right] e^{i\beta_{4k}x},\end{aligned} \tag{39}$$

where β_{jk} $(j = 1, 2, 3, 4)$ and ω_k can be solved from the following algebraic equations:

$$\begin{aligned}& v_f^4(\beta_{1k} + \beta_{2k} + \beta_{3k} + \beta_{4k}) = \gamma_0^2 - 1, \\ & \beta_{1k}\beta_{2k} + \beta_{1k}\beta_{3k} + \beta_{1k}\beta_{4k} + \beta_{2k}\beta_{3k} + \beta_{2k}\beta_{4k} + \beta_{3k}\beta_{4k} = 0, \\ & v_f^4(\beta_{1k}\beta_{2k}\beta_{3k} + \beta_{1k}\beta_{2k}\beta_{4k} + \beta_{1k}\beta_{3k}\beta_{4k} + \beta_{2k}\beta_{3k}\beta_{4k}) = 2\gamma_0\omega_k, \\ & v_f^4\beta_{1k}\beta_{2k}\beta_{3k}\beta_{4k} = -\omega_k^2\end{aligned} \tag{40}$$

and the transcendental equation

$$\begin{aligned}& (\beta_{1k}^2 - \beta_{2k}^2)(\beta_{3k}^2 - \beta_{4k}^2)\lfloor e^{i(\beta_{1k}+\beta_{2k})} + e^{i(\beta_{3k}+\beta_{4k})}\rfloor + (\beta_{2k}^2 - \beta_{4k}^2)(\beta_{3k}^2 - \beta_{1k}^2)\lfloor e^{i(\beta_{1k}+\beta_{3k})} \\ & \quad + e^{i(\beta_{2k}+\beta_{4k})}\rfloor + (\beta_{1k}^2 - \beta_{4k}^2)(\beta_{2k}^2 - \beta_{3k}^2)[e^{i(\beta_{2k}+\beta_{3k})} - e^{i(\beta_{1k}+\beta_{4k})}] = 0.\end{aligned} \tag{41}$$

Consider an axially moving beam with $v_f = 0.8$ and $\gamma = 2.0$. The first two natural frequencies and coefficients in corresponding eigenfunctions (39), numerically solved from Eqs. (40) and (41), are $\omega_1 = 5.3692$, $\beta_{11} = 3.9906$, $\beta_{21} = -1.2424 + 2.4397i$, $\beta_{31} = -1.2424 - 2.4397i$, $\beta_{41} = -1.5058$ and $\omega_2 = 30.1200$, $\beta_{12} = 7.4497$, $\beta_{22} = -1.2497 + 6.0726i$, $\beta_{32} = -1.2497 - 6.0726i$, $\beta_{42} = -4.9503$.

In summation parametric resonance, Eq. (22) gives $c_{11} = 45.8597$, $c_{22} = 709.7023$, $d_{12} = 1.2427 + 0.7843i$, and $d_{21} = 0.2948 + 0.1860i$. In the case that c_{kk} is real, Eq. (22) reduces to

$$\left[\frac{\sigma}{2}(c_{nn}^R - c_{mm}^R)\right]^2 + (c_{nn}^R + c_{mm}^R)^2\left[\frac{\sigma^2}{4} + \alpha^2(c_{nn}^R c_{mm}^R) - \gamma_1^2 \operatorname{Re}(d_{nm}\bar{d}_{mn})\right] = 0. \tag{42}$$

Therefore, the instability region is given as

$$-2\sqrt{\frac{\gamma_1^2 \operatorname{Re}(d_{nm}\bar{d}_{mn}) - \alpha^2 c_{nn}^R c_{mm}^R}{1 + \kappa^2}} < \sigma < 2\sqrt{\frac{\gamma_1^2 \operatorname{Re}(d_{nm}\bar{d}_{mn}) - \alpha^2 c_{nn}^R c_{mm}^R}{1 + \kappa^2}}, \tag{43}$$

where

$$\kappa = \frac{c_{nn}^{R} - c_{mm}^{R}}{c_{nn}^{R} + c_{mm}^{R}}. \tag{44}$$

The instability region exists on the condition that $c_{nn}^{R} c_{mm}^{R}$ and $\mathrm{Re}(d_{nm}\bar{d}_{mn})$ have the same sign and the axial speed variation amplitude is large enough, namely,

$$\gamma_1 > \alpha\sqrt{\frac{c_{nn}^{R} c_{mm}^{R}}{\mathrm{Re}(d_{nm}\bar{d}_{mn})}}. \tag{45}$$

The stability boundaries for the summation resonance of first two modes in plane $\sigma - \gamma_1$ are shown in Fig. 1 for $\alpha = 0$, 0.0005, 0.001. The increasing viscosity coefficient makes the stability boundaries move towards the increasing direction of γ_1 in plane (ω, γ_1) and the instability regions become narrow. That is, the larger viscosity coefficient leads to the larger instability threshold of γ_1 for given σ, and the smaller instability range of σ for given γ_1.

In principal parametric resonance, Eq. (35) gives $d_{11} = -1.0456 + 1.1879\mathrm{i}$, $d_{22} = -0.4182 + 0.9776\mathrm{i}$. The instability region is

$$-2\sqrt{\gamma_1^2 |d_{nn}|^2 - \alpha^2 c_{nn}^{R^2}} < \sigma < 2\sqrt{\gamma_1^2 |d_{nn}|^2 - \alpha^2 c_{nn}^{R^2}}. \tag{46}$$

The instability region exists on the condition that the axial speed variation amplitude is beyond a critical value,

$$\gamma_1 > \frac{\alpha |c_{nn}^{R}|}{|d_{nn}|}. \tag{47}$$

The stability boundaries for the first and second principal resonance in plane $\sigma - \gamma_1$ are shown, respectively, in Fig. 2 for $\alpha = 0$, 0.02, 0.05 and Fig. 3 for $\alpha = 0$, 0.001, 0.002. In both cases, the increasing viscosity coefficient makes the stability boundaries move towards the increasing direction of γ_1 in plane (ω, γ_1) and the instability regions become narrow.

In difference parametric resonance, Eq. (37) gives $c_{11} = 45.8597$, $c_{22} = 741.7379$ $d_{12} = -3.6139 - 2.2809\mathrm{i}$, $d_{21} = 0.5997 + 0.6081\mathrm{i}$. For real c_{11} and c_{22}, the stability boundary is given

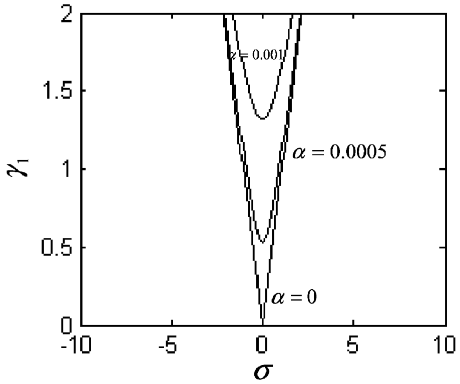

Fig. 1. The stability boundaries for the summation resonance of beams with simple supports.

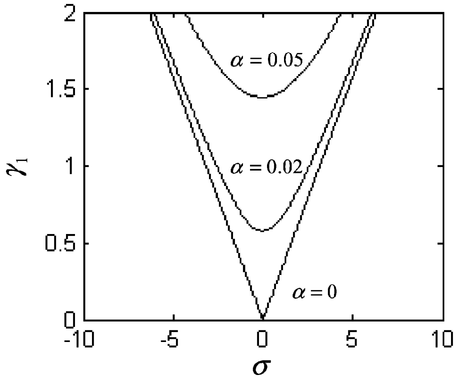

Fig. 2. The stability boundaries for the first principal resonance with simple supports.

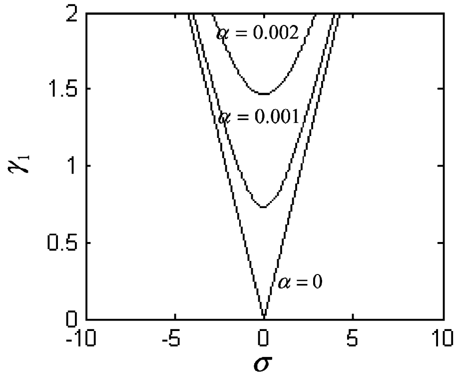

Fig. 3. The stability boundaries for the second principal resonance with simple supports.

by Eq. (42). In this example, $\mathrm{Re}(d_{nm}\bar{d}_{mn})$ is negative. Thus there is no instability region in the difference resonance.

To depict the stability boundaries in the same scale, the different viscosity coefficients are chosen in Figs. 1–3. These figures indicate that the stability boundary for the summation resonance is most sensitive to the change of the viscosity coefficient, while the stability boundary in the first principal resonance is most insensitive.

5. Stability boundaries of beams with fixed supports

For an axially moving beam with simple supports, the boundary conditions in dimensionless form are

$$u(0,t) = u(1,t) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0. \tag{48}$$

Under the boundary conditions (45), the eigenfunction corresponding the kth natural frequency ω_k is [5]

$$\phi_k(x) = e^{i\beta_{1k}x} - \frac{(\beta_{4k}-\beta_{1k})(e^{i\beta_{3k}}-e^{i\beta_{1k}})}{(\beta_{4k}-\beta_{2k})(e^{i\beta_{3k}}-e^{i\beta_{2k}})} e^{i\beta_{2k}x} - \frac{(\beta_{4k}-\beta_{1k})(e^{i\beta_{3k}}-e^{i\beta_{1k}})}{(\beta_{4n}-\beta_{3n})(e^{i\beta_{3k}}-e^{i\beta_{3k}})} e^{i\beta_{2k}x}$$
$$- \left[1 - \frac{(\beta_{4k}-\beta_{1k})(e^{i\beta_{3n}}-e^{i\beta_{1k}})}{(\beta_{4k}-\beta_{2k})(e^{i\beta_{3k}}-e^{i\beta_{2k}})} - \frac{(\beta_{4k}-\beta_{1k})(e^{i\beta_{3k}}-e^{i\beta_{1k}})}{(\beta_{4k}-\beta_{3k})(e^{i\beta_{3k}}-e^{i\beta_{3k}})}\right] e^{i\beta_{4k}x}, \quad (49)$$

where β_{jk} ($j = 1,2,3,4$) and ω_k can be solved from Eq. (40) and the following transcendental equation:

$$(\beta_{1k}-\beta_{2k})(\beta_{3k}-\beta_{4k})\lfloor e^{i(\beta_{1k}+\beta_{2k})} + e^{i(\beta_{3k}+\beta_{4k})}\rfloor + (\beta_{2k}-\beta_{4k})(\beta_{3k}-\beta_{1k})\lfloor e^{i(\beta_{1k}+\beta_{3k})}$$
$$+ e^{i(\beta_{2k}+\beta_{4k})}] + (\beta_{1k}-\beta_{4k})(\beta_{2k}-\beta_{3k})[e^{i(\beta_{2k}+\beta_{3k})} - e^{i(\beta_{1k}+\beta_{4k})}] = 0 \quad (50)$$

Consider an axially moving beam with $v_f = 0.8$ and $\gamma = 4.0$. The first two natural frequencies and coefficients in corresponding eigenfunctions (49), numerically solved from Eqs. (40) and (50), are $\omega_1 = 6.8647$, $\beta_{11} = 6.6676$, $\beta_{21} = -2.4953 + 2.5344i$, $\beta_{31} = -2.4953 - 2.5344i$, $\beta_{41} = -1.6771$ and $\omega_2 = 43.3456$, $\beta_{12} = 10.2236$, $\beta_{22} = -2.4997 + 6.9798i$, $\beta_{32} = -2.4997 - 6.9798i$, $\beta_{42} = -5.2241$.

In summation parametric resonance, Eq. (22) gives $c_{11} = 203.4929$, $c_{22} = 1893.0621$, $d_{12} = -0.1772 - 0.2642i$, and $d_{21} = -0.0601 - 0.0895i$. The stability boundaries in the summation resonance of first two modes in plane $\sigma - \gamma_1$ are illustrated in Fig. 4 for $\alpha = 0, 0.0005, 0.001$. In principal parametric resonance, Eq. (35) gives $d_{11} = 1.5272 - 0.6178i$, $d_{22} = 0.7776 - 0.7987i$. The stability boundaries for the first and second principal resonance in plane $\sigma - \gamma_1$ are illustrated, respectively, in Fig. 5 for $\alpha = 0, 0.005, 0.01$ and Fig. 6 for $\alpha = 0, 0.0005, 0.001$. In all figures, the instability regions draft towards the increasing direction of the amplitude with the increase of the viscosity coefficient. The stability boundary in the first principal resonance is less sensitive to the change of the viscosity coefficient. In difference parametric resonance, Eq. (37) gives $c_{11} = 203.4929$, $c_{22} = 483.0170$, $d_{12} = 2.1967 + 3.2696i$, $d_{21} = -4.0192 + 0.3636i$. There is no instability region in the difference resonance.

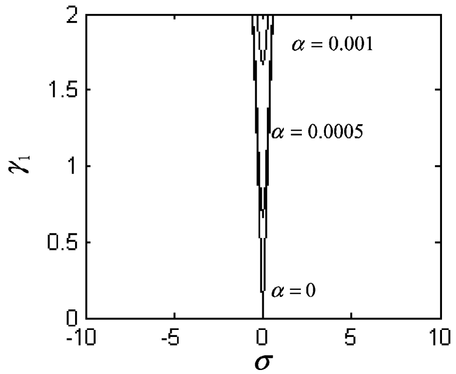

Fig. 4. The stability boundaries for the summation resonance of beams with fixed supports.

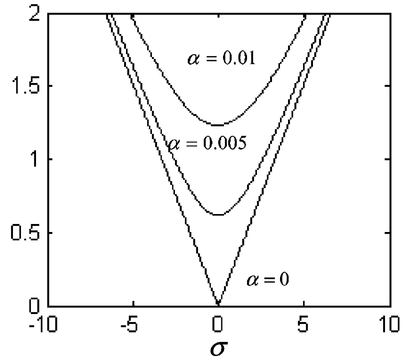

Fig. 5. The stability boundaries for the first principal resonance with fixed supports.

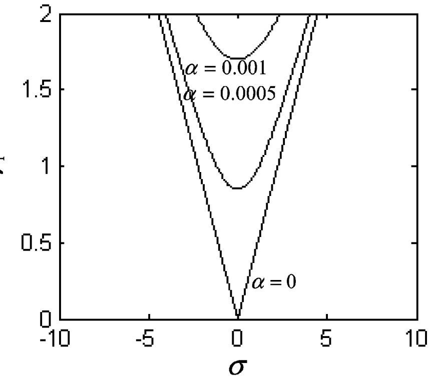

Fig. 6. The stability boundaries for the second principal resonance with fixed supports.

6. Conclusions

Transverse stability is studied for axially moving viscoelastic beams with the speed that is harmonically fluctuating about a constant mean value. Such a parametric vibration system can be cast into an autonomous continuous gyroscopic system under a small time dependent perturbation. The method of multiple scales is applied to a partial-differential equation governing the transverse parametric vibration. The stability boundary is derived from the solvability condition. Axially accelerating beams with simple supports and fixed supports are numerically investigated. Numerical results demonstrate that instability occurs if the axial speed fluctuation frequency is close to the sum of any two natural frequencies (summation parametric resonance) or two times of a natural frequency (principal parametric resonance) of the unperturbed system. A detuning parameter is used to quantify the deviation between the speed fluctuation frequency and the sum of two natural frequencies or the multiple of a natural frequency. The stability boundaries are numerically determined in the axial speed fluctuation detuning parameter–amplitude plane for varying viscosity coefficient. With the increase of the viscosity coefficient, the lager instability threshold of speed fluctuation amplitude becomes large for given detuning parameter, and the instability range of the detuning parameter becomes small for given speed fluctuation amplitude. In addition, the viscosity coefficient influents more on the stability boundary in higher order principal parametric resonance.

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