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Wave component analysis of energy flow in complex structures – Part I: a deterministic model

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Abstract

A wave-based method is developed for the analysis of vibrational energy flow in built-up structures. The structure is divided into a network of interconnected subsystems and its response to external forcing is expressed as the time-averaged energies of these subsystems. The flow of energy between subsystems is described in terms of generalised 'wave components'. These propagate perpendicularly to cross-sections of the structure—generally defined at the interfaces between subsystems and the junctions between subsystems—and have characteristic patterns of amplitude variation over the cross-sections which correspond to local eigenfunctions. Reflection and transmission of wave components is described in terms of two global scattering matrices, which are systematically constructed from local coefficients associated with individual subsystems and junctions. The method forms the deterministic basis for an alternative, statistical approach to the analysis of vibrations in complex, uncertain structures, which is described in two companion papers.

1. Introduction

The analysis of structural vibrations is most commonly carried out using the finite element method. This approach is limited in higher frequency applications, however, by the need to include sufficiently many elements to accurately model the shortest wavelength deformations of

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the structure. In these situations, the numerical error can be significant and the computational cost prohibitive.

Although alternative approaches are available, their application is generally limited to special types of structures. The direct-dynamic stiffness method of Langley [1] and the wave-based approach taken by Miller and von Flotow [2], and Beale and Accorsi [3], for example, are typically applied to frameworks of one-dimensional components such as rods and beams. In these methods, the dynamic properties of structural components are derived from the governing equations without approximation. Another alternative, statistical energy analysis (SEA), achieves a reduction in computational cost through a range of approximations. It provides estimates of the time- and ensemble-averaged response of the structure at relatively low spatial resolution [4,5], but can generally be applied only to structures in which the subsystems are reverberant, the modal overlap is high and the coupling between subsystems is weak.

The approach described in this paper can be applied to built-up structures involving two- and three-dimensional subsystems such as plates and acoustic cavities, and is in some sense a generalisation of the wave-based method of Miller and von Flotow [2]. It is deterministic and based on an assumption that the properties of the structure are known exactly. The approach also provides the basis for an alternative, statistical, wave-based analysis method for complex, uncertain structures, which is described in two companion papers [6,7]. The first of these deals with the extension of the deterministic approach to accommodate the uncertain geometric and material properties of complex structures by assuming that the structure is drawn from an ensemble of structures which differ randomly in detail. Expressions are derived for the statistics of responses over the ensemble. The second paper concerns the application of the statistical method to example structures of two coupled plates.

Broad features of the deterministic approach, including the division of the structure into connected subsystems and the use of power and subsystem energies as primary response variables, are described in Section 2. The concept of generalised, energy-bearing ‘wave components’, which are central to the approach, is introduced in Section 3. These components are defined at cross-sections of the structure in a way that ensures that the total energy flow at any cross-section is the sum of flows associated with individual wave components. The relationships between the amplitudes of wave components throughout the structure are described in terms of subsystem and junction scattering matrices. The properties of these matrices and details concerning the generation of wave components by externally applied forces are considered in Section 4. These are then used in Section 5 to find the input and junction powers associated with rain-on-the-roof excitation of the structure. Comparison of predicted responses with results derived by the finite element method are presented for an example two-plate structure in Section 6. Further details concerning the approach can be found in Ref. [8].

2. The energy flow model

The approach involves the notional division of the structure into subsystems and junctions, approximately in the manner which is familiar in SEA. Stationary, random external forces act on subsystems and the forces acting on different subsystems are assumed to be uncorrelated. The vibrational response of the structure is described in terms of the time-averaged subsystem energies and the energy flows through junctions.

An energy balance equation of the form

$$P_{\text{in},X} = P_{\text{junc},X} + P_{\text{diss},X} \tag{1}$$

can be written for each subsystem X, where $P_{\text{in},X}$ and $P_{\text{diss},X}$ are input and dissipated powers, respectively, and $P_{\text{junc},X}$ is the total power lost by subsystem X to neighbouring subsystems via junctions. It is assumed that no energy is dissipated in the junctions. To a good approximation, the power dissipated in the subsystem is given by [9]

$$P_{\text{diss},X} = \omega\eta_X E_X, \tag{2}$$

where E_X is the subsystem energy and η_X is the subsystem loss factor. It is assumed in this relationship that the kinetic and potential energies are approximately equal and that the subsystem is lightly damped. It follows from these equations that the subsystem energies can be found if the input and junction powers are known.

3. Wave components

The flow of energy through the structure can be described in terms of wave components which propagate and are reflected and transmitted at inhomogeneities, most often at boundaries and junctions. These flows are evaluated at 'cross-sections' of the structure, typically defined at the interfaces between subsystems and junctions, and take the form of a point, line or surface, as appropriate for the dimensionality of the subsystem and the type of wave motion. The total power P_{junc} lost by any subsystem to immediately neighbouring subsystems can be found from the amplitudes of the wave components at cross-sections of the subsystem at the junctions.

At low frequencies, the wavelengths of deformations at any cross-section are generally large compared with the dimensions of the cross-section. Motion over the cross-section can then be described by a single variable and the structure can be described locally as 'dynamically one-dimensional'. This is the situation, for example, for flexural waves in a thin beam or sound in a narrow acoustic duct.

At higher frequencies, the wavelengths are small compared with the dimensions of the cross-section and the wave field comprises the fundamental wave associated with one-dimensional propagation, together with waves which are reflected from structural boundaries as they propagate towards and away from the cross-section. The total field near the cross-section is then a superposition of wave components, which it is convenient to define in such a way that the total cross-sectional energy flow is the sum of flows associated with individual wave components. Components of this kind can be found for many cross-sections by the method of 'separation of variables' which involves factorisation of the wave equation in several independent spatial variables into two differential equations, each in fewer independent variables [10]. The two equations are associated, respectively, with variations of the field across the cross-section and normal to the cross-section.

3.1. A regular structure: wave components in a plate strip

Consider the example, illustrated in Fig. 1, of a thin, infinitely long, simply supported plate strip of width d. A line cross-section is defined perpendicular to the edges of the strip. The transverse displacement $w(x,y)$ of the plate, assuming a time dependence of the form $\exp(\mathrm{i}\omega t)$, is governed by

$$\nabla^4 w - k^4 w = 0, \tag{3}$$

where $k = \sqrt[4]{m\omega^2/B}$ is the flexural wavenumber, m is the mass per unit area of plate material and B is its flexural stiffness.

Following the method of separation of variables, the displacement can be written as

$$w(x,y) = F(x)\Psi(y), \tag{4}$$

where each factor depends on only one of the spatial coordinates. It then follows from Eq. (3) that

$$\mathrm{d}^2F/\mathrm{d}x^2 + k_x^2 F = 0 \tag{5}$$

and

$$\mathrm{d}^2\Psi/\mathrm{d}y^2 + k_y^2 \Psi = 0, \tag{6}$$

where the two constants of separation correspond to components of wavenumber which satisfy $k_x^2 + k_y^2 = \pm k^2$.

Since the displacements and bending moments vanish along the edges of the strip, Eq. (6) has eigenfunction solutions of the form

$$\Psi_i(y) = \sqrt{2/d}\,\sin(k_{yi}y), \quad i = 1, 2, \ldots, \tag{7}$$

with $k_{yi} = i\pi/d$. Solutions which satisfy $k_{xi}^2 + k_{yi}^2 = -k^2$ for any k_{yi}, or which satisfy $k_{xi}^2 + k_{yi}^2 = k^2$ with $k_{yi} > k$, correspond to near-field wave components which generally propagate little energy over distances greater than about a wavelength. Their contribution to the total field will be ignored here. The functions Ψ_i are orthogonal over the width of the strip and normalised so that $\int_0^d \Psi_i \Psi_j \,\mathrm{d}y = \delta_{ij}$.

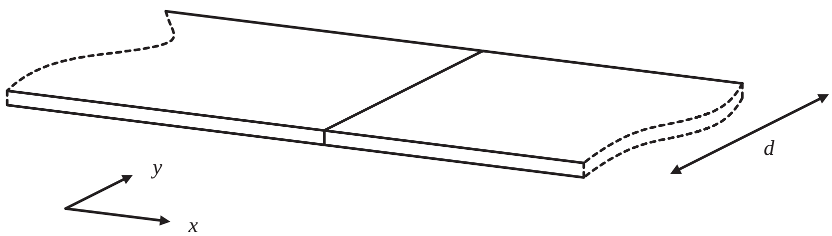

Fig. 1. Infinite, simply supported plate strip with line cross-section.

Eq. (5) has solutions of the form $e^{\pm ik_x x}$ so that the total field in the strip (without near-field contributions) can be expressed as a sum over wave components as

$$w(x, y) = \sqrt{2/d} \sum_{i=1}^{N} (w_i^+ e^{-ik_{xi}x} + w_i^- e^{ik_{xi}x}) \sin(k_{yi}y), \tag{8}$$

where $w_i^+$and w_i^- are, respectively, the amplitudes of positive- and negative-going wave components and N is the largest integer smaller than kd/π.

Since the physical properties of the strip are constant along its axis, the sinusoidal variation of a wave component across the strip given by Eq. (7) is preserved as it propagates, and no energy is transferred to other wave components. The time-averaged energy flow carried by a wave component which has displacement amplitude w_i is given by

$$P_i = Z_i \omega^2 w_i^2 / 2, \tag{9}$$

where

$$Z_i = 2Bk^2 k_{xi}/\omega \tag{10}$$

is the characteristic impedance of the wave component. Orthogonality of the shape functions Ψ_i ensures that the total energy flow through the line cross-section shown in Fig. 1 is the sum of powers associated with individual wave components. It is often convenient to refer to the 'power' amplitude a_i of a wave component, defined so that $P_i = |a_i|^2/2$.

The effects of material damping on wave component propagation can be taken into account by assuming a complex-valued bending stiffness $B(1 + i\eta)$ for the plate material. Since it is assumed that $\eta \ll 1$, the x-component of the wavenumber becomes

$$k_{xi} \approx k\sqrt{1 - (k_{yi}/k)^2 - i\eta/2}. \tag{11}$$

This quantity usually has a small negative imaginary part which leads to a slow reduction in the amplitude of the wave component as it propagates. The rate of attenuation with distance increases with increasing k_{yi} [11].

3.2. General structures

Extension of the results above to other types of 'regular' structure in which the wave field at the cross-sections is separable follows straightforwardly. Examples include structures involving acoustic cavities [12] and thick plates of uniform cross-section. The results also hold approximately if the structure is mildly irregular. In this case, the departure of structural surfaces from coordinate surfaces near the cross-section is slow, the identity of wave components are preserved as they propagate and no significant energy is transferred between them. Similar observations have been made in analyses of wave fields in a range of geophysical, ocean acoustic and other applications (see, for example, Ref. [13]). In the presence of strong irregularity near the cross-section, however, it can no longer be assumed that the total energy flow there is the sum of the flows of individual wave components.

It is assumed in the remaining discussion that irregularity at cross-sections is small, and that cross-sections are sufficiently well separated that any energy flow associated with interactions between near-fields can be neglected.

4. Wave component scattering and excitation

The factors which determine the amplitudes of wave components at any cross-section can be broadly divided into those associated with the scattering properties of the structure and those concerned with the nature and distribution of the externally applied forces. Each will be considered in turn.

4.1. Scattering

Energy is lost from any given subsystem either by the effects of damping or by transmission via junctions to adjacent subsystems. The magnitude of the power lost by transmission is the sum of the energy flows through the cross-sections at the boundaries between the subsystem and adjacent junctions. These flows can be found from the amplitudes of wave components emerging from the subsystem which, if there is no external forcing of the subsystem, can be found in turn from a knowledge of the amplitudes of wave components incident from outside and the reflection and transmission coefficients which characterise the scattering properties of the subsystem.

4.1.1. Subsystem scattering matrices

Consider, for example, the subsystem shown in Fig. 2, which has boundaries with the remaining structure at cross-sections labelled A and B. Let the power amplitudes of the wave components incident on and emerging from the subsystem at A and associated with the shape function $\Psi_{A,i}$ be a_i^- and a_i^+, respectively. The corresponding amplitudes associated with $\Psi_{B,i}$ at B are b_i^- and b_i^+.

In general, energy entering the subsystem at A can be reflected back to A, be transmitted to B or be dissipated in the subsystem. When wave components are incident only at A, the amplitudes of components emerging at A and B can be expressed as $\mathbf{a}^+ = \mathbf{R}_A\mathbf{a}^-$ and $\mathbf{b}^+ = \mathbf{T}_{BA}\mathbf{a}^-$, respectively, where $\mathbf{a}^- = [a_i^-]$, and $\mathbf{R}_A$ and $\mathbf{T}_{BA}$ are matrices of subsystem reflection and transmission

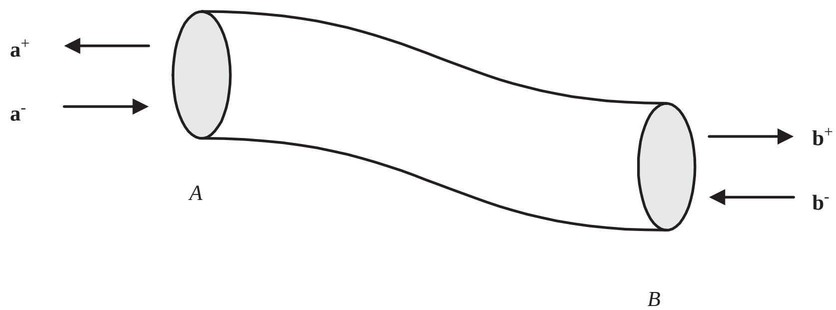

Fig. 2. Subsystem with cross-sections at A and B.

coefficients. Analogous matrices, $\mathbf{R}_B$ and $\mathbf{T}_{AB}$, can be defined for wave components incident on the subsystem at the cross-section B.

For the more general situation in which wave components are incident from both A and B, it is convenient to group the amplitudes of incident and emerging wave components at the two cross-sections together and to define a square scattering matrix for the subsystem so that

$$\begin{bmatrix} \mathbf{a}^+ \\ \mathbf{b}^+ \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{a}^- \\ \mathbf{b}^- \end{bmatrix}, \tag{12}$$

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{R}_A & \mathbf{T}_{AB} \\ \mathbf{T}_{BA} & \mathbf{R}_B \end{bmatrix}. \tag{13}$$

Corresponding matrices for subsystems involving greater numbers of cross-sections can be accommodated straightforwardly by augmentation of $\mathbf{S}$. It follows from the principle of reciprocity that $\mathbf{S}$ is always symmetric and, for subsystems without damping, that $\mathbf{S}$ is also unitary. If damping is present, $\mathbf{S}$ is no longer unitary and its singular values are less than unity.

4.1.2. Junction scattering matrices

The scattering of wave components incident on junctions can similarly be described in terms of a junction scattering matrix $\mathbf{T}$. In this case, for example,

$$\begin{bmatrix} \mathbf{c}^- \\ \mathbf{d}^- \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{c}^+ \\ \mathbf{d}^+ \end{bmatrix}. \tag{14}$$

(The superscripts '$-$' and '$+$' will be used throughout this paper to denote to wave components incident on and emerging from subsystems, respectively. The convention is reversed for wave components incident on and emerging from junctions.) Since junctions are assumed to be conservative, $\mathbf{T}$ is both symmetric and unitary.

4.1.3. Global scattering matrices

A global description of wave component reflection and transmission throughout the structure can be found by augmenting vectors of wave amplitudes for incident and emerging wave components at *all* cross-sections. The corresponding global subsystem and junction scattering matrices can then be constructed for the entire structure from the entries of the scattering matrices of individual subsystems and junctions.

Consider, for example, the coupled three-plate structure shown in Fig. 3. Cross-sections between the plates and the two junctions are numbered 1–4. An alternative representation of the structure is shown in Fig. 4, which has been derived from the original by re-arrangement of junctions and subsystems to the left and right, respectively, of a 'global cross-section' made up of all local cross-sections. Vectors of wave component amplitudes are defined for each subsystem or junction in terms of the vectors of amplitudes at cross-sections so that, for example,

$$\mathbf{a}_A^+ = \mathbf{a}_1^+, \quad \mathbf{a}_B^+ = \begin{bmatrix} \mathbf{a}_2^+ \\ \mathbf{a}_3^+ \end{bmatrix} \quad \text{and} \quad \mathbf{a}_C^+ = \mathbf{a}_4^+, \tag{15}$$

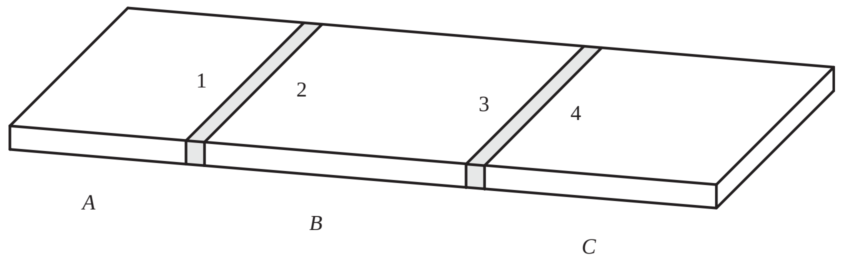

Fig. 3. Three inline plates. Junctions are shown shaded.

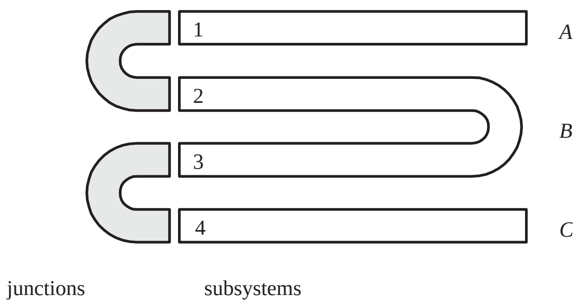

Fig. 4. Global junction–subsystem representation of three inline plate structure.

where $\mathbf{a}_i^+$ is the amplitude of the component emerging from the subsystem at cross-section i. Similar definitions can be written for $\mathbf{a}_A^-$, $\mathbf{a}_B^-$ and $\mathbf{a}_C^-$.

Two global vectors are then constructed, following the ordering suggested by Fig. 4, for the two sets of wave components:

$$\mathbf{a}^+ = \begin{bmatrix} \mathbf{a}_A^+ \\ \mathbf{a}_B^+ \\ \mathbf{a}_C^+ \end{bmatrix} \quad \text{and} \quad \mathbf{a}^- = \begin{bmatrix} \mathbf{a}_A^- \\ \mathbf{a}_B^- \\ \mathbf{a}_C^- \end{bmatrix}. \tag{16}$$

The construction of global subsystem and junction scattering matrices then follows naturally by blockwise augmentation of the individual scattering matrices. Subsystem scattering matrices for the present structure are given by

$$\mathbf{S}_A = \mathbf{S}_{11}, \quad \mathbf{S}_B = \begin{bmatrix} \mathbf{S}_{22} & \mathbf{S}_{23} \\ \mathbf{S}_{23}^{\mathrm{T}} & \mathbf{S}_{33} \end{bmatrix} \quad \text{and} \quad \mathbf{S}_C = \mathbf{S}_{44}, \tag{17}$$

where $\mathbf{S}_{ij}$ denotes the scattering matrix for wave components travelling through the subsystem from cross-section j to cross-section i. Because the ordering of cross-sections in Fig. 4 is according to their association with subsystems, the global subsystem scattering matrix is assembled by placing individual subsystem matrices blockwise along the diagonal to give

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_A & & \\ & \mathbf{S}_B & \\ & & \mathbf{S}_C \end{bmatrix}. \tag{18}$$

(Matrix entries not shown explicitly are equal to zero.) In global terms, and ignoring, for the moment, wave components generated by the action of external forces on the subsystems, the amplitudes of wave components emerging from all subsystems are given in terms of incident wave component amplitudes by

$$\mathbf{a}^+ = \mathbf{S}\mathbf{a}^-. \tag{19}$$

The global junction scattering matrix can be found by inspection of Fig. 4, and is given by

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & & \\ \mathbf{T}_{12}^{\mathrm{T}} & \mathbf{T}_{22} & & \\ & & \mathbf{T}_{33} & \mathbf{T}_{34} \\ & & \mathbf{T}_{34}^{\mathrm{T}} & \mathbf{T}_{44} \end{bmatrix}. \tag{20}$$

Since no external forces act on the junctions, the amplitudes of wave components emerging from junctions are given by

$$\mathbf{a}^- = \mathbf{T}\mathbf{a}^+. \tag{21}$$

It may be noted that the global junction scattering matrix $\mathbf{T}$ represents the process of wave component redistribution which occurs in the junctions, 'as seen' from the subsystems, and that the global subsystem scattering matrix $\mathbf{S}$ represents the corresponding process in the subsystems 'as seen' from the junctions.

4.2. Excitation

4.2.1. Excitation wave components

External forces on a given subsystem generate wave components in addition to those which enter the subsystem via junctions. In this case, Eq. (19) takes the form

$$\mathbf{a}^+ = \mathbf{S}\mathbf{a}^- + \mathbf{e}, \tag{22}$$

where $\mathbf{a}^-$ are the amplitudes of wave components incident from outside and $\mathbf{e}$ are the amplitudes of directly excited wave components. Eq. (22) holds both for local subsystem scattering matrices and for the corresponding global matrix. For the three-plate structure, for example, the global vector of excitation wave component amplitudes can be defined in terms of the corresponding subsystem vectors as

$$\mathbf{e} = \begin{bmatrix} \mathbf{e}_A \\ \mathbf{e}_B \\ \mathbf{e}_C \end{bmatrix}, \tag{23}$$

where the local vectors are given by

$$\mathbf{e}_A = \mathbf{e}_1, \quad \mathbf{e}_B = \begin{bmatrix} \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_C = \mathbf{e}_4. \tag{24}$$

Together, Eqs. (21) and (22) imply that

$$\mathbf{a}^+ = (\mathbf{I} - \mathbf{ST})^{-1}\mathbf{e}, \tag{25}$$

where $\mathbf{I}$ is the identity matrix. The role of the matrix product $\mathbf{ST}$ in characterising the dynamic properties of the structure can be illustrated by noting that wave components, which are incident on the junctions of the structure with amplitudes $\mathbf{e}$, pass through the junctions and the subsystems in a single circuit of the structure to return to their point of origin with amplitudes $\mathbf{STe}$. Wave components which have completed n circuits of the structure return with amplitudes $(\mathbf{ST})^n\mathbf{e}$. Re-writing Eq. (25) as

$$\mathbf{a}^+ = [\mathbf{I} + (\mathbf{ST}) + (\mathbf{ST})^2 + (\mathbf{ST})^3 + \ldots]\mathbf{e} \tag{26}$$

then demonstrates that the wave components which emerge from subsystems can be interpreted as the cumulative result of these circulating components.

It will be convenient in later discussion to consider the decomposition

$$(\mathbf{I} - \mathbf{ST})^{-1} = \mathbf{A}/\varDelta \tag{27}$$

in which $\mathbf{A}$ is the adjoint of $(\mathbf{I} - \mathbf{ST})$ and $\varDelta$ is its determinant.

4.2.2. Excitation by a point force

The response of a structure to an externally applied time-harmonic point force is of fundamental interest in determining its response to more general force distributions. A point force generates wave components which propagate away from the ‘forcing cross-section’, which is defined to include the point of application. For the case of a point force of amplitude F applied at location (x_0, y_0) on a uniform plate strip, for example, a line cross-section is defined along the line $x = x_0$. From the requirements of continuity of displacement and slope and of equilibrium of shear forces and moments along the line, the power amplitudes of the generated wave components are found to be [8]

$$q_i = \frac{F}{2\sqrt{Z_i}}\, \Psi_i(y_0). \tag{28}$$

The vector of excitation amplitudes at the cross-sections of the excited subsystem can be expressed in terms of the amplitudes $\mathbf{q}$ of wave components injected at the forcing location by

$$\mathbf{e} = \mathbf{Eq}, \tag{29}$$

where the matrix $\mathbf{E}$ characterises the propagation path between the cross-section at the forcing location and the cross-sections at neighbouring junctions. When the forcing cross-section divides the subsystem into two parts,

$$\mathbf{E} = \begin{bmatrix} \mathbf{K}_l \\ \mathbf{K}_r \end{bmatrix}, \tag{30}$$

where $\mathbf{K}_l$ and $\mathbf{K}_r$ are transmission matrices associated with wave components travelling from the forcing cross-section to junctions to the left and right, respectively.

In particular, for a uniform plate strip subsystem connected to the rest of the structure at each of its two ends, $\mathbf{K}_l$ and $\mathbf{K}_r$ are diagonal with

$$\mathbf{K}_l = \mathrm{diag}[\mathrm{e}^{-\mathrm{i}k_{xi}x}\mathrm{e}^{-\mu_i x/2l}] \tag{31}$$

and

$$\mathbf{K}_r = \mathrm{diag}[\mathrm{e}^{-\mathrm{i}k_{xi}(l-x)}\mathrm{e}^{-\mu_i(1-x/l)/2}], \tag{32}$$

where l is the length of the subsystem, x is the distance between the forcing location and the junction on the left, and $\mu_i = \mu_0/\sqrt{1-(k_{yi}/k)^2}$ is an attenuation parameter with $\mu_0 = kl\eta/2$. If the plate strip subsystem is connected to the rest of the structure only by its end on the left, say, then

$$\mathbf{E} = \mathbf{K}_l[\mathbf{I} + \mathbf{K}_r\mathbf{R}_r\mathbf{K}_r], \tag{33}$$

where $\mathbf{R}_r$ is the reflection matrix associated with the boundary on the right. The subsystem scattering matrices for strips connected at both ends and at the left end only are, respectively,

$$\mathbf{S} = \begin{bmatrix} & \mathbf{K}_l\mathbf{K}_r \\ \mathbf{K}_l\mathbf{K}_r & \end{bmatrix} \tag{34}$$

and

$$\mathbf{S} = \mathbf{K}_l\mathbf{K}_r\mathbf{R}_r\mathbf{K}_r\mathbf{K}_l. \tag{35}$$

5. Global wave component energy flows

As described in Section 2, response quantities such as the subsystem energies can be found directly from the input powers and the energy flows through junctions. These powers are derived in this section in terms of the external forces applied to the structure and the properties of the structure characterised by the scattering matrices $\mathbf{S}$ and $\mathbf{T}$.

5.1. Input power

The time-averaged input power delivered to a subsystem by a time-harmonic point force of amplitude F is given by $P_{\mathrm{in}} = \mathrm{Re}\{F^* v\}/2$, where v is the velocity response at the driving point. The velocity is given in terms of the vector $\mathbf{d}$ of power amplitudes of wave component responses by

$$v = \mathbf{\Psi}^{\mathrm{T}}(y_0)\mathbf{Z}^{-1/2}\mathbf{d}, \tag{36}$$

where $\mathbf{\Psi}(y_0)$ is the vector of wave component shape functions evaluated at y_0, the excitation location on the forcing cross-section, and $\mathbf{Z}^{-1/2} = \mathrm{diag}\left[1/\sqrt{Z_i}\right]$. In completely general circumstances, $\mathbf{d}$ comprises the direct contribution $\mathbf{q}$ of the injected wave components (given by Eq. (28)), together with the contributions of components which have been reflected either within the subsystem or from more distant parts of the structure outside the subsystem [8].

If the subsystem is regular, reflection occurs only at the boundaries of the subsystem and

$$\mathbf{d} = [\mathbf{E}^{\mathrm{T}}\mathbf{C}/\Delta + \mathbf{I}]\mathbf{q}, \tag{37}$$

where $\mathbf{E}$ is the matrix introduced in Section 4.2 and $\mathbf{C}/\Delta$ is the square, symmetric sub-matrix of $\mathbf{T}(\mathbf{I}-\mathbf{ST})^{-1}$ corresponding to wave components defined at cross-sections on the boundaries of the subsystem.

The input power is then given by

$$P_{\mathrm{in}} = F^2\mathbf{\Psi}^{\mathrm{T}}(y_0)\mathbf{Z}^{-1/2}\,\mathrm{Re}\{\mathbf{E}^{\mathrm{T}}\mathbf{CE}/\Delta + \mathbf{I}\}\mathbf{Z}^{-1/2}\mathbf{\Psi}(y_0)/4, \tag{38}$$

where the matrix $\mathbf{E}^{\mathrm{T}}\mathbf{CE}/\Delta$ is the effective reflection matrix for the whole structure, 'as seen' from the forcing cross-section. The magnitude of its contribution depends strongly on the magnitude of the determinant Δ, particularly if Δ is close to zero. Maxima with respect to frequency in the magnitude of $1/\Delta$ correspond to resonance frequencies of the structure, and the magnitudes of these maxima are strongly determined by the level of damping in the structure.

In more general irregular subsystems, the situation may be complicated by significant reflection at inhomogeneities within the directly excited subsystem. The input power may also be large at frequencies corresponding to resonances which involve phase closure over paths entirely within that subsystem.

5.1.1. Rain-on-the-roof

The response of a structure to point forcing at high frequencies can be very sensitive to the location of the driving point. Since this location can often not be specified exactly, it is common to consider the average of the responses to excitation at each possible forcing location on the subsystem. This corresponds to 'rain-on-the-roof' excitation.

The input power which results from rain-on-the-roof excitation can be found for regular subsystems from Eq. (38) by averaging, firstly, over point force locations y_0 in the forcing cross-section, and then over all cross-section positions x_0. The first average is given by

$$\langle P_{\mathrm{in}}\rangle_{y_0} = F^2\,\mathrm{trace}(\mathbf{Z}^{-1/2}\,\mathrm{Re}\{\mathbf{E}^{\mathrm{T}}\mathbf{CE}/\Delta + \mathbf{I}\}\mathbf{Z}^{-1/2})/4S, \tag{39}$$

where S is the length or surface area of the cross-section at x_0 as appropriate, and trace() denotes the trace of the matrix argument. It can be shown that, with increasing subsystem width or length, this power oscillates about

$$\langle P_{\mathrm{in}}\rangle_{\infty} = F^2\sum_{i=1}^{N} 1/(4SZ_i). \tag{40}$$

This corresponds also to the input power that would be delivered to an infinitely extended subsystem. For a uniform plate strip, it takes the value

$$\langle P_{\mathrm{in}} \rangle_\infty = \frac{F^2 \omega}{16 B k^2}. \tag{41}$$

5.2. Junction power

5.2.1. Incident powers at junctions

The total energy flow through any junction is the difference between the powers incident on and transmitted from the junction. The incident power can be written in terms of the amplitudes of incident wave components. The amplitudes of wave components incident on all junctions, given by Eq. (25), can be written as

$$\mathbf{a}^+ = \mathbf{A}\mathbf{e}/\Delta, \tag{42}$$

where, according to the convention described earlier, the entries of $\mathbf{a}^+$ are ordered blockwise in a way that reflects their association with particular subsystems and where $\mathbf{A}$ and Δ are defined in Eq. (27). For the particular group of wave components incident from subsystem X, a sub-vector of amplitudes can be defined which has the form

$$\mathbf{a}_X^+ = \mathbf{A}_{X\bullet}\mathbf{e}/\Delta, \tag{43}$$

where $\mathbf{A}_{X\bullet}$ is used to denote the rows of $\mathbf{A}$ corresponding to wave components in X.

The vector of wave component powers incident on all junctions of subsystem X is then given by

$$\mathbf{P}_{\mathrm{inc},X} = \mathrm{diag}(\mathbf{A}_{X\bullet}\mathbf{e}\mathbf{e}^{\mathrm{H}}\mathbf{A}_{X\bullet}^{\mathrm{H}})/2|\Delta|^2, \tag{44}$$

where diag() denotes the vector formed from the diagonal entries of the matrix argument. If, in addition, only subsystem Y is excited directly, so that the vector of excitation amplitudes is zero everywhere except for a block $\mathbf{e}_Y$, this incident power becomes

$$\mathbf{P}_{\mathrm{inc},XY} = \mathrm{diag}(\mathbf{A}_{XY}\mathbf{e}_Y\mathbf{e}_Y^{\mathrm{H}}\mathbf{A}_{XY}^{\mathrm{H}})/2|\Delta|^2, \tag{45}$$

where $\mathbf{A}_{XY}$ is the sub-matrix made up of the rows of $\mathbf{A}$ which correspond to wave components in X and the columns which correspond to wave components in Y. The 'direct' power associated with excitation wave components for point force excitation is found from Eqs. (28) and (29) to be given by

$$\mathbf{e}_Y\mathbf{e}_Y^{\mathrm{H}}/2 = F^2\mathbf{E}\mathbf{Z}^{-1/2}\mathbf{\Psi}(y)\mathbf{\Psi}^{\mathrm{T}}(y)\mathbf{Z}^{-1/2}\mathbf{E}^{\mathrm{H}}/8. \tag{46}$$

The magnitude of the incident power given in Eq. (45) depends strongly on the magnitude of the determinant Δ and may be large for structures at resonance.

As before, the value of $\mathbf{e}_Y\mathbf{e}_Y^{\mathrm{H}}$ corresponding to rain-on-the-roof excitation can be found by averaging Eq. (46) over all point force locations. For the plate strip subsystem connected to the rest of the structure at both ends

$$\langle \mathbf{e}_Y\mathbf{e}_Y^{\mathrm{H}} \rangle_{\mathrm{rain}} = \frac{F^2}{4S}\begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}, \tag{47}$$

where $\mathbf{D} = \mathrm{diag}[(1 - \mathrm{e}^{-\mu_i})/Z_i\mu_i]$. For the plate strip subsystem connected to the rest of the structure at one end only

$$\langle \mathbf{e}_Y \mathbf{e}_Y^{\mathrm{H}} \rangle_{\mathrm{rain}} = F^2 \left(\mathbf{D} + \mathrm{diag}\left[\mathrm{e}^{-\mu_i} \sum_j |r_{ij}|^2 (1 - \mathrm{e}^{-\mu_j})/Z_j\mu_j \right] \right) \Big/ 4S, \tag{48}$$

where $\mathbf{R}_r = [r_{ij}]$ is the reflection matrix associated with the end of the subsystem opposite the junction. The first term in this equation corresponds to power associated with wave components which have travelled directly from the forcing cross-section to the junction, while the second is associated with those which have been reflected at the end of the subsystem.

5.2.2. Junction powers

The total energy flow $P_{\mathrm{junc},X}$ from subsystem X through junctions is the difference between the powers carried by incident and emerging wave components at all the cross-sections in X, and is given by

$$P_{\mathrm{junc},X} = [\mathbf{a}_X^{+\mathrm{H}} \mathbf{a}_X^{+} - \mathbf{a}_X^{-\mathrm{H}} \mathbf{a}_X^{-}]/2. \tag{49}$$

Since the sum of junction powers over every subsystem in the structure is zero and since the excitations in different subsystems are uncorrelated, it is sufficient to consider only the junction powers for subsystems which are not themselves directly driven by external forces. For each such subsystem, $\mathbf{e}_X = 0$ and $\mathbf{a}_X^{+} = \mathbf{S}_X \mathbf{a}_X^{-}$, where $\mathbf{S}_X$ is the subsystem scattering matrix, so that

$$P_{\mathrm{junc},X} = \mathbf{a}_X^{+\mathrm{H}} [\mathbf{I} - (\mathbf{S}_X \mathbf{S}_X^{\mathrm{H}})^{-1}] \mathbf{a}_X^{+}/2. \tag{50}$$

If there is no damping in the subsystem, $\mathbf{S}_X \mathbf{S}_X^{\mathrm{H}} = \mathbf{I}$, and the junction power $P_{\mathrm{junc},X}$ is zero. In practice, off-diagonal entries of the product $(\mathbf{S}_X \mathbf{S}_X^{\mathrm{H}})^{-1}$ are usually negligibly small compared to unity, even for relatively heavily damped subsystems. By defining a diagonal matrix $\mathbf{D}_X^2$ as that obtained by setting off-diagonal entries of $(\mathbf{S}_X \mathbf{S}_X^{\mathrm{H}})^{-1}$ to zero, the total junction power can be written to good approximation as

$$P_{\mathrm{junc},X} \approx \mathbf{P}_{\mathrm{inc},X}^{\mathrm{T}} \, \mathrm{diag}(\mathbf{I} - \mathbf{D}_X^2). \tag{51}$$

5.3. Summary

The discussion above may be summarised as follows. The input power associated with rain-on-the-roof excitation of a subsystem is found by averaging the input power due to a harmonic point force (Eq. (38)) over all locations on a 'forcing cross-section', and then by averaging over all such cross-sections in the directly excited subsystem. A similar procedure is used to obtain the power, due to rain-on-the-roof excitation, which is incident on all the junctions of any given subsystem. For an indirectly excited subsystem, the junction power can be expressed, as in Eq. (51), in terms of the incident power and the subsystem reflection coefficients. The magnitude of the junction power in the directly excited subsystem is the sum of junction powers in all the other subsystems. Eq. (1) then allows the subsystem energies to be found from the power dissipated in each subsystem, which is in turn found from the junction and input powers via Eq. (2).

6. Example

The approach presented in this paper is illustrated below for the case of a simple two-plate structure. A comparison is made between predictions of the response of the structure found using the present approach and that derived from a finite element model.

6.1. System model

The structure, shown in Fig. 5, comprises two coupled plates which are uniform and rectangular and have the nominal properties given in Table 1. The plates are rigidly joined together, and out-of-plane motion along the coupling is constrained by a line translational spring of stiffness $10^8\ \mathrm{N/m^2}$, which extends over part of the width of the plates. Outside edges of the structure are simply supported. In-plane motion of the plates is neglected.

6.2. Wave components and wave component scattering

Flexural waves are generated in the structure by time-harmonic rain-on-the-roof forcing applied to plate A. The plates then act as plate-strip waveguides of the kind described in Section 3.1, where the wave field is expressed in terms of wave components obtained by Fourier decomposition along the junction between the plates.

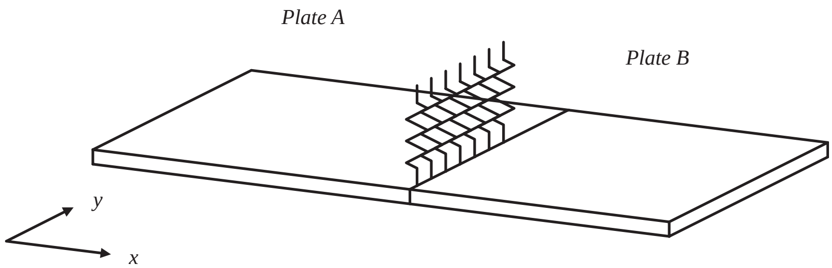

Fig. 5. Rectangular plate structure with line translational spring.

Table 1
Nominal plate properties

Property	Nominal value
Length, plate A	840 mm
Length, plate B	680 mm
Width	520 mm
Thickness	1.0 mm
Density	$8000\ \mathrm{kg/m^3}$
Young's modulus	$2 \times 10^{11}\ \mathrm{N/m^2}$
Poisson's ratio	0.3
Loss factor	0.005

A global junction–subsystem representation of the structure is shown in Fig. 6. If wave component transmission and reflection at the junction is described by the matrices $\mathbf{T}_{AB}$, $\mathbf{R}_A$ and $\mathbf{R}_B$ (these are derived explicitly for this structure in Ref. [8]), a global junction scattering matrix can be constructed, which has the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{R}_A & \mathbf{T}_{AB} \\ \mathbf{T}_{AB}^{\mathrm{T}} & \mathbf{R}_B \end{bmatrix}. \tag{52}$$

It follows from the physical symmetry of the junction that $\mathbf{T}_{AB}$ is square and symmetric and that $\mathbf{R}_A = \mathbf{R}_B$.

The scattering properties of the junction vary considerably with the length of the region over which the spring is attached to the plates. If the spring spans their full width, the structure is uniform over its width and is separable in the sense described earlier. Wave components are then confined to dynamically one-dimensional waveguides, between which there is no exchange of energy.

If the spring spans only part of the width of the plates, the structure is no longer uniform and energy can be transferred between wave components of different trace wavenumber. The degree of non-uniformity of the structure can be crudely controlled by varying the length of the spring. Fig. 7 shows an example of the scattering matrix $\mathbf{T}$ for a regular junction in which the spring spans the full width of the plates, and for an irregular junction in which the spring spans 0.61 of the width, both at 200 Hz. There are four propagating wave components in each case. The figure clearly shows that the four sub-matrices $\mathbf{R}_A$, $\mathbf{T}_{AB}$, $\mathbf{T}_{AB}^{\mathrm{T}}$ and $\mathbf{R}_B$ are diagonal for the case of the

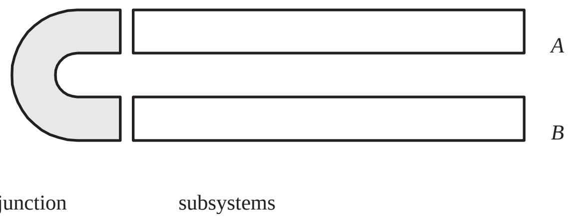

Fig. 6. Global junction–subsystem representation of two plate structure shown in Fig. 5.

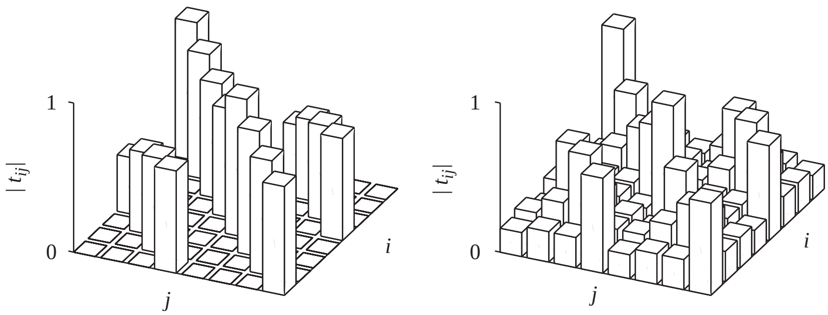

Fig. 7. Magnitudes of the entries t_{ij} of the junction scattering matrices for regular (left) and irregular (right) junctions. Entry t_{11} is drawn in the lower-most corner.

regular junction, and that there are significant off-diagonal entries for the case of the irregular junction.

The global subsystem scattering matrix is given by

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_B \end{bmatrix}, \tag{53}$$

where, since the plates are rectangular and uniform, $\mathbf{S}_A$ and $\mathbf{S}_B$ are diagonal, local scattering matrices of the form given by Eq. (35).

6.3. Subsystem energies

Fig. 8 shows estimates of the energy levels of the plates in a structure incorporating the irregular junction described above. These have been obtained using Eqs. (39) and (51) and the relationships given in Section 2. Also shown are results derived from a finite element model consisting of approximately 6500 rectangular thin shell elements whose large dimension is about $\lambda/20$ at 200 Hz. The energies associated with rain-on-the-roof excitation were obtained from the finite element model natural frequencies and mode shapes by a procedure outlined in Ref. [14]. The computational cost in calculating the finite element response estimates was very much greater than that incurred by the method described in this paper.

The two estimates are generally in close agreement at low frequencies. A systematic difference, increasing at higher frequencies, is the result of the increasing inaccuracy of the finite element method as element dimensions become comparable to bending wavelengths. Larger discrepancies

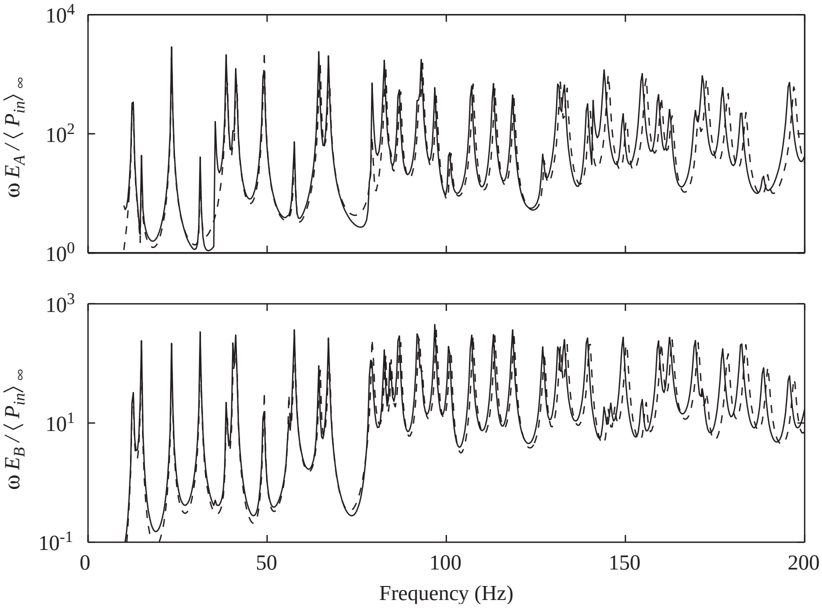

Fig. 8. Normalised energies of plates A and B based on the wave component approach (——) and the finite element method (- - - -).

near 35, 79 and 140 Hz are associated with the effects of almost 'cut on' wave components. Their contributions, neglected in this analysis, become less significant as the frequency and the number of propagating wave components increases.

7. Concluding remarks

In this paper, a wave component approach has been developed for the analysis of energy flow in a broad class of structures, limited only by the requirement that the wave field at cross-sections where the flow is to be evaluated be at least approximately separable. The approach represents a generalisation of earlier works which dealt generally with frameworks of one-dimensional elements, and is significantly less computationally expensive than traditional finite element methods when wavelengths are smaller than typical structural dimensions.

Application of the method is particularly simple for structures which involve only regular subsystems and junctions since the derivation of scattering matrices is then straightforward. Less regular subsystems and junctions may more conveniently be approximated by further sub-division into piecewise regular elements. It may also be possible, in applications involving regular and irregular elements, to use a mixture of the present and finite element or other methods.

Although the present approach is fundamentally deterministic, it is acknowledged that statistical approaches such as SEA may be more appropriate for applications involving higher frequency excitation. However, traditional SEA is limited in many applications of interest by uncertain accuracy, and there is considerable interest in the development of more rigorously derived statistical approaches. The method presented in this paper is used as the basis for one such approach which is described two companion papers [6,7].

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