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Wave component analysis of energy flow in complex structures – Part II: ensemble statistics

E.C.N. Wester[a,*], B.R. Mace[b]

[a]*Industrial Research Limited, P.O. Box 2225, Auckland, New Zealand*
[b]*ISVR, University of Southampton, Highfield, Southampton SO17 1BJ, UK*

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Abstract

A wave-based method is presented for the analysis of high-frequency vibrations in complex structures. The response of the structure to external forcing is described in terms of generalised, energy-bearing wave components, and the structure is represented by global subsystem and junction wave component scattering matrices, **S** and **T**. Uncertainty in the properties of the structure is taken into account by assuming that the structure is drawn from an ensemble of structures that differ randomly in detail. A 'scalar random phase' ensemble is defined in terms of random eigenvalues of the product **ST** of the scattering matrices, and analytical expressions are derived for the average and variance of the energy responses over this ensemble. The scalar random phase ensemble is thought to be a reasonable approximation to many practical ensembles and the approach provides a means for estimating response statistics at relatively low computational cost.

1. Introduction

Of the techniques which are used to predict the vibrational response of mechanical structures to external forcing, most are based on idealised mathematical models and an assumption that details of the structure and applied forces are known exactly. However, these approaches are not

*Corresponding author. Tel.: +64 9 920 3100; fax: +64 9 920 3116.
E-mail addresses: e.wester@irl.cri.nz (E.C.N. Wester), brm@isvr.soton.ac.uk (B.R. Mace).

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appropriate for 'complex' structures involving high-frequency excitation and significant structural uncertainty.

The limitations of traditional approaches arise, in part, from the short-wavelength nature of the response of the structure at high frequencies and the sensitivity of this response to variations in structural detail. Since the accuracy with which this detail can be specified or easily measured is always limited in practice, and since some approximation is inevitably present in the equations upon which the model is based, differences between the structure and its mathematical model can lead to significant differences between actual and predicted responses.

A further difficulty relates to the spatial resolution of results provided by 'exact' methods. Although the response is calculated at locations which are typically much less than a wavelength apart, this resolution cannot be justified in terms of the uncertainty inherent in the modelling process. The process therefore involves data which are informationally redundant and wasted computational effort.

Traditional approaches are also limited by the potentially very large volumes of data and computational expense of analyses at high frequencies. For many problems of interest, the computational cost of the task is prohibitive or simply beyond the capacity of modern computers.

It follows from these concerns that an ideal analysis approach should satisfy a 'conservation of information' principle where the amount of information in the predicted response is consistent with that available in the input to the modelling process. The method should also be computationally efficient and not, for example, involve the calculation of informationally unjustified detail at any stage.

Statistical Energy Analysis (SEA) [1,2] is the best known of the methods which have been developed for complex structures. In SEA, only limited account is taken of structural detail—the structure is notionally divided into subsystems and the response of the system to external forcing is described in terms of the temporal and spatial averages of subsystem energies. Each of the subsystems is assumed to have been drawn from an ensemble of subsystems which differ randomly in detail, and the principal quantity of interest is generally the average response over the ensemble. However, the validity of various assumptions which form the basis of traditional SEA is often unclear in any given situation and has been the subject of considerable investigation (see Refs. [2,3], for example).

An alternative approach is presented in this paper, which avoids a number of assumptions of traditional SEA. In Section 2, the response of the structure to external forcing is given in terms of energy-bearing 'wave components' which, as described in a companion paper [4], propagate through the structure and undergo reflection and transmission at junctions and subsystems. Relationships between the amplitudes of wave components at the various cross-sections throughout the structure are expressed in terms of global subsystem and junction scattering matrices $\mathbf{S}$ and $\mathbf{T}$, which are systematically assembled from local reflection and transmission coefficients. The dynamic properties of the structure are described both in terms of the matrix product $\mathbf{ST}$ and of the eigenvalues and eigenvectors of $\mathbf{ST}$.

Structural uncertainty is modelled in Section 3 by assuming that the structure at hand is drawn from an ensemble of random structures. Variations in structural properties lead to changes in the spectral properties of $\mathbf{ST}$ and the relationship between these kinds of variation is examined in Section 4, in detail for a simple structure comprising two subsystems and qualitatively for more general structures. The dependence of the response of the structure on variations in the

eigenvalues and eigenvectors is investigated in Section 5. It is observed that the energy response of the structure is particularly sensitive to changes in the phases of the eigenvalues of **ST**.

An ensemble, referred to as the 'scalar random phase' ensemble, is defined in terms of certain assumed distributions of eigenvalue phases in Section 6. Analytical expressions are derived for the average and variance of the energy responses over this ensemble. The scalar random phase ensemble is thought to be a reasonable approximation to many practical ensembles and allows response statistics to be found at relatively low computational cost. Application of the method to an example structure of two coupled plates is described in a second companion paper [5], where good agreement is demonstrated between predicted statistics and those found by numerical Monte Carlo methods.

2. The deterministic wave component model

The deterministic wave component model which forms the basis of the present approach is described in detail in a companion paper [4]. Its key features are reviewed in this section.

As in traditional SEA, the structure is divided into subsystems and external forces acting on different subsystems are assumed to be random, stationary and uncorrelated. The vibrational response of the structure is described in terms of the time-averaged energies of the subsystems and the energy flows between subsystems. For each subsystem X, an energy balance equation can be written, which has the form

$$P_{\mathrm{in},X} = P_{\mathrm{junc},X} + P_{\mathrm{diss},X}, \tag{1}$$

where $P_{\mathrm{in},X}$ and $P_{\mathrm{diss},X}$ are input and dissipated powers, respectively, and $P_{\mathrm{junc},X}$ is the total power lost by subsystem X to neighbouring subsystems via junctions. It is assumed that no energy is dissipated in the junctions. The energy which is dissipated in any subsystem is given, to a good approximation, by

$$P_{\mathrm{diss},X} = \omega \eta_X E_X, \tag{2}$$

where E_X is the energy of the subsystem and η_X is the subsystem loss factor.

The flow of energy through the structure is described in terms of propagating, energy-bearing wave components which can be defined at cross-sections of the structure using the method of separation of variables [6] in such a way that the total cross-sectional energy flow is the sum of individual wave component flows. It is assumed that energy flow associated with interactions between near-fields can be neglected.

The relationship between wave components at the various cross-sections in the structure can be expressed in terms of scattering matrices. If the vectors $\mathbf{a}^+$ and $\mathbf{a}^-$ denote the power amplitudes of wave components leaving and entering any given subsystem via junctions, and if $\mathbf{e}$ denotes the vector of power amplitudes of 'excitation' wave components generated within the subsystem by external forces, then a subsystem scattering matrix $\mathbf{S}$ can be defined so that

$$\mathbf{a}^+ = \mathbf{S}\,\mathbf{a}^- + \mathbf{e}. \tag{3}$$

Junctions can be similarly characterised by a scattering matrix $\mathbf{T}$ so that

$$\mathbf{a}^- = \mathbf{T}\,\mathbf{a}^+. \tag{4}$$

(The superscripts '+' and '−' are used throughout this paper to denote to wave components incident on junctions and incident on subsystems, respectively.)

A compact description of wave component reflection and transmission throughout the structure is found by considering augmented amplitude vectors for incident and emerging wave components at *all* cross-sections. Global subsystem and junction scattering matrices are then constructed for the entire structure from the entries of the scattering matrices of individual subsystems and junctions. With $\mathbf{S}$ and $\mathbf{T}$ redefined as global scattering matrices, Eqs. (3) and (4) can also be interpreted as describing global scattering.

It follows from the global forms of Eqs. (4) and (3) that

$$\mathbf{a}^{+} = (\mathbf{I} - \mathbf{ST})^{-1}\,\mathbf{e}, \tag{5}$$

where $\mathbf{I}$ is the identity matrix. In terms of $\mathbf{A}$, the adjoint of $(\mathbf{I} - \mathbf{ST})$, and Δ, its determinant,

$$\mathbf{a}^{+} = \mathbf{A}\,\mathbf{e}\,/\Delta. \tag{6}$$

The amplitudes of the wave components emerging from all subsystems are expressed in Eq. (5) as the product of two factors of which the first, $(\mathbf{I} - \mathbf{ST})^{-1}$, represents the dynamic properties of the whole structure and the second, $\mathbf{e}$, represents the details of excitation.

A sub-vector of amplitudes corresponding to the group of wave components directly incident on junctions from within subsystem X can be defined so that

$$\mathbf{a}_X^{+} = \mathbf{A}_{X\bullet}\,\mathbf{e}\,/\Delta, \tag{7}$$

where $\mathbf{A}_{X\bullet}$ is used to denote the rows of $\mathbf{A}$ associated with wave components in X. A vector of the wave component powers incident on junctions from X can then be written as

$$\mathbf{P}_{\mathrm{inc},X} = \mathrm{diag}(\mathbf{a}_X^{+}\,\mathbf{a}_X^{+\mathrm{H}})/\,2 \tag{8}$$

and, with a sub-vector $\mathbf{a}_X^{-}$ defined for the amplitudes of components leaving X, the power lost from X via junctions can be written as

$$\mathbf{P}_{\mathrm{junc},X} = \mathrm{diag}(\mathbf{a}_X^{+}\,\mathbf{a}_X^{+\mathrm{H}} - \mathbf{a}_X^{-}\,\mathbf{a}_X^{-\mathrm{H}})/\,2. \tag{9}$$

If the subsystem is not directly driven by external forces, $\mathbf{e}_X = \mathbf{0}$, and

$$\mathbf{a}_X^{+} = \mathbf{S}_X\mathbf{a}_X^{-}, \tag{10}$$

where $\mathbf{S}_X$ is the subsystem scattering matrix. The total junction power is then

$$P_{\mathrm{junc},X} = \mathbf{a}_X^{+\mathrm{H}}[\mathbf{I} - (\mathbf{S}_X\mathbf{S}_X^{\mathrm{H}})^{-1}]\mathbf{a}_X^{+}/\,2. \tag{11}$$

In almost all systems of practical interest, the matrix $(\mathbf{S}_X\mathbf{S}_X^{\mathrm{H}})^{-1}$ is strongly diagonally dominant. By defining $\mathbf{D}_X^2$ as the diagonal matrix obtained by setting off-diagonal entries of $(\mathbf{S}_X\mathbf{S}_X^{\mathrm{H}})^{-1}$ to zero, it then follows that the junction power for the indirectly driven subsystem is given to good approximation by

$$P_{\mathrm{junc},X} \approx \mathbf{P}_{\mathrm{inc},X}^{\mathrm{T}}\,\mathrm{diag}(\mathbf{I} - \mathbf{D}_X^2). \tag{12}$$

If external forces act only on a single subsystem, the junction power for this subsystem can be found from the knowledge that the sum of junction powers over all subsystems is zero.

The response energy E_X can thus be found in terms of the excitation wave component amplitudes and the scattering matrices $\mathbf{S}$ and $\mathbf{T}$—the incident and junction powers are found from

the wave component amplitudes through Eqs. (7), (8) and (12), and E_X is found through $P_{\mathrm{diss},X}$ given by Eq. (2) and the power balance equation (1).

3. Ensembles of structures

Uncertainty in the properties of complex structures leads naturally to the concept of an ensemble of structures with random properties and to a probabilistic description of structural responses. This uncertainty can be formally quantified in a joint probability density function $f(\mathbf{x})$ which specifies the probability of occurrence of any combination of the uncertain structural parameters $\mathbf{x} = [x_i]$ in the ensemble. The kth moment of a response quantity such as the energy flow $P(\mathbf{x})$ is then given in principle by

$$P^{(k)} = \int P^k(\mathbf{x}) f(\mathbf{x}) \, \mathrm{d}\mathbf{x}. \tag{13}$$

For most practical applications, the number of uncertain parameters is large, derivation or measurement of the joint probability density function is not practicable and evaluation of the integral in Eq. (13) is not possible, or is computationally very expensive. However, the main requirements of the probabilistic approach are only that the chosen ensemble reproduce the probability of occurrence of particular energy flows in real-world structures, and that it lead to mathematical expressions of tractable form. Alternative approaches are therefore possible—for example, those in which the ensemble is defined instead in terms of the joint probability density function of *dynamic* properties of the structure. In traditional SEA, the ensemble is associated with a particular distribution of natural frequencies. Elsewhere, structural uncertainty has been described in terms of the random phases of wave propagation and scattering coefficients [7]. In the approach taken here, ensembles are viewed in terms of the variations in the eigenvalues and eigenvectors of **ST**.

4. Spectral decomposition

An ensemble of structures can be described in terms of the wave component model by random variations in the system matrix **ST**. Since this matrix can be decomposed as

$$\mathbf{ST} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}, \tag{14}$$

where **V** is a square matrix in which each column is an eigenvector of **ST** and $\mathbf{\Lambda} = \mathrm{diag}[\lambda_i]$ is the corresponding diagonal matrix of eigenvalues, it follows that the ensemble can also be described in terms of variations in eigenvalues and eigenvectors.

In this section, the characteristics of the variations in eigenvalues and eigenvectors associated with realistic ensemble variations in structural properties are examined. An elementary structure comprising two one-dimensional subsystems is firstly investigated in detail, and then qualitative observations are made for more general structures.

4.1. Two coupled one-dimensional subsystems

Consider a structure comprising two coupled regular, dynamically one-dimensional [4] subsystems as illustrated in Fig. 1. A system matrix can be constructed for this structure, which has the form

$$\mathbf{ST} = \begin{bmatrix} s_A r_A & s_A t \\ s_B t & s_B r_B \end{bmatrix}, \tag{15}$$

where s_A and s_B are subsystem reflection coefficients, r_A and r_B are junction reflection coefficients and t is the junction transmission coefficient.

The two products $s_A r_A$ and $s_B r_B$ are propagation factors associated with the passage of wave components over circuits of subsystems A and B, respectively. If the coupling is assumed to be conservative, these products can be expressed as

$$s_A r_A = |s_A| R e^{i(\theta-\delta/2)} \qquad \text{and} \qquad s_B r_B = |s_B| R e^{i(\theta+\delta/2)} \tag{16}$$

where θ is the 'common' component of phase, δ is the phase separation and $R = |r_A| = |r_B|$. The two eigenvalues of $\mathbf{ST}$ are given by

$$\lambda_{1,2} = e^{i\theta}\left(\alpha \pm \sqrt{\alpha^2 - |s_A s_B|}\right), \tag{17}$$

where

$$\alpha = R\left(|s_A| e^{-i\delta/2} + |s_B| e^{i\delta/2}\right)/2. \tag{18}$$

It has been observed elsewhere [7] that typical changes in structural parameters are generally associated with only small relative variations in the magnitudes of $s_A r_A$ and $s_B r_B$, but with variations in their phases which are often large compared with 2π. It is of interest, therefore, to examine the dependence of the eigenvalues on the phases of $s_A r_A$ and $s_B r_B$.

4.1.1. Magnitudes of eigenvalues

The magnitudes of the two eigenvalues are independent of the common phase θ. Magnitude extrema occur, as illustrated in Fig. 2, when $s_A r_A$ and $s_B r_B$ are in anti-phase and $\delta = \pi$. The example corresponds to $\mu_A = 0.1$ and $\mu_B = 0.01$, where the subsystem attenuation parameter μ is defined so that $|s| = \exp(-\mu)$. The difference between the two extreme magnitudes is given by

$$|\lambda_{\max}| - |\lambda_{\min}| = R\,||s_A| - |s_B||. \tag{19}$$

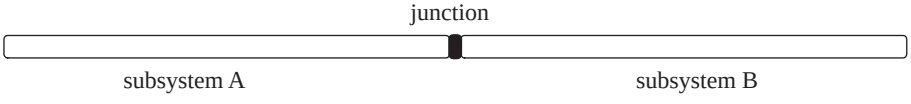

Fig. 1. Two coupled uniform, one-dimensional subsystems, A and B.

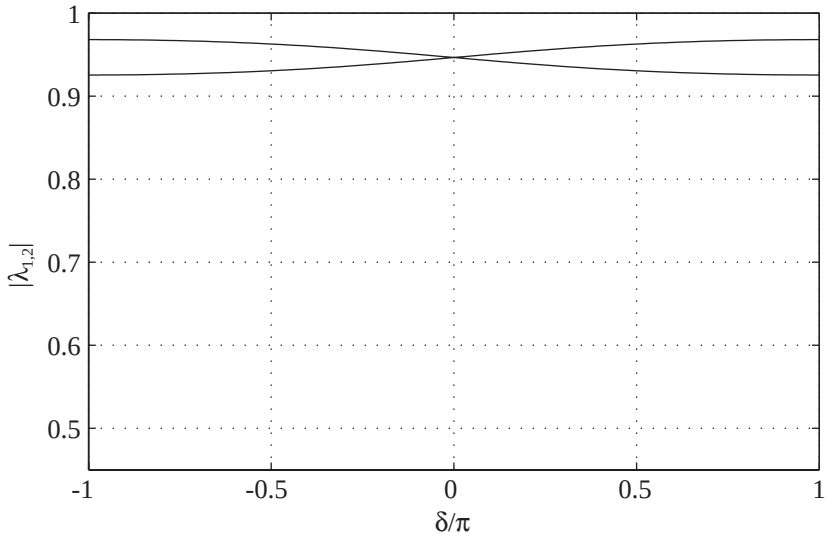

Fig. 2. Eigenvalue magnitude as a function of phase separation δ for a structure with $\mu_A = 0.1$, $\mu_B = 0.01$ and $R = 0.5$. (Note the origin of the vertical scale.)

For the special group of structures which have $|s_A| = |s_B| = |s|$, the eigenvalues given in Eq. (17) can be expressed as

$$\lambda_{1,2} = \mathrm{e}^{\mathrm{i}\,\theta}\,|s|\left[R\cos(\delta/2) \pm \mathrm{i}\sqrt{1 - R^2\cos^2(\delta/2)}\right] \tag{20}$$

or, since the factor in large brackets has unit magnitude, as

$$\lambda_{1,2} = \mathrm{e}^{\mathrm{i}\,\theta}\,|s|\,\mathrm{e}^{\pm\mathrm{i}\,\phi/2}, \tag{21}$$

where ϕ is the phase angle between the two eigenvalues. For these structures, $|\lambda_{1,2}| = |s|$ and the magnitudes of the eigenvalues are independent of variations in the phases of $s_A r_A$ and $s_B r_B$ and independent of R. If the structure has no damping, $|s_A| = |s_B| = 1$ and $|\lambda_{1,2}| = 1$.

4.1.2. Phases of eigenvalues

It follows from Eq. (17) that the common phase θ of $s_A r_A$ and $s_B r_B$ corresponds also to the common phase of the two eigenvalues. The phase separation ϕ between the eigenvalues is given by

$$\cos\,\phi = (|\lambda_1 + \lambda_2|^2 - (|\lambda_1|^2 + |\lambda_2|^2))\,/\,2\,|\lambda_1\lambda_2|. \tag{22}$$

For structures which have $|s_A| = |s_B| = |s|$, this equation can be written as

$$\cos(\phi/2) = R\cos(\delta/2). \tag{23}$$

Examples of the dependence of the phase separation of the eigenvalues on the phase difference between $s_A r_A$ and $s_B r_B$ are shown for these structures in Fig. 3. The maximum possible separation between the eigenvalues occurs when $\delta = \pi$ and $s_A r_A$ and $s_B r_B$ are in anti-phase. The eigenvalues

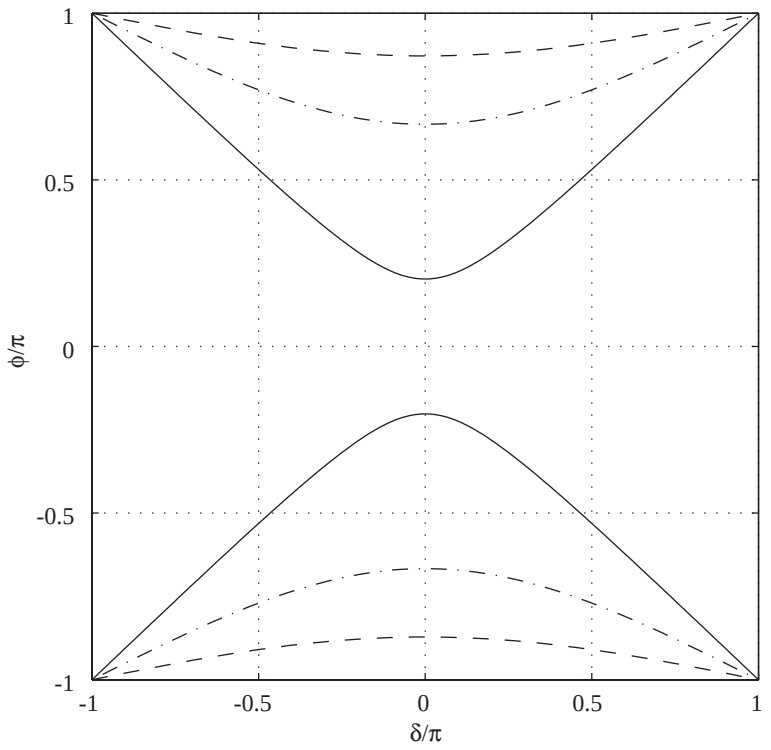

Fig. 3. Phase spacing ϕ between eigenvalues as a function of phase difference δ between $s_A r_A$ and $s_B r_B$ for a structure with $R = 0.95$ (——), $R = 0.5$ (- · - · - · -) and $R = 0.2$ (- - - -).

cannot approach each other arbitrarily closely and have a minimum phase separation $\phi_{\min} = 2\cos^{-1} R$, which occurs when $\delta = 0$ and $s_A r_A$ and $s_B r_B$ have the same phase.

The phase spacing between eigenvalues in structures with $|s_A| = |s_B|$ is determined only by the magnitude of the junction reflection coefficient R. At one extreme when $R = 1$, the subsystems are weakly connected, the phases of the two pairs of eigenvalues are independent with

$$\lambda_1 = s_A r_A \qquad \text{and} \qquad \lambda_2 = s_B r_B, \tag{24}$$

and all phase separations are possible. Each eigenvalue is then uniquely associated with one of the subsystems. At the other extreme, when $R \approx 0$, the connection between subsystems is strong and the eigenvalues,

$$\lambda_{1,2} \approx \pm \sqrt{s_A s_B t^2}, \tag{25}$$

are always in anti-phase. The quantity $s_A s_B t^2$ is the propagation factor associated with the passage of a wave component over a circuit of the whole structure.

For more general structures in which $|s_A|$ and $|s_B|$ are not the same, the distribution of possible phase spacings between eigenvalues depends on R and, to an often smaller extent, on $|s_A|$ and $|s_B|$. The observations made above apply qualitatively to these structures also.

4.1.3. Eigenvectors

For structures with $|s_A| = |s_B|$, the eigenvector matrix is given by

$$\mathbf{V} = \begin{bmatrix} X & -Y\,\mathrm{e}^{-\mathrm{i}\,\delta/2} \\ Y\,\mathrm{e}^{\mathrm{i}\,\delta/2} & X \end{bmatrix}, \tag{26}$$

where X and Y satisfy

$$X^2,\ Y^2 = \tfrac{1}{2}\left(1 \pm \frac{R\,\sin(\delta/2)}{\sqrt{1 - R^2\cos^2(\delta/2)}}\right). \tag{27}$$

The amplitudes of the eigenvectors in the two subsystems are functions of both R and δ, but do not depend on $|s|$ or the level of damping in the structure. When $|s_A| \neq |s_B|$, damping has an effect on these shapes which is generally small. The eigenvectors are independent of the common phase θ, even when $|s_A| \neq |s_B|$.

Examples of the dependence of X and Y on the phase difference δ between $s_A r_A$ and $s_B r_B$ are shown in Fig. 4 for a range of values of the reflection coefficient R. For structures in which R is small, the subsystems are strongly connected, eigenvector amplitudes are approximately equal in the two subsystems and the eigenvectors are 'global' in nature. This is the case for all values of δ. For structures in which R is not small, the eigenvector amplitudes tend, for most values of δ, to be large in one subsystem and small in the other. The eigenvectors are then 'localised' in one or other of the two subsystems. The eigenvector forms are complementary in that if one eigenvector has large amplitude in A, then the other has small amplitude there and vice versa. Localisation is

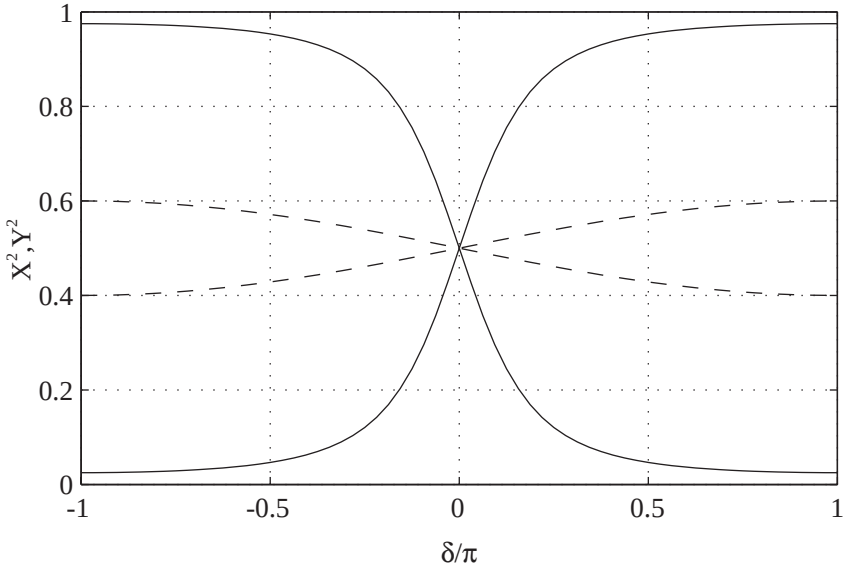

Fig. 4. Dependence of eigenvector magnitudes on the phase difference δ, for $R = 0.95$ (——) and $R = 0.2$ (- - - -).

strongest when the separation between eigenvalues is a maximum, and the eigenvector amplitudes are equal in the two subsystems when the eigenvalues are at their closest.

4.1.4. Summary

The observations made in this section can be summarised as follows. It has been noted in Ref. [7] that a realistic ensemble of two one-dimensional subsystems can be defined by assuming that the magnitudes of $s_A r_A$ and $s_B r_B$ are constant but that their phases are independent random variables, uniformly distributed in $[-\pi, \pi]$. In terms of the spectral parameters of the matrix **ST**, it is observed that, in most cases, the magnitudes of eigenvalues and the eigenvectors vary little over this ensemble, while the common eigenvalue phase is random and uniformly distributed in $[-\pi, \pi]$. If the junction reflection coefficient $R \approx 1$, the phases of the eigenvalues are approximately independent and the eigenvectors are generally localised. If $R \approx 0$, on the other hand, the two eigenvalues are relatively rigidly locked in anti-phase and the eigenvectors are generally global in nature.

4.2. General structures

For more general structures, the characteristics of variations in the eigenvalues and eigenvectors of **ST** associated with typical ensemble variations in the structural properties are difficult to quantify. However, many of the observations made in the previous section for two coupled one-dimensional subsystems are found, at least qualitatively, to hold for more complex systems also.

4.2.1. Common eigenvalue phase

As for the simple case of two subsystems, it is useful to consider the variations in the phases of the n eigenvalues of a more general structure in terms of a common phase θ and $n-1$ eigenvalue phase separations. A common variation in the phases of the eigenvalues has the form $e^{i\theta}\mathbf{\Lambda}$, and is associated with changes in the system matrix which satisfy $\mathbf{V}(e^{i\theta}\mathbf{\Lambda})\mathbf{V}^{-1} = e^{i\theta}\mathbf{ST}$. This change leaves the magnitudes of all the entries of **ST** unchanged, but increases the phase of each entry by θ.

4.2.2. Eigenvalue phase separation

In the earlier example involving pairs of one-dimensional subsystems, it was observed that the two eigenvalues cannot approach each other arbitrarily closely as structural properties vary. For perturbations of more general structures involving greater numbers of eigenvalues, the eigenvalues appear to display a 'mutual repulsion' analogous to that which has been observed in other contexts where eigenvalues correspond to natural vibration frequencies of rooms and engineering structures, or where the eigenvalues correspond to the energy levels of heavy nuclei (see, for example, Refs. [1,8–10]).

Although considerable research has been carried out in establishing the statistical properties of eigenvalues of various random matrix ensembles (see Ref. [11], for example), most is relevant here only for a restricted class of mechanical structures.

One of these is the circular unitary ensemble (CUE), which comprises the group of all unitary matrices of a given, high order. In the present context, such matrices are associated with undamped structures involving many wave components. Its relevance is limited, however, by the

large 'volume' of the ensemble which, due to the inclusion of every possible unitary system matrix of a given order, encompasses an extremely wide range of structural variation.

The eigenvalues of matrices in the CUE have unit magnitude and can be written as $e^{i\phi_i}$, $i = 1, ..., n$. Dyson [10], who carried out the first detailed investigation of this ensemble, made extensive use of an analogy involving electrostatic charges on a thin circular conducting wire. The charges, identified by the angles $\phi_1, \ldots, \phi_n$, were assumed to repel each other according to the Coulomb law of two-dimensional electrostatics. The probability distributions of the eigenvalue phases and the electrostatic charges are the same, reinforcing the concept of eigenvalue 'repulsion'.

A further probability distribution of interest is that associated with structures in which there is no interaction between the eigenvalues. The nearest-neighbour eigenvalue phase-spacing probability density function for this ensemble is given by $f(s) = (1 - s/n)^{n-2}(n-1)/n$, where $s = \phi/\phi_{\text{avg}}$ [12]. When the number of eigenvalues n is large and the circular nature of the ensemble becomes less significant, the probability density function becomes $f(s) = e^{-s}$. This is the more familiar form of the Poisson distribution corresponding to the situation in which the expected value of s is 1.

The distribution of eigenvalue phase spacings given by the CUE is in some sense intermediate between the two extremes in which eigenvalues are either rigidly spaced, or, as in the circular Poisson ensemble, completely unconstrained. It is thought that the distribution of spacings in the CUE is representative of those of the matrix **ST** corresponding to structures which are in some sense 'maximally irregular'.

4.2.3. Eigenvectors

Considerably less information is available concerning the statistical variations of eigenvectors in random matrix ensembles. This is generally because the detailed ensemble behaviour of eigenvectors has not been relevant to the problems investigated so far and, indeed, this is to some extent the case here also. However, a number of general observations can be made regarding the sensitivities of eigenvectors to changes in structural detail, which are based on the results of standard perturbational analysis methods [13].

It is found that the eigenvector $\mathbf{v}_i$ associated with eigenvalue λ_i is relatively insensitive to structural changes if λ_i is far from other eigenvalues and the 'condition' of each eigenvalue is small. The condition of an eigenvalue is large when **ST** is 'near' to a matrix which has a multiple eigenvalue. The condition is a minimum, and the eigenvector sensitivity is relatively low, when **ST** is a normal matrix. This includes structures without damping, since for these **ST** is unitary and therefore normal, but also includes the structures comprising two one-dimensional subsystems with $|s_A| = |s_B|$ described earlier.

4.3. Spectral parameters and structural variations

The sensitivity of eigenvalues and eigenvectors to structural changes is determined, in large part, by what may be termed the 'connection strength' within the structure. This refers to the ease with which energy passes between subsystems and corresponds loosely to the magnitude of the transmission coefficient of the junction which joins the subsystems.

Consider an idealised structure in which most parts of the structure are weakly connected to all others. The interaction between eigenvalues is then weak and the phase spacing between them is distributed approximately according to the circular Poisson ensemble. Eigenvectors associated with these structures are generally strongly localised and can be relatively sensitive to perturbations of the structure.

In an idealised structure involving strong connections on the other hand, the repulsion between eigenvalues is strong and there is relatively little variation in eigenvalue spacing. The amplitudes of the eigenvectors of these structures are distributed approximately uniformly over the structure and, overall, the sensitivity of all the spectral parameters to structural perturbations is generally small.

5. Ensemble variations of energy flow

Ensemble variations in structural properties, which can be expressed in terms of variations in the eigenvalues and eigenvectors of **ST**, lead to variations in structural responses. In this section, previously derived equations describing structural responses are recast in a form that highlights the roles of the spectral parameters. It is found that variations in response are due, in large part, to the sensitivity of the response to changes in the common component of the eigenvalue phases.

It follows from Eq. (12) that the power lost by subsystem X to neighbouring subsystems when subsystem Y is excited can be expressed as

$$P_{\text{junc},XY} = \mathbf{P}_{\text{inc},XY}^{\text{T}} \operatorname{diag}(\mathbf{I} - \mathbf{D}_X^2), \tag{28}$$

where $\mathbf{P}_{\text{inc},XY}$ is the vector of powers incident on the junctions of X due to excitation of Y. The diagonal entries of $\mathbf{D}_X^2$ correspond to column sums of the squares of the magnitudes of the entries of $\mathbf{S}_X^{-1}$, where $\mathbf{S}_X$ is the scattering matrix for subsystem X. These magnitudes are largely determined by the level of damping in X and are relatively insensitive to structural perturbations. It can be assumed, therefore, that $\mathbf{D}_X^2$ is constant over the ensemble and that ensemble variations in $P_{\text{junc},XY}$ arise only through the effects of ensemble variations in the incident powers $\mathbf{P}_{\text{inc},XY}$.

The global vector of powers incident on all junctions is given by

$$\mathbf{P}_{\text{inc}} = \operatorname{diag}\left(\mathbf{A}\,\mathbf{e}\mathbf{e}^{\text{H}}\,\mathbf{A}^{\text{H}}\right) / 2\,|\Delta|^2, \tag{29}$$

where $\mathbf{A}/\Delta = (\mathbf{I} - \mathbf{ST})^{-1}$. Eq. (29) can be re-written in terms of the eigenvalues and eigenvectors of **ST** as

$$\mathbf{P}_{\text{inc}} = \tfrac{1}{2}\operatorname{diag}\left(\mathbf{V}\,(\mathbf{I} - \boldsymbol{\Lambda})^{-1}\mathbf{V}^{-1}\,\mathbf{e}\mathbf{e}^{\text{H}}\,\mathbf{V}^{-1\text{H}}(\mathbf{I} - \boldsymbol{\Lambda})^{-1\text{H}}\,\mathbf{V}^{\text{H}}\right). \tag{30}$$

The powers incident on the junctions of subsystem X due to excitation of subsystem Y are then

$$\mathbf{P}_{\text{inc},XY} = \tfrac{1}{2}\operatorname{diag}\left(\mathbf{V}_{X\bullet}\,(\mathbf{I} - \boldsymbol{\Lambda})^{-1}\,\mathbf{U}_{Y\bullet}^{\text{H}}\,\mathbf{e}_Y\mathbf{e}_Y^{\text{H}}\,\mathbf{U}_{Y\bullet}\,(\mathbf{I} - \boldsymbol{\Lambda})^{-1\text{H}}\,\mathbf{V}_{X\bullet}^{\text{H}}\right), \tag{31}$$

where $\mathbf{U}^{\text{H}} = \mathbf{V}^{-1}$ and where $\mathbf{V}_{X\bullet}$ and $\mathbf{U}_{Y\bullet}$ denote the rows of $\mathbf{V}$ and $\mathbf{U}$ associated with wave components in subsystems X and Y, respectively. By further defining the eigenvalue-dependent vector $\mathbf{g} = \operatorname{diag}(\mathbf{I} - \boldsymbol{\Lambda})^{-1}$ and collecting $\mathbf{g}$ and its Hermitian transpose $\mathbf{g}^{\text{H}}$, Eq. (31) can be

re-written as

$$\mathbf{P}_{\mathrm{inc},XY} = \tfrac{1}{2}\mathrm{diag}(\mathbf{V}_{X\bullet}[(\mathbf{U}_{Y\bullet}^{\mathrm{H}}\mathbf{e}_Y\mathbf{e}_Y^{\mathrm{H}}\mathbf{U}_{Y\bullet}) \circ (\mathbf{g}\mathbf{g}^{\mathrm{H}})]\mathbf{V}_{X\bullet}^{\mathrm{H}}), \tag{32}$$

where '$\circ$' denotes the Hadamard or entry-wise matrix product.

The incident power associated with a particular wave component x in X is

$$P_{\mathrm{inc},xY} = \tfrac{1}{2}\sum_{y\in Y}|e_y|^2\,\mathbf{V}_{x\bullet}[(\mathbf{U}_{y\bullet}^{\mathrm{H}}\mathbf{U}_{y\bullet}) \circ (\mathbf{g}\mathbf{g}^{\mathrm{H}})]\mathbf{V}_{x\bullet}^{\mathrm{H}}, \tag{33}$$

where it has been noted that for suitably large subsystems driven by rain-on-the-roof excitation, $\mathbf{e}_Y\mathbf{e}_Y^{\mathrm{H}}$ is diagonal. Eigenvector-related factors can be gathered to give

$$P_{\mathrm{inc},xY} = \tfrac{1}{2}\sum_{y\in Y}|e_y|^2\sum_{i,j=1}^{n}[(\mathbf{U}_{y\bullet}\circ\mathbf{V}_{x\bullet}^{*})^{\mathrm{H}}(\mathbf{U}_{y\bullet}\circ\mathbf{V}_{x\bullet}^{*})]\circ(\mathbf{g}\mathbf{g}^{\mathrm{H}}), \tag{34}$$

where the summation indicated by $\sum_{ij}$ is understood to be over all the entries of its $n\times n$ matrix summand, n is the order of $\mathbf{ST}$ and * denotes the complex conjugate.

Each of the terms $[(\mathbf{U}_{y\bullet}\circ\mathbf{V}_{x\bullet}^{*})^{\mathrm{H}}(\mathbf{U}_{y\bullet}\circ\mathbf{V}_{x\bullet}^{*})]\circ(\mathbf{g}\mathbf{g}^{\mathrm{H}})$ in Eq. (34) quantifies the ease with which energy is transmitted between wave components x and y, and is formed from the entry-wise product of an eigenvector-dependent matrix and an eigenvalue-dependent matrix. The latter, $\mathbf{g}\mathbf{g}^{\mathrm{H}}$, is common to all wave component pairs, while the former depends on the specific participation of the wave components in the eigenvectors. In the following sections, the influences of eigenvalue- and eigenvector-related factors will be investigated.

5.1. Eigenvalue-related influences

Eqs. (28) and (34) show, respectively, that junction and incident powers depend on the eigenvalues of $\mathbf{ST}$ only through the Hermitian outer product $\mathbf{g}\mathbf{g}^{\mathrm{H}}$, where

$$\mathbf{g}\mathbf{g}^{\mathrm{H}} = \left[\frac{1}{(1-\lambda_i)(1-\lambda_j^{*})}\right] \tag{35}$$

and $i,j = 1,\ldots,n$. In this section, ensemble variations in the individual entries of this matrix and ensemble variations in the overall form of the matrix are examined in turn.

5.1.1. Ensemble characteristics of individual entries of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$

The entries of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ are highly sensitive to the locations of the pair of eigenvalues λ_i, λ_j in the complex eigenvalue plane and, in particular, to their proximity to the singularity at $\lambda = 1$. The sensitivity of the energy flow to ensemble variations in structural properties is due in great part to the often highly nonlinear dependence of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$, via the eigenvalues, on these variations.

As noted in Section 4, the magnitudes of the eigenvalues generally vary little over the ensemble. Since the sensitivity of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ to structural variations is most closely associated with the sensitivity of the phases of the eigenvalues, it is convenient to assume that the magnitudes of eigenvalues are

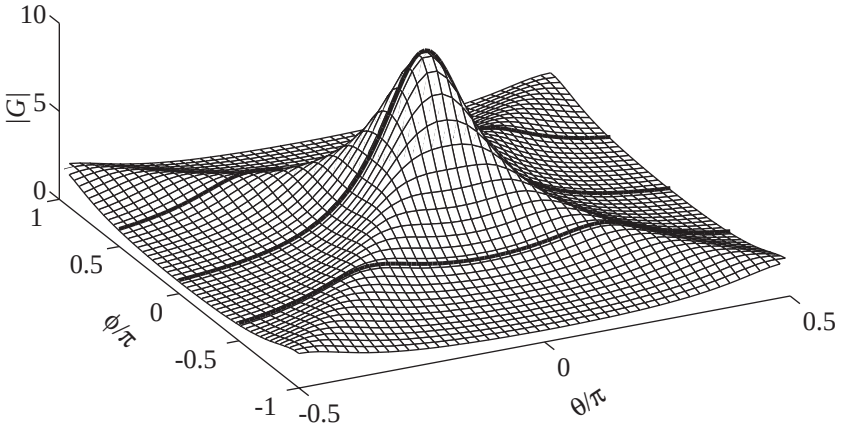

Fig. 5. Magnitude of G as a function of the common phase θ and the phase spacing ϕ for $|\lambda_i| = |\lambda_j| = 0.7$. The loci of points corresponding to common phase variations with $\phi = 0$ and $\phi = \pm\pi/2$ are also shown.

constant and to investigate the function

$$G(\phi_i, \phi_j) = \frac{1}{(1 - |\lambda_i| e^{i\,\phi_i})(1 - |\lambda_j| e^{-i\,\phi_j})} \tag{36}$$

over the plane defined by the two eigenvalue phases ϕ_i and ϕ_j. As in Section 4, the phases of eigenvalues can be expressed in terms of common and separation components defined, respectively, by $\theta = (\phi_i + \phi_j)/2$ and $\phi = \phi_i - \phi_j$. Eq. (36) can then be re-written as

$$G(\theta, \phi) = \frac{1}{1 - (|\lambda_i| e^{i\,\theta} + |\lambda_j| e^{-i\,\theta})\, e^{i\,\phi/2} + |\lambda_i||\lambda_j| e^{i\,\phi}}. \tag{37}$$

Common-phase variations in the system matrix **ST** affect only the common phase of the eigenvalues and, since the eigenvectors are unaffected, influence energy flows only through variations in $\mathbf{g}\mathbf{g}^{\mathrm{H}}$. In terms of the function $G(\theta, \phi)$, these variations correspond to constant ϕ and have loci in the θ, ϕ plane which form lines parallel to the θ-axis. Examples are shown in Fig. 5.

The characteristics of ensemble variations in the phase separation ϕ between pairs of eigenvalues are determined by the strength of eigenvalue repulsion, which is loosely determined by the strength of connection between wave components in the structure. When the eigenvalue repulsion is strong, eigenvalue phase separations vary little over the ensemble and general ensemble phase variations are closely approximated by common phase variations alone. In structures which have weaker eigenvalue repulsion, the loci may form broader bands parallel to the θ-axis.

5.1.2. Ensemble characteristics of the full matrix $\mathbf{g}\mathbf{g}^{\mathrm{H}}$

The qualitative features of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ as a whole depend on both the variations of individual entries *and* the correlations between entries. These are illustrated in this section by reference to two idealised example structures, (a) and (b), over which the damping, connection strength and eigenvalue repulsion are assumed uniform. All eigenvalues have the same magnitude, given in

terms of an attenuation parameter μ by $|\lambda| = \exp(-\mu)$, and the repulsion between any pair of eigenvalues is the same. In structure (a), the eigenvalue repulsion is very strong and the phase separation between adjacent eigenvalues, given by $2\pi/n$, is constant over the ensemble. The common component of phase is random and uniformly distributed in $[-\pi, \pi]$. In structure (b), the eigenvalue repulsion is very weak and the individual eigenvalue phases are independent random variables, uniformly distributed in $[-\pi, \pi]$. The effects of different damping levels on the form of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ will be examined for each type of structure.

In structures with heavy to moderate damping, ensemble variations in the entries of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ and the difference in magnitude between the largest and smallest entry are generally small. Examples are shown in Fig. 6. The form shown in Fig. 6(a), in which $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ is dominated by diagonal or near-diagonal entries corresponding to the small number of adjacent eigenvalues which have phases closest to zero, is typical of structures with strong eigenvalue repulsion. For structures with weak eigenvalue repulsion (Fig. 6(b)), the phases of the eigenvalues are approximately independent and correlation between the entries of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ over the ensemble is reduced. For most ensemble members, the magnitudes of the entries are smaller than the maximum observed when the eigenvalue repulsion is strong. For some, and possibly many ensemble members, no eigenvalue phase is near zero and the system is far from resonance.

In lightly damped structures, the function G has a distinct, narrow peak and the magnitudes of the entries of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ vary considerably over the ensemble. When the eigenvalue repulsion is strong, $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ is dominated by diagonal entries corresponding to the small number of eigenvalues close to unity. This is illustrated in Figs. 7(a), 8(a) and 9(a), for structures at, close to and far from resonance, respectively. When the eigenvalue repulsion is weak, the possibility arises of there being more than one eigenvalue with a phase near zero and the presence, as shown in Fig. 7(b), of a number of diagonal and corresponding off-diagonal entries with large magnitudes. The probability that more than one eigenvalue phase is close to zero is generally small, however, and the example in Fig. 7(b) is not typical—most 'at resonance' ensemble members have only a single eigenvalue with zero phase. Weak repulsion also leads to the possibility, illustrated in Fig. 9(b), of

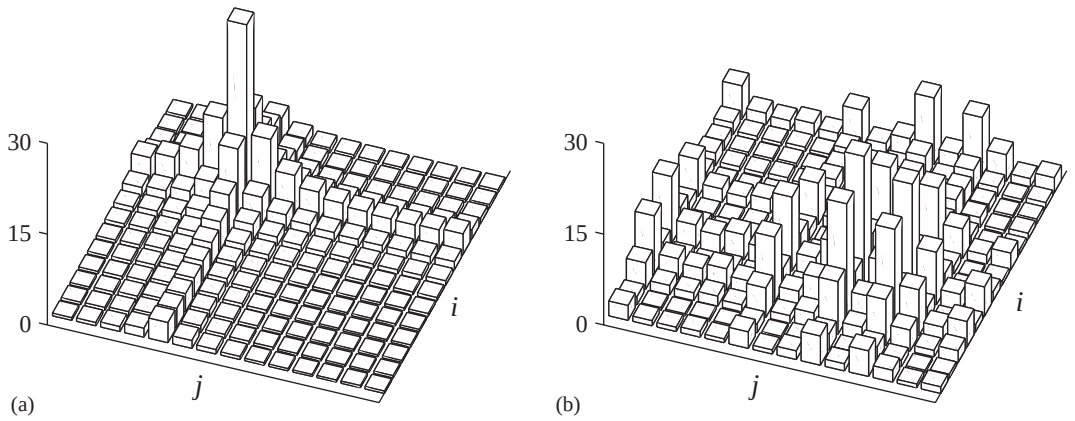

Fig. 6. Example magnitudes of the entries of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ corresponding to moderate damping ($\mu = 0.2$) in structures with (a) strong eigenvalue repulsion and (b) weak eigenvalue repulsion.

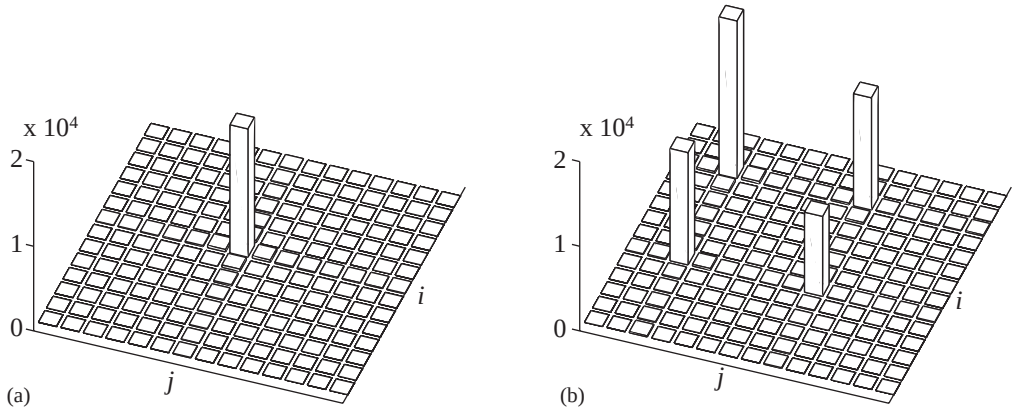

Fig. 7. Example magnitudes of the entries of $\mathbf{gg}^{\mathrm{H}}$ corresponding to light damping ($\mu = 0.001$) in *at resonance* structures with (a) strong eigenvalue repulsion and (b) weak eigenvalue repulsion.

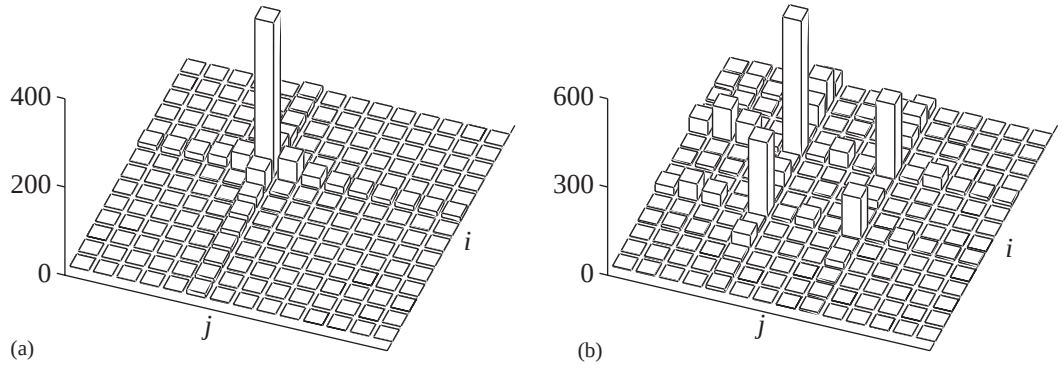

Fig. 8. Example magnitudes of the entries of $\mathbf{gg}^{\mathrm{H}}$ corresponding to light damping ($\mu = 0.001$) in structures *not far from resonance* with (a) strong eigenvalue repulsion and (b) weak eigenvalue repulsion.

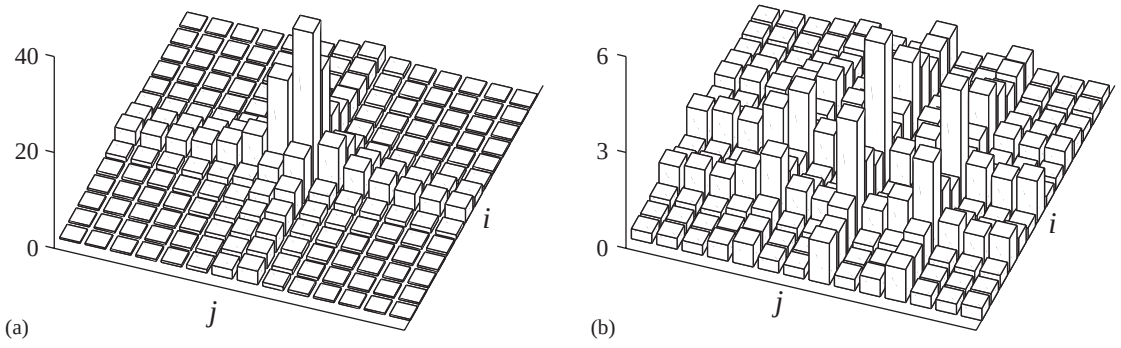

Fig. 9. Example magnitudes of the entries of $\mathbf{gg}^{\mathrm{H}}$ corresponding to light damping ($\mu = 0.001$) in structures *far from resonance* with (a) strong eigenvalue repulsion and (b) weak eigenvalue repulsion.

structures in which none of the eigenvalues have phases near zero. This situation, in which *all* entries of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ have very small magnitudes, cannot occur if the eigenvalue repulsion is strong.

Although structures close to and far from resonance (Figs. 8 and 9) occur frequently in the ensemble, the average ensemble response is dominated by the large contributions associated with resonant structures (Fig. 7).

5.2. Eigenvector-related influences

Eq. (34) shows that the energy flow depends on the eigenvectors of $\mathbf{ST}$ through the factor

$$(\mathbf{U}_{y\bullet} \circ \mathbf{V}^{*}_{x\bullet})^{\mathrm{H}} (\mathbf{U}_{y\bullet} \circ \mathbf{V}^{*}_{x\bullet}). \tag{38}$$

By definition, $\mathbf{U}^{\mathrm{H}} = \mathbf{V}^{-1}$, so that for normal system matrices, $\mathbf{U} = \mathbf{V}$. This last equality holds for all systems without damping and approximately for systems with light damping. It also holds exactly for the pair of coupled one-dimensional subsystems considered earlier, for all levels of damping, if the magnitudes of the two subsystem reflection coefficients s_A and s_B are the same.

If it is assumed that $\mathbf{U}$ and $\mathbf{V}$ are approximately equal, then

$$(\mathbf{U}_{y\bullet} \circ \mathbf{V}^{*}_{x\bullet})^{\mathrm{H}} \approx [v^{*}_{yj}\, v_{xj}], \qquad j = 1, 2, \ldots n, \tag{39}$$

where $\mathbf{V} = [v_{ij}]$. The magnitudes of the entries in this vector then reflect the extent of the involvement of *both* wave components, x in the responding subsystem X and y in the directly excited subsystem Y. This situation is analogous to that in traditional modal descriptions of dynamic response, where the contribution made by a mode to the total response depends on the amplitudes of the mode at both the excitation and response locations.

5.3. Discussion

As noted above, the sensitivity of the energy flow to variations in structural properties can be largely attributed to the sensitivity of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$, through the phases of the eigenvalues, to these variations. Changes in the magnitudes of the eigenvalues and in the eigenvectors generally have smaller effects.

For structures in which the eigenvalue repulsion is strong, variations in the phases of the eigenvalues involve almost exclusively variations in the common component of phase, θ. Eigenvalue repulsion in a structure is largely determined by the strengths of connections between wave components and has its greatest influence on the response of moderately or lightly damped structures. It is one of the factors that determine the likely distribution of the magnitudes of the entries in $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ for any ensemble member, and the number of eigenvalues which may then play a significant role in determining the response of the structure. This number varies more widely when the eigenvalue repulsion is weak.

6. The scalar random phase ensemble

In the sections above, the energy response of a structure has been described in terms of a system matrix $\mathbf{ST}$ and its eigenvalues and eigenvectors. Uncertainty in the properties of the structure then leads to the concept of an ensemble of structures over which the eigenvalues and eigenvectors of

ST are assumed to vary randomly. The relationship between variations in structural properties and the eigenvalues and eigenvectors, and the dependence of the response of the structure on variations in the eigenvalues and eigenvectors, were then described in Sections 4 and Section 5.

In this section, an ensemble, the 'scalar random phase' ensemble, is defined in terms of the spectral properties of its member structures. Response statistics over this ensemble are believed to closely approximate those of many realistic ensembles when significant uncertainty is involved. Statistical moments of the response over this ensemble can be found at low computational cost by analytical, or mostly analytical, evaluation of the integral in Eq. (13).

6.1. Definition of the scalar random phase ensemble

It has been observed in Section 5 that, for the majority of structures, large variations in energy flow are most closely associated with changes in the phases of eigenvalues and, particularly, with changes in the common component of these phases.

If the subsystems are fairly uniform, then significant scattering of wave components occurs only at the junctions between subsystems, and eigenvalues are typically associated with the propagation of wave components over distances as large, or larger than, a single subsystem. For complex structures of the kind being considered here, subsystem dimensions are generally much larger than the wavelengths of the wave components. Since ensemble variations in these dimensions are also significant compared to these wavelengths, ensemble variations in phases of the eigenvalues are generally large compared to 2π. The energy flow depends only on the phases modulo 2π, however, so that the distribution of phase of any individual eigenvalue over the ensemble may be expected to be random and approximately uniformly distributed in $[-\pi, \pi]$.

It may also be anticipated that the common component of eigenvalue phase is random and uniformly distributed in $[-\pi, \pi]$. Although this does not follow strictly from the observation that individual phases are distributed in this way (because the phases are generally correlated over the ensemble), a number of observations can be made which support the validity of this proposal.

For structures in which the eigenvalue repulsion is very strong, the phase spacing between eigenvalues is approximately constant over the ensemble and the assumption of the uniform distribution of the common phase in $[-\pi, \pi]$ follows immediately. For structures in which the eigenvalue repulsion is very weak, on the other hand, the eigenvalue phases are independent random variables. It can be shown that if the damping is not too heavy and the system matrix **ST** is approximately unitary, then the common eigenvalue phase is approximately uniformly distributed in $[-\pi, \pi]$.

An ensemble of structures is therefore proposed, in which system matrices have the form $e^{i\theta}\mathbf{ST}$, where **ST** is the nominal system matrix associated with the structure at hand and θ is random and uniformly distributed in $[-\pi, \pi]$. This ensemble, over which the magnitudes of the eigenvalues and the eigenvectors are assumed not to vary, will be referred to as the 'scalar random phase', or 'scalar' ensemble.

6.2. Estimation of moments

As described in Section 3, an ensemble of structures can be described in terms of the joint probability density function $f(\mathbf{x})$ of its uncertain parameters $\mathbf{x} = [x_i]$. Although the kth moment of a response quantity $P(\mathbf{x})$ can be found in principle from this density function through Eq. (13), direct evaluation of the integral in this equation is impractical for most structures. An alternative

approach is described here which involves separation of the integral into two parts, of which one can be evaluated analytically and the other can be found by numerical methods.

It is possible to transform the parameters $\mathbf{x}$ into an alternative representation of the form $(\theta, \boldsymbol{\vartheta})$, where θ is the scalar common phase and the vector $\boldsymbol{\vartheta}$ corresponds to parameters orthogonal to θ. With this transformation, the general expression for the moments of $P(\mathbf{x})$ can be written as

$$P^{(k)} = \int \int P^k(\theta, \boldsymbol{\vartheta}) f(\theta, \boldsymbol{\vartheta}) \, \mathrm{d}\theta \, \mathrm{d}\boldsymbol{\vartheta}. \tag{40}$$

If θ is uniformly distributed in $[-\pi, \pi]$, then $f(\theta, \boldsymbol{\vartheta}) = f(\boldsymbol{\vartheta}) / 2\pi$ and Eq. (40) becomes

$$P^{(k)} = \int \langle P^k(\theta, \boldsymbol{\vartheta}) \rangle_\theta f(\boldsymbol{\vartheta}) \, \mathrm{d}\boldsymbol{\vartheta}, \tag{41}$$

where

$$\langle P^k(\theta, \boldsymbol{\vartheta}) \rangle_\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} P^k(\theta, \boldsymbol{\vartheta}) \, \mathrm{d}\theta. \tag{42}$$

The quantity $\langle P^k(\theta, \boldsymbol{\vartheta}) \rangle_\theta$ is the average of $P^k(\theta, \boldsymbol{\vartheta})$ over the scalar ensemble and Eq. (41) gives the expected value of $\langle P^k(\theta, \boldsymbol{\vartheta}) \rangle_\theta$ over the remaining parameters $\boldsymbol{\vartheta}$.

Of greatest interest here are the average and the variance of energy flow over the ensemble. It will be demonstrated in the following sections that Eq. (42) can be evaluated analytically for low-order moments of the response over the scalar ensemble.

6.3. Ensemble-averaged energy flow

Variations in energy flow over the scalar ensemble are associated solely with variations in the outer product $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ through changes in the common component θ of the eigenvalue phases. The average of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ over θ in the interval $[-\pi, \pi]$ is given by

$$\langle \mathbf{g}\mathbf{g}^{\mathrm{H}} \rangle_\theta = \left[\frac{1}{1 - \lambda_i \lambda_j^*} \right]. \tag{43}$$

The scalar ensemble average of the incident power in subsystem X due to excitation in Y, found by substituting Eq. (43) into Eq. (32), is given by

$$\langle \mathbf{P}_{\mathrm{inc},\,XY} \rangle_\theta = \tfrac{1}{2} \mathrm{diag}(\mathbf{V}_{X\bullet} [(\mathbf{U}_{Y\bullet}^{\mathrm{H}} \mathbf{e}_Y \mathbf{e}_Y^{\mathrm{H}} \mathbf{U}_{Y\bullet}) \circ \langle \mathbf{g}\mathbf{g}^{\mathrm{H}} \rangle_\theta] \mathbf{V}_{X\bullet}^{\mathrm{H}}). \tag{44}$$

Since the junction power lost by subsystem X to neighbouring subsystems is given by

$$P_{\mathrm{junc},\,XY} = \mathbf{P}_{\mathrm{inc},XY}^{\mathrm{T}} \, \mathrm{diag}(\mathbf{I} - \mathbf{D}_X^2) \tag{45}$$

and $\mathbf{D}_X^2$ is assumed constant over the ensemble, the scalar ensemble average of this power is

$$\langle P_{\mathrm{junc},\,XY} \rangle_\theta = \langle \mathbf{P}_{\mathrm{inc},\,XY}^{\mathrm{T}} \rangle_\theta \, \mathrm{diag}(\mathbf{I} - \mathbf{D}_X^2). \tag{46}$$

The ensemble averages of other energy-related quantities over the scalar ensemble can be found in a similar way.

Ensembles other than the scalar ensemble may involve more general variations including variations in the eigenvalue phase separations, the eigenvalue magnitudes and the eigenvectors.

Since, for many structures, it is not possible to characterise these variations or to analytically evaluate the expectation of the scalar ensemble average with respect to the uncertain parameters $\boldsymbol{\vartheta}$ as expressed in Eq. (41), it is anticipated that numerical techniques will be required.

6.4. Variance of energy flow

The variance of the response quantity P is given in terms of first and second moments by $\mathrm{Var}[P] = P^{(2)} - P^{(1)\,2}$. Response quantities such as the junction power or the subsystem energies are linear functions of the incident powers, $\mathbf{P}_{\mathrm{inc}}$. Since these functions are assumed not to vary over the ensemble, their second moments are related through constants to the covariance matrix of the incident powers. For example, consider again the junction power lost by subsystem X to neighbouring subsystems given by Eq. (45). The squared junction power can be written as

$$P^2_{\mathrm{junc},\,XY} = \mathrm{diag}(\mathbf{I} - \mathbf{D}_X^2)^{\mathrm{T}}\,\mathbf{L}\,\mathrm{diag}(\mathbf{I} - \mathbf{D}_X^2), \tag{47}$$

where $\mathbf{L} = \mathbf{P}_{\mathrm{inc},XY}\,\mathbf{P}^{\mathrm{T}}_{\mathrm{inc},XY}$ is the incident power covariance matrix. Since $\mathbf{D}_X^2$ is assumed constant over the ensemble, the second moment of the junction power can be found from the expected value of $\mathbf{L}$ through Eq. (47).

Variations in $\mathbf{L}$ over the scalar ensemble arise from variations in the common phases of eigenvalues in $\mathbf{P}_{\mathrm{inc},XY}$ through the matrix $\mathbf{g}\mathbf{g}^{\mathrm{H}}$. The incident power associated with an individual wave component x in subsystem X can be found from Eq. (32) to be given by

$$P_{\mathrm{inc},\,xY} = \tfrac{1}{2}\mathbf{V}_{x\bullet}\,[\,(\mathbf{y}\mathbf{y}^{\mathrm{H}}) \circ (\mathbf{g}\mathbf{g}^{\mathrm{H}})\,]\,\mathbf{V}^{\mathrm{H}}_{x\bullet}, \tag{48}$$

where $\mathbf{y} = \mathbf{U}^{\mathrm{H}}_{Y\bullet}\,\mathbf{e}_Y$. It then follows that the entries of $\mathbf{L} = [l_{ij}]$ have the form

$$l_{ij} = \mathbf{V}_{i\bullet}\,[(\mathbf{y}\mathbf{y}^{\mathrm{H}}) \circ (\mathbf{g}\mathbf{g}^{\mathrm{H}})]\,\mathbf{V}^{\mathrm{H}}_{i\bullet}\,\mathbf{V}_{j\bullet}\,[(\mathbf{y}\mathbf{y}^{\mathrm{H}}) \circ (\mathbf{g}\mathbf{g}^{\mathrm{H}})]\,\mathbf{V}^{\mathrm{H}}_{j\bullet}/4. \tag{49}$$

A number of operations involving the 'vec' operator (which 'stacks' the columns of its matrix argument) and the Kronecker product[1] [14] can be used to gather eigenvalue- and eigenvector-related factors. Applying the 'vec' operator to both sides of Eq. (49) and using a standard result relating to Kronecker products[2] yields

$$l_{ij} = (\mathbf{V}^*_{j\bullet} \otimes \mathbf{V}_{i\bullet})\,([(\mathbf{g}\mathbf{g}^{\mathrm{H}})^{\mathrm{T}} \otimes (\mathbf{g}\mathbf{g}^{\mathrm{H}})] \circ [(\mathbf{y}\mathbf{y}^{\mathrm{H}})^{\mathrm{T}} \otimes (\mathbf{y}\mathbf{y}^{\mathrm{H}})])\,(\mathbf{V}^*_{j\bullet} \otimes \mathbf{V}_{i\bullet})^{\mathrm{H}}/4, \tag{50}$$

where it has been noted that $\mathrm{vec}(\mathbf{V}^{\mathrm{H}}_{i\bullet}\,\mathbf{V}_{j\bullet}) = (\mathbf{V}^*_{j\bullet} \otimes \mathbf{V}_{i\bullet})^{\mathrm{H}}$.

Since the scalar ensemble only involves variations of $\mathbf{g}\mathbf{g}^{\mathrm{H}}$, variations in l_{ij} over the same ensemble depend only on $(\mathbf{g}\mathbf{g}^{\mathrm{H}})^{\mathrm{T}} \otimes (\mathbf{g}\mathbf{g}^{\mathrm{H}})$, which, if $\mathbf{g}\mathbf{g}^{\mathrm{H}}$ is of order $n \times n$, is of order $n^2 \times n^2$. Each

[1] The Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is defined as the block matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

where $\mathbf{A} = [a_{ij}]$ is of order $m \times n$.

[2] For matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{Y}$, $\mathrm{vec}(\mathbf{A}\mathbf{Y}\mathbf{B}) = (\mathbf{B}^{\mathrm{T}} \otimes \mathbf{A})\,\mathrm{vec}\mathbf{Y}$.

entry of this larger matrix depends on four (not necessarily distinct) eigenvalues and has the general form

$$\frac{1}{(1-\lambda_i)(1-\lambda_j^*)(1-\lambda_k)(1-\lambda_l^*)}, \tag{51}$$

where, for entry (p,q) of $(\mathbf{g}\mathbf{g}^{\mathrm{H}})^{\mathrm{T}} \otimes (\mathbf{g}\mathbf{g}^{\mathrm{H}})$,

$$p = (j-1)n + k \qquad \text{and} \qquad q = (i-1)n + l, \tag{52}$$

with $i,j,k,l = 1,\ldots,n$ and $p,q = 1,\ldots,n^2$.

Since only variation in the common phase of the eigenvalues is of interest, it is convenient to consider the function

$$f(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{(1-\lambda_i \mathrm{e}^{\mathrm{i}\theta})(1-\lambda_j^* \mathrm{e}^{-\mathrm{i}\theta})(1-\lambda_k \mathrm{e}^{\mathrm{i}\theta})(1-\lambda_l^* \mathrm{e}^{-\mathrm{i}\theta})}. \tag{53}$$

The average of this function over θ in the interval $[-\pi,\pi]$ is given by

$$\langle f(\mathrm{e}^{\mathrm{i}\theta})\rangle_\theta = \frac{1-\lambda_i\,\lambda_j^*\,\lambda_k\,\lambda_l^*}{(1-\lambda_i\,\lambda_j^*)(1-\lambda_k\,\lambda_j^*)(1-\lambda_i\,\lambda_l^*)(1-\lambda_k\,\lambda_l^*)}. \tag{54}$$

The scalar ensemble average $\langle(\mathbf{g}\mathbf{g}^{\mathrm{H}})^{\mathrm{T}} \otimes (\mathbf{g}\mathbf{g}^{\mathrm{H}})\rangle_\theta$ then follows from the index relationships given in Eq. (52). This average replaces $(\mathbf{g}\mathbf{g}^{\mathrm{H}})^{\mathrm{T}} \otimes (\mathbf{g}\mathbf{g}^{\mathrm{H}})$ in Eq. (50) to give the corresponding averages of $\mathbf{L}$ and the squared junction power given in Eq. (47). The variance of the junction power over the scalar ensemble is then given by

$$\mathrm{Var}[P_{\mathrm{junc},\,XY}]_\theta = \langle P_{\mathrm{junc},\,XY}^2\rangle_\theta - \langle P_{\mathrm{junc},\,XY}\rangle_\theta^2. \tag{55}$$

As with the estimation of the average of the energy flow over the general ensemble from the averages over scalar ensembles, the variance over the general ensemble can be found from the expected value of scalar ensemble variances, as given in Eq. (41). It is again anticipated that numerical methods will be required for this task.

7. Concluding remarks

A computationally efficient method has been presented for the estimation of the response statistics of complex structures. The vibration field is described in terms of wave components, and the wave component transmission and reflection characteristics of the structure are quantified in a pair of subsystem and junction global scattering matrices, $\mathbf{S}$ and $\mathbf{T}$. Uncertainty is introduced through the concept of an ensemble of structures which differ randomly in detail, and the eigenproperties of the matrix product $\mathbf{ST}$ are assumed to be random variables.

Examination of the relationships between typical ensemble variations in the eigenproperties of $\mathbf{ST}$ and the response of any structure indicates that, of all the eigenproperties, response variations are most closely associated with variations in the common component of the eigenvalue phases. A 'scalar random phase' ensemble is proposed in which the common phase of the eigenvalues is random and uniformly distributed in $[-\pi,\pi]$, and the other eigenproperties remain constant. Low-order statistical moments of structural response over this ensemble are thought to closely resemble

those of many real ensembles, particulary those involving strong eigenvalue repulsion (for example, those with strong irregular connections between irregular subsystems).

Analytical expressions are given for the average and variance of energy responses over the scalar ensemble. For ensembles of structures less well represented by this ensemble (those with less strong eigenvalue repulsion, for example), the analytical estimates of average and variance can be improved by Monte Carlo simulation with a relatively small sample size. The computational cost of this improvement is expected to be very small because, although low-order moments of the response statistics are dominated by the effects of the generally small number of resonant members of the ensemble, the analytical scalar ensemble estimates vary relatively slowly and can be quickly numerically integrated. A strong similarity exists between averaging over the scalar ensemble and averaging with respect to frequency [12].

In principle, the method can be applied to a wide variety of structures, so long as the cross-sections at which energy flows are evaluated are not so close together that there is significant power transport associated with near-field interactions. A more important limitation in practice may be that associated with the determination of scattering matrices for structures of general form. In view of the fundamental uncertainty involved in the applications for which this approach is designed, detailed calculation of these matrices is not warranted.

Application of the present approach to regular and irregular two-plate structures, and good agreement between the averages and variances of energy flows predicted by the method and the results of numerical simulations, are demonstrated in a companion paper [5].

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