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On the stability properties of a Van der Pol–Duffing oscillator that is driven by a real noise

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Abstract

In this paper, we consider a Van der Pol–Duffing oscillator that is excited parametrically by a small intensity real noise, which is assumed to be an integrable function of an n-dimensional Ornstein–Uhlenbeck vector process that is an output of a linear filter system. The stability properties include the moment Lyapunov exponent $g(p, x_0)$ and the maximal Lyapunov exponent, and the stability in probability are examined. To study a model of enhanced generality, we remove both the detailed balance condition and the strong mixing condition. In the case of an arbitrary finite real number p, we employ the perturbation method and a spectrum representation of the Fokker–Planck operator of the linear filter system to construct asymptotic expansions of the pth moment Lyapunov exponent and the top Lyapunov exponent. The same methods are also used for a nonlinear stochastic system to obtain the FPK (Fokker–Planck–Kolmogonov) equation for the amplitude process, which is identical to the one that is derived from the stochastic averaging method in the case of a broadband noise excitation. On the basis of this FPK equation, we also examine the almost-sure stability condition of the Ito stochastic differential equation for the amplitude process, which matches the result that is derived from the maximal Lyapunov exponent. Finally, the method proposed by Lin and Cai (Probabilistic Structural Dynamics, Advanced Theory and

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Application, McGraw-Hill, New York, 1995) is adopted to examine the stability in probability of the amplitude process for the nonlinear Ito differential equation.

1. Introduction

The maximal Lyapunov exponent has been effectively employed as an important index in defining the stochastic bifurcation point for a random dynamical system in the probability 1 sense, which is called the dynamic bifurcation point [1,2]. This is mainly attributed to the fact that the Lyapunov exponent characterizes the exponential rate of change of the response of a random dynamical system, and therefore the sample or the almost-sure stability of the stationary solution of a random dynamical problem depends on the sign of the maximal Lyapunov exponent. A general method for exactly evaluating the maximal Lyapunov exponent of a linear Ito stochastic differential equation was first presented by Khasminskii [3], and has been successfully employed for two-dimensional Ito stochastic systems by Kozin and Prodromou [4], Nishioka [5], Ariaratnam and Xie [6], and many other researchers.

In the case of ergodic and real noise excitations, there are some results that refer to the asymptotic expansions of top Lyapunov exponents, most of which are due to Arnold et al. [7], Namachchivaya and Roessel [8], Doyle and Namachchivaya [9], and Liu and Liew [10–12]. For a real noise system that does not meet the strong mixing condition, the stochastic averaging method is not available, and one has to resort to a perturbation method [7]. However, even for an almost-sure stable system, there is a probability that the mean square response for the system may still exceed some threshold and may grow exponently, which implies that the mean square response is unstable.

Let $x(t,x_0)$ be a solution to a random dynamical system. To describe the exponential growth rate of its pth ($p>0$) moment, we can define the moment Lyapunov exponent as

$$g(p,x_0) = \lim_{t\to\infty} \frac{1}{t} \log E\|x(t,\, x_0)\|^p, \quad p \in \mathrm{R}, \tag{1}$$

which implies that if $g(p,x_0)<0$, then $E\|x(t,x_0)\|^p \to 0$ as $t\to\infty$, whereas if $g(p,x_0)>0$, then $E\|x(t,x_0)\|^p \to \infty$ as $t\to\infty$. In Ref. [13], it has been shown that, under the conditions specified, the limit in Eq. (1) exists and is independent of x_0. It can then be expressed as $g(p)$, which is a convex analytic function of $p\in\mathrm{R}$, $g(p)/p$ is increasing, and

$$\lambda = \left.\frac{\partial g}{\partial p}\right|_{p=0} = \lim_{t\to\infty} \frac{1}{t} \log \|x(t,\, x_0)\|^p, \quad p \in \mathrm{R}, \tag{2}$$

which is the maximal Lyapunov exponent.

In accordance with the large-deviation theory [2,14], if there is a non-zero solution δ of the equation $g(p)=0$, then it is unique and is called the stability index. It has been shown that if the trivial solution $x=0$ of an Ito linear stochastic differential equation is almost-sure stable, then the probability of exit from the ball $\|x\|<r$ has the order of $\|x\|^\delta$ for $x\to 0$ for any $r>0$, and the solution $x=0$ is exponentially p-stable for $p<\delta$ and exponentially p-unstable for $p>\delta$, which results in the stability condition for the pth moment.

Comparatively, the problems that arise in the asymptotic analysis of moment Lyapunov exponents become much more complicated than those that arise in the analysis of maximal Lyapunov exponents, and furthermore there are not as many existing results for moment Lyapunov exponents as for maximal Lyapunov exponents. For a white and real noises excited two-dimensional system and a system of two coupled oscillators that is driven by a real noise, Arnold et al. [15] and Namachchivaya et al. [16] obtain the asymptotic expansions of the pth moment Lyapunov exponents in the case of small noise intensity and a small p. The idea is that one can obtain the asymptotic expansion in powers of the small noise intensity for the maximal Lyapunov exponent $\lambda = \partial g(0)/\partial p$. The same is also true for $\partial^n g(0)/\partial p^n$. Finally, one can obtain the Taylor series of the pth moment Lyapunov exponent in terms of a small p. These formulas, however, are rather complicated, and the approximation is only valid for a small p, which does not allow us to compute, for example, the stability index.

Khasminskii and Moshchuk [14] consider a two-dimensional system with small white noise excitations. They prove that for a system in which the system matrix has two purely imaginary eigenvalues, the pth moment Lyapunov exponent possesses an asymptotic expansion in terms of the small noise intensity for a finite value of p. For a system of two coupled oscillators that is driven by real noises, Namachchivaya and Roessel [17] obtain the asymptotic expansion of the moment Lyapunov exponent for a finite p. In Ref. [17], the extension of the perturbation method that was introduced by Arnold et al. [7] is applied.

In this paper, we consider a Van der Pol–Duffing oscillator that is excited parametrically by a small intensity real noise, which is assumed to be an integrable function of an n-dimensional Ornstein–Uhlenbeck vector process that is an output of a linear filter system. A detailed study is carried out on the stability properties, including the pth moment Lyapunov exponent $g(p,x_0)$, the maximal Lyapunov exponent, and the stability in probability. In this work, we propose a model of enhanced generality that removes the detailed balance condition and also the strong mixing condition that is the prerequisite for the stochastic averaging method. To tackle the difficulties encountered, for an arbitrary finite p, the perturbation method and a spectrum representation of the Fokker–Planck operator for the linear filter system are employed to construct the asymptotic expansion of the pth moment Lyapunov exponent and the top Lyapunov exponent. Using the same methods for a nonlinear stochastic system, we obtain the FPK equation of the amplitude process, which is identical to that which is derived from the stochastic averaging method in the case of a broadband noise excitation. Based on this FPK equation, we can examine the almost-sure stability condition of the amplitude process, which matches the result that is derived from the expression of the maximal Lyapunov exponent. To investigate the stability in probability of the amplitude process, the method proposed by Lin and Cai [1] is adopted in this study.

2. Spectral analysis for a linear filter system

In this section, we review the existing results for the spectral analysis of an n-dimensional linear filter system. Consider a general linear filter system, which is governed by the following stochastic differential system:

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \dot{\mathbf{W}}(t), \tag{3}$$

where $\mathbf{A}=(a_{ij})_{n\times n}$; a_{ij} are the real or complex numbers, $\dot{\mathbf{W}}(t)$ is an n-dimensional zero-mean Gaussian white noise with $E(\dot{\mathbf{W}}(t+\tau)\dot{\mathbf{W}}(t))=\mathbf{V}\delta(\tau)$, $\mathbf{V}=(v_{ij})_{n\times n}$ is a symmetric, non-negative defined constant matrix, and $\mathbf{u}=(u_1, u_2,\dots,u_n)^{\mathrm{T}}$ is an Ornstein–Uhlenbeck vector process, which is in fact a zero-mean stationary Gaussian diffusion process. The matrix $\mathbf{A}$ is assumed to have a complete set of eigenvalues $\alpha_1,\dots,\alpha_n$, along with the corresponding eigenvectors $\mathbf{e}_1,\dots,\mathbf{e}_n$, which means that $\alpha_{i\neq\alpha_j}$ $(i\neq j)$. Furthermore, as in Ref. [18], the following two conditions are assumed in the present study:

(a) Each eigenvalue α_i is assumed to possess a negative real part, i.e., $\mathrm{R}(\alpha_i)<0$ $(i=1,2,\dots,n)$.
(b) $(\mathbf{A},\tilde{\mathbf{V}})$ is a controllable pair, i.e., $\mathrm{rank}(\tilde{\mathbf{V}},\mathbf{A}\tilde{\mathbf{V}},\dots,\mathbf{A}^{n-1}\tilde{\mathbf{V}})=n$, where $\mathbf{V}=\tilde{\mathbf{V}}\tilde{\mathbf{V}}^{\mathrm{T}}$.

In fact, the first condition assures that the equilibrium solution $\mathbf{u}=\mathbf{0}$ for the relevant deterministic system is Lyapunov asymptotically stable.

For the diffusion process $\mathbf{u}(t)$, the differential generator (backward Kolmogorov operator) $L_{\mathbf{u}}$ and its adjoint, the Fokker–Planck operator $L_{\mathbf{u}}^*$, are, respectively, given by

$$L_{\mathbf{u}}=a_{ij}u_j\frac{\partial}{\partial u_i}+\frac{1}{2}v_{ij}\frac{\partial^2}{\partial u_i\partial u_j},\quad L_{\mathbf{u}}^*=\frac{\partial}{\partial u_i}[a_{ij}u_j]-\frac{1}{2}v_{ij}\frac{\partial^2}{\partial u_i\partial u_j},\tag{4}$$

where the repeated indices indicate the usual summation. Correspondingly, the Kolmogorov backward equation, the FPK equation, and their initial conditions are

$$\left[\frac{\partial}{\partial t_0}+L_{\mathbf{u}_0}\right]q(\mathbf{u},t\,|\,\mathbf{u}_0,t_0)=0,\quad \left[\frac{\partial}{\partial t}+L_{\mathbf{u}}^*\right]p(\mathbf{u},t\,|\,\mathbf{u}_0,t_0)=0,$$
$$q(\mathbf{u},t\,|\,\mathbf{u}_0,t)=\delta(\mathbf{u}_0-\mathbf{u}),\quad p(\mathbf{u},t_0\,|\,\mathbf{u}_0,t_0)=\delta(\mathbf{u}-\mathbf{u}_0).\tag{5}$$

For the system that is described in Eq. (3), the stationary probability density function for $\mathbf{u}(t)$, which is the solution to the degenerate FPK equation $\partial p(\mathbf{u},t\,|\,\mathbf{u}_0,t_0)/\partial t=0$, is

$$p_s(\mathbf{u})=N\exp[-\tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}_{\mathbf{u}}^{-1}\mathbf{u}],\quad N=(2\pi)^{-n/2}[\det\mathbf{K}_{\mathbf{u}}]^{1/2},\tag{6}$$

where N is the normalization constant, and $\mathbf{K}_{\mathbf{u}}=\langle\mathbf{u}(t)(\mathbf{u}(t))^{\mathrm{T}}\rangle$ is the covariance matrix, which is the solution of the steady-state variance equation

$$\mathbf{A}\mathbf{K}_{\mathbf{u}}+\mathbf{K}_{\mathbf{u}}\mathbf{A}^{\mathrm{T}}+\mathbf{V}=0.\tag{7}$$

In this study, $\mathbf{U}=(\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n)$ is assumed to be the relevant eigenmatrix of $\mathbf{A}$, which leads to $\mathbf{D}=\mathbf{U}^{-1}\mathbf{A}\mathbf{U}=\mathrm{diag}[\alpha_1,\alpha_2,\dots,\alpha_n]$.

The eigenvalue problems that correspond to the two operators arise as

$$L_{\mathbf{u}}\psi_\lambda(\mathbf{u})=\lambda\psi_\lambda(\mathbf{u}),\quad L_{\mathbf{u}}^*\psi_{\lambda'}^*(\mathbf{u})=\lambda'\psi_{\lambda'}^*(\mathbf{u}).\tag{8}$$

It can be verified that the spectrum of the operators $L_{\mathbf{u}}$ and $L_{\mathbf{u}}^*$ is discrete, and that the operators possess the same set of eigenvalues.

According to Roy [19], the solutions to the associated eigenvalue problem of the backward Kolmogorov operator $L_{\mathbf{u}}$ contain two parts:

(i) Each of the eigenvalues can be expressed as $\lambda_{\mathbf{m}}=m_1\alpha_1+\cdots+m_n\alpha_n$, where $\mathbf{m}=(m_1,m_2,\dots,m_n)$, $m=m_1+m_2+\cdots+m_n$, in which m_i $(i=1,2,\dots,n)$ are the non-negative integers.

(ii) The corresponding eigenfunction is found to be an element of the set of multivariate Hermite polynomials, i.e.,

$$\psi_{\mathbf{m}}(\mathbf{u}) = G_{\mathbf{m}}(\mathbf{v}) = (-1)^m \exp\left[\frac{1}{2}\mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{v}\right] \frac{\partial^m}{\partial w_1^{m_1}\partial w_2^{m_2}\cdots\partial w_n^{m_n}} \exp\left[-\frac{1}{2}\mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{v}\right],$$
$$\mathbf{v} = \mathbf{U}^{-1}\mathbf{u}, \quad \mathbf{C} = \mathbf{U}^{\mathrm{T}}\mathbf{K}_{\mathbf{u}}^{-1}\mathbf{U} = \mathbf{K}_{\mathbf{v}}^{-1}, \quad \mathbf{w} = \mathbf{U}^{\mathrm{T}}\mathbf{K}_{\mathbf{u}}^{-1}\mathbf{u}. \tag{9}$$

To determine the function of $\psi^*_{\mathbf{m}}(\mathbf{u})$, which is the eigenfunction of $L^*_{\mathbf{u}}$ and corresponds to the same eigenvalue $\lambda_{\mathbf{m}}$, Roy [19] shows that if the stochastic system that is described in Eq. (3) satisfies the detailed balance condition (see Ref. [20])

$$p(\mathbf{u}', \tau \mid \mathbf{u}, 0)p_s(\mathbf{u}) = p(\varepsilon\mathbf{u}, \tau \mid \varepsilon\mathbf{u}', 0)p_s(\mathbf{u}'), \quad p_s(\mathbf{u}) = p_s(\varepsilon\mathbf{u}), \tag{10}$$

then $\psi^*_{\mathbf{m}}(\mathbf{u})$ can be expressed as

$$\psi^*_{\mathbf{m}}(\varepsilon\mathbf{u}) = \psi^*_0(\mathbf{u})\psi_{\mathbf{m}}(\mathbf{u}) = (-1)^m \frac{\partial^m}{\partial w_1^{m_1}\cdots\partial w_n^{m_n}}\psi^*_0(\mathbf{u}),$$
$$\psi^*_0(\mathbf{u}) = \psi^*_0(\varepsilon\mathbf{u}) = N\exp\left[-\frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}_{\mathbf{u}}^{-1}\mathbf{u}\right]$$
$$= N\exp\left[-\frac{1}{2}\mathbf{v}^{\mathrm{T}}\mathbf{K}_{\mathbf{v}}^{-1}\mathbf{v}\right] = N\exp\left[-\frac{1}{2}\mathbf{w}^{\mathrm{T}}\mathbf{K}_{\mathbf{v}}\mathbf{w}\right]. \tag{11}$$

In fact, $\psi^*_{\mathbf{m}}(\mathbf{u})$ can also be expressed as

$$\psi^*_{\mathbf{m}}(\mathbf{u}) = \psi^*_0(\mathbf{u})\prod_{i=1}^{n}(\mathbf{u}_i^{-1}\mathbf{u})^{m_i}, \tag{12}$$

where $\mathbf{u}_i^{-1}$ is the ith row vector of $\mathbf{U}^{-1}$, which is the inverse matrix of $\mathbf{U}$.

From the results in Liberzon and Brockett [18], we know that under the above conditions (a) and (b), the detailed balance condition can be removed, and then $\psi^*_{\mathbf{m}}(\mathbf{u})$ can be expressed as

$$\psi^*_{\mathbf{m}}(\mathbf{u}) = \psi_0(\mathbf{u})\prod_{i=1}^{n}(\mathbf{u}_i^{\mathrm{T}}\mathbf{u})^{m_i}, \tag{13}$$

where $\mathbf{u}_i^{\mathrm{T}}$ is the ith row vector of $\mathbf{U}^{\mathrm{T}}$, which is the transpose matrix of $\mathbf{U}$. With this conclusion, it can be easily verified that if the system matrix $\mathbf{A}$ is real and symmetric, without the condition of detailed balance, Eq. (12) is also tenable. Therefore, in this paper, we remove the condition of detailed balance.

In fact, the explicit expressions of the eigenfunctions for $L_{\mathbf{u}}$ and $L^*_{\mathbf{u}}$ are not necessary. We assume in this paper that the system that is described in Eq. (3) is defined on domain D, which is a bounded closed set in R^n with its entire boundary ∂D. Being the solutions to the FPK equation and the Kolmogorov backward equation, respectively, on ∂D, both $p(\mathbf{u}, t \mid \mathbf{u}_0, t_0)$ and $q(\mathbf{u}, t \mid \mathbf{u}_0, t_0)$ are assumed to satisfy the boundary conditions

$$\mathbf{n}\cdot\mathbf{G}(\mathbf{u}, t \mid \mathbf{u}_0, t_0) = 0, \quad \text{or} \quad p(\mathbf{u}, t \mid \mathbf{u}_0, t_0) = 0, \quad \mathbf{u} \in \partial D,$$
$$n_i v_{ij}\frac{\partial}{\partial u_{j0}}q(\mathbf{u}, t \mid \mathbf{u}_0, t_0) = 0, \quad \text{or} \quad q(\mathbf{u}, t \mid \mathbf{u}_0, t_0) = 0, \quad \mathbf{u}_0 \in \partial D, \tag{14}$$

that correspond to ∂D which is called a reflective or absorbing boundary. In Eq. (14), $\mathbf{n}$ is a unit vector that is normal to ∂D and

$$G_i(\mathbf{u}, t \mid \mathbf{u}_0, t_0) = a_{ij} u_j p(\mathbf{u}, t \mid \mathbf{u}_0, t_0) - \frac{1}{2} v_{ij} \frac{\partial}{\partial u_j} p(\mathbf{u}, t \mid \mathbf{u}_0, t_0). \tag{15}$$

Furthermore, $\psi_{\mathbf{m}}(\mathbf{u})$ and $\psi^*_{\mathbf{m}}(\mathbf{u})$ are also assumed to satisfy the same boundary conditions, which ensure that $\psi_{\mathbf{m}}(\mathbf{u})$ and $\psi^*_{\mathbf{m}}(\mathbf{u})$ are bi-orthogonally normal [20]

$$\langle \psi_{\mathbf{m}1}(\mathbf{u}), \psi^*_{\mathbf{m}2}(\mathbf{u}) \rangle = \int_D \psi_{\mathbf{m}1}(\mathbf{u}) \psi^*_{\mathbf{m}2}(\mathbf{u}) \, d\mathbf{u} = \delta_{\mathbf{m}1,\mathbf{m}2} = \begin{cases} 1, & \mathbf{m}1 = \mathbf{m}2, \\ 0, & \mathbf{m}1 \neq \mathbf{m}2. \end{cases} \tag{16}$$

With these results, the transition probability density of the process $\mathbf{u}(t)$ can be written as

$$p(\mathbf{u}, \tau \mid \mathbf{u}') = \sum_{m_1=0,\ldots,m_n=0}^{\infty} \exp[\lambda_{\mathbf{m}} \tau] \psi_{\mathbf{m}}(\mathbf{u}') \psi^*_{\mathbf{m}}(\mathbf{u}), \quad \tau \geqslant 0. \tag{17}$$

This yields the expression of $\mathbf{R}_{\mathbf{u}}(\tau)$ and the covariance matrix of $\mathbf{u}(t)$:

$$\begin{aligned} \mathbf{R}_{\mathbf{u}}(\tau) &= \int_D d\mathbf{u} \int_D d\mathbf{u}' [\mathbf{u}^{\mathrm{T}} \mathbf{u}' p(\mathbf{u}, \tau | \mathbf{u}') p_s(\mathbf{u}')] \\ &= \sum_{m_1=0,\ldots,m_n=0}^{\infty} \langle \mathbf{u} \psi^*_0(\mathbf{u}), \psi_{\mathbf{m}}(\mathbf{u}) \rangle [\langle \mathbf{u}, \psi^*_{\mathbf{m}}(\mathbf{u}) \rangle]^{\mathrm{T}} \exp[\lambda_{\mathbf{m}} \tau], \end{aligned} \tag{18}$$

from which we obtain the spectral density function matrices

$$\begin{aligned} \mathbf{S}_{\mathbf{u}}(\omega) &= 2 \int_0^{\infty} \mathbf{R}_{\mathbf{u}}(\tau) \cos(\omega \tau) \, d\tau \\ &= - \sum_{m_1=0,\ldots,m_n=0}^{\infty} \langle \mathbf{u} \psi^*_0(\mathbf{u}), \psi_{\mathbf{m}}(\mathbf{u}) \rangle [\langle \mathbf{u}, \psi^*_{\mathbf{m}}(\mathbf{u}) \rangle]^{\mathrm{T}} \frac{2\lambda_{\mathbf{m}}}{\lambda^2_{\mathbf{m}} + \omega^2}, \\ \mathbf{\Phi}_{\mathbf{u}}(\omega) &= 2 \int_0^{\infty} \mathbf{R}_{\mathbf{u}}(\tau) \sin(\omega \tau) \, d\tau \\ &= - \sum_{m_1=0,\ldots,m_n=0}^{\infty} \langle \mathbf{u} \psi^*_0(\mathbf{u}), \psi_{\mathbf{m}}(\mathbf{u}) \rangle [\langle \mathbf{u}, \psi^*_{\mathbf{m}}(\mathbf{u}) \rangle]^{\mathrm{T}} \frac{2\omega}{\lambda^2_{\mathbf{m}} + \omega^2}. \end{aligned} \tag{19}$$

For a scalar stochastic function $f(\mathbf{u})$, which is an integrable function of $\mathbf{u}$ in the sense that $\int_D [f(\mathbf{u})]^2 \psi^*_0(\mathbf{u}) \, d\mathbf{u} < +\infty$, then

$$E[f(\mathbf{u})] = \int_D f(\mathbf{u}) \psi^*_0(\mathbf{u}) \, d\mathbf{u} = 0. \tag{20}$$

The covariance and the spectral density function for $f(\mathbf{u})$ can be obtained as

$$\begin{aligned} R_f(\tau) &= \int_D d\mathbf{u} \int_D d\mathbf{u}' [f(\mathbf{u}) f(\mathbf{u}') p(\mathbf{u}, \tau | \mathbf{u}') p_s(\mathbf{u}')] \\ &= \sum_{m_1=0,\ldots,m_n=0}^{\infty} \langle f(\mathbf{u}) \psi^*_0(\mathbf{u}), \psi_{\mathbf{m}}(\mathbf{u}) \rangle \langle f(\mathbf{u}), \psi^*_{\mathbf{m}}(\mathbf{u}) \rangle \exp[\lambda_{\mathbf{m}} \tau], \end{aligned}$$

$$\begin{aligned} S_f(\omega) &= 2\int_0^{\infty} R_f(\tau)\cos(\omega\tau)\,\mathrm{d}\tau \\ &= -\sum_{m_1=0,\ldots,m_n=0}^{\infty} \langle f(\mathbf{u})\psi_0^*(\mathbf{u}), \psi_{\mathbf{m}}(\mathbf{u})\rangle \langle f(\mathbf{u}), \psi_{\mathbf{m}}^*(\mathbf{u})\rangle \frac{2\lambda_{\mathbf{m}}}{\lambda_{\mathbf{m}}^2+\omega^2}, \\ \Phi_f(\omega) &= 2\int_0^{\infty} R_f(\tau)\sin(\omega\tau)\,\mathrm{d}\tau \\ &= -\sum_{m_1=0,\ldots,m_n=0}^{\infty} \langle f(\mathbf{u})\psi_0^*(\mathbf{u}), \psi_{\mathbf{m}}(\mathbf{u})\rangle \langle f(\mathbf{u}), \psi_{\mathbf{m}}^*(\mathbf{u})\rangle \frac{2\omega}{\lambda_{\mathbf{m}}^2+\omega^2}. \end{aligned} \tag{21}$$

3. Van der Pol–Duffing oscillator excited parametrically by real noise

In this section, we consider a deterministic nonlinear Van der Pol–Duffing oscillator that is driven parametrically by a real noise process $f(\mathbf{u})$, i.e.,

$$\ddot{x} - \varepsilon^2\beta\dot{x} + \omega^2 x + \varepsilon^2\gamma x^2\dot{x} + \varepsilon^2\delta\dot{x}^3 = \varepsilon(\omega\sigma_1 x + \sigma_2\dot{x})f(\mathbf{u}), \tag{22}$$

where β is the damping constant, ω is the natural frequency, γ and δ are real parameters, $f(\mathbf{u})$ is an integrable function of $\mathbf{u}(t)$, which is defined in Eq. (3), and the parameters σ_1 and σ_2 represent the noise intensities.

To investigate the stability properties for such a system, an appropriate transformation of the original system should be undertaken. With the transformation

$$x = a\cos\phi, \quad \dot{x} = -a\omega\sin\phi, \quad \phi = \varphi + \omega t, \quad \phi, \varphi \in [0, \pi], \tag{23}$$

we can obtain a set of differential equations that govern the amplitude process a, phase process ϕ, and the noise process $\mathbf{u}$:

$$\dot{a} = a_\varepsilon(a,\phi), \quad \dot{\phi} = \phi_\varepsilon(a,\phi), \quad \dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \dot{\mathbf{W}}(t), \tag{24}$$

where

$$\begin{aligned} a_\varepsilon(a,\phi) &= \varepsilon a_1(a,\phi)f(\mathbf{u}) + \varepsilon^2 a_2(a,\phi), \\ \phi_\varepsilon(\phi) &= \omega + \varepsilon\phi_1(a,\phi)f(\mathbf{u}) + \varepsilon^2\phi_2(a,\phi), \\ a_1(a,\phi) &= \tfrac{1}{2}\sigma_2 a[1-\cos(2\phi)] - \tfrac{1}{2}\sigma_1 a\sin 2\phi, \\ a_2(a,\phi) &= \tfrac{1}{2}\beta a[1-\cos(2\phi)] \\ &\quad - \tfrac{1}{8}a^3\{(\omega^2\delta-\gamma)\cos(4\phi) - 4\omega^2\delta\cos(2\phi) + (3\omega^2\delta+\gamma)\}, \\ \phi_1(a,\phi) &= \tfrac{1}{2}\sigma_2\sin 2\phi - \tfrac{1}{2}\sigma_1[1+\cos(2\phi)], \\ \phi_2(a,\phi) &= \frac{\beta}{2}\sin 2\phi + \frac{1}{8}a^2\{-2(\omega^2\delta+\gamma)\sin(2\phi) + (\omega^2\delta-\gamma)\sin(4\phi)\}. \end{aligned} \tag{25}$$

4. Moment Lyapunov exponent and maximal Lyapunov exponent

4.1. Formulation

In this section, we derive the asymptotic expansions of the moment Lyapunov exponent and the top Lyapunov exponent for the system that is described in Eq. (22).

Consider the linearization of Eq. (22)

$$\ddot{x} - \varepsilon^2 \beta \dot{x} + \omega^2 x = \varepsilon(\omega\sigma_1 x + \sigma_2 \dot{x}) f(\mathbf{u}). \tag{26}$$

The following transformation:

$$x = a\cos\phi, \quad \dot{x} = -a\omega \sin\phi, \quad \rho = \ln a, \quad \phi = \varphi + \omega t, \quad \phi, \varphi \in [0, \pi] \tag{27}$$

yields a set of equations for the arguments of a, ϕ, and the noise process $\mathbf{u}(t)$;

$$\dot{\rho} = \rho_{l\varepsilon}(\phi), \quad \dot{\phi} = \phi_{l\varepsilon}(\phi), \quad \dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \dot{\mathbf{W}}(t), \tag{28}$$

where

$$\begin{aligned}
&\rho_{l\varepsilon}(\phi) = \varepsilon\rho_{l1}(\phi) f(\mathbf{u}) + \varepsilon^2 \rho_{l2}(\phi),\\
&\phi_{l\varepsilon}(\phi) = \omega + \varepsilon\phi_{l1}(\phi) f(\mathbf{u}) + \varepsilon^2 \phi_{l2}(\phi),\\
&\rho_{l1}(\phi) = \sigma_2 \sin^2\phi - \tfrac{1}{2}\sigma_1 \sin 2\phi, \quad \rho_{l2}(\phi) = \beta \sin^2\phi,\\
&\phi_{l1}(\phi) = \tfrac{1}{2}\sigma_2 \sin 2\phi - \sigma_1 \cos^2\phi, \quad \phi_{l2}(\phi) = \frac{\beta}{2}\sin 2\phi.
\end{aligned} \tag{29}$$

As $\phi(t)$ and $\mathbf{u}(t)$ are both independent of the variable ρ, the vector process $(\phi(t), \mathbf{u}(t))$ forms a diffusive process of dimension $(n+1)$ with the generator (backward Kolmogorov operator)

$$\begin{aligned}
&L_\varepsilon = L_0 + \varepsilon L_1 + \varepsilon^2 L_2,\\
&L_0 = L_{\mathbf{u}} + \omega\frac{\partial}{\partial\phi}, \quad L_1 = f(\mathbf{u})\phi_{l1}\frac{\partial}{\partial\phi}, \quad L_2 = \phi_{l2}\frac{\partial}{\partial\phi}
\end{aligned} \tag{30}$$

and the adjoint operator (Fokker–Planck operator)

$$\begin{aligned}
&L_\varepsilon^* = L_0^* + \varepsilon L_1^* + \varepsilon^2 L_2^*,\\
&L_0^* = L_{\mathbf{u}}^* - \omega\frac{\partial}{\partial\phi}, \quad L_1^* = -f(\mathbf{u})\frac{\partial}{\partial\phi}\phi_{l1}, \quad L_2^* = -\frac{\partial}{\partial\phi}\phi_{l2}.
\end{aligned} \tag{31}$$

The moment Lyapunov exponent $g_{\varepsilon,p}$ is the principal simple eigenvalue for the operator L_p [13,14], i.e.,

$$L_p f_{\varepsilon,p} = g_{\varepsilon,p} f_{\varepsilon,p}, \tag{32}$$

where the L_p is defined as

$$\begin{aligned}
L_p &= L_\varepsilon + p\rho_\varepsilon\\
&= L_0 + \varepsilon(L_1 + pf(\mathbf{u})\rho_{l1}) + \varepsilon^2(L_2 + p\rho_{l2}).
\end{aligned} \tag{33}$$

Although the moment Lyapunov exponent is an important characteristic in the analysis of the exponential growth rate of the pth moment of the solution process for a linear stochastic system, it is almost impossible to obtain an explicit expression for it. Therefore, for some stochastic linear systems that are close to deterministic, efforts have been made to determine the asymptotic expressions of the maximal Lyapunov exponents and the moment Lyapunov exponents. Two-dimensional real noise and white noise systems are investigated by Arnold et al. [15]. The asymptotical expressions of the moment Lyapunov exponents are obtained for powers of small noise intensity and a small p. Two coupled oscillations that are driven by a real noise are considered by Namachchivaya et al. [16]. They obtain the small noise expansions for the moment Lyapunov exponents for the powers of a small p. Therefore, as pointed out by Khasminskii and Moshchuk [14], an approximation that is valid for small values of p does not allow us to determine the stability index. For two-dimensional Ito stochastic differential equations for finite values of p when the system matrix is assumed to have two purely imaginary eigenvalues, Khasminskii and Moshchuk [14] prove that the pth moment Lyapunov exponent can be expressed asymptotically for the powers of ε that represent the small noise intensity. The asymptotic series expressions for the finite pth moment Lyapunov exponents of two coupled oscillators that are driven by real noises are obtained by Namachchivaya and Roessel [17], who assume that an infinitesimal generator of the noise has an isolated simple zero eigenvalue.

In this section, we determine the asymptotic expansion of the pth moment Lyapunov exponent of the system that is described in Eq. (26) for the powers of small ε for any finite pth moment. To consider a model of enhanced generality, the strong mixing condition and the detailed balance condition are removed. To tackle the complexity that is encountered in the present work, a perturbation method and the results of the spectral analysis of the Fokker–Planck operator of a linear filter system are employed. We show that the results that are obtained match those in Ref. [15], which are for a small pth moment.

4.2. Asymptotic analysis

As the present system matrix possesses a pair of purely imaginary eigenvalues, according to Khasminskii and Moshchuk [14], we can assume that

$$\begin{aligned} g_{\varepsilon,p} &= g_{0,p} + \varepsilon g_{1,p} + \cdots + \varepsilon^n g_{n,p} + \cdots, \\ f_{\varepsilon,p}(\phi) &= f_{0,p}(\phi) + \varepsilon f_{1,p}(\phi) + \cdots + \varepsilon^n f_{\varepsilon,p}(\phi) + \cdots . \end{aligned} \tag{34}$$

Substituting Eq. (34) into Eq. (32) leads to the following recursive equations:

$$\begin{aligned} &\varepsilon^0 : L_0 f_{0,p} = g_{0,p} f_{0,p}, \\ &\varepsilon^1 : L_0 f_{1,p} = g_{0,p} f_{1,p} + g_{1,p} f_{0,p} - p f(\mathbf{u})\rho_1 f_{0,p} - L_1 f_{0,p}, \\ &\varepsilon^2 : L_0 f_{2,p} = g_{0,p} f_{2,p} + g_{1,p} f_{1,p} + g_{2,p} f_{0,p} - (L_2 + p\rho_2) f_{0,p} - (L_1 + p f(\mathbf{u})\rho_1) f_{1,p}, \\ &\varepsilon^3 : L_0 f_{3,p} = g_{0,p} f_{3,p} + g_{1,p} f_{2,p} + g_{2,p} f_{1,p} + g_{3,p} f_{0,p} - (L_2 + p\rho_2) f_{1,p} - (L_1 + p f(\mathbf{u})\rho_1) f_{2,p}. \\ &\qquad \vdots \end{aligned} \tag{35}$$

We consider first the equation of order ε^0, i.e.,

$$\left[L_{\mathbf{u}} + \omega \frac{\partial}{\partial \phi}\right] f_{0,p}(\phi, \mathbf{u}) = g_{0,p} f_{0,p}(\phi, \mathbf{u}). \tag{36}$$

As $\rho_{l0}(\phi) = 0$, from the definition of $g_{0,p}$ we know that $g_{0,p} = 0$, and therefore Eq. (36), along with the periodic boundary condition of the solution function $f_{0,p}(\phi, \mathbf{u})$, reduces to

$$\left[L_{\mathbf{u}} + \omega \frac{\partial}{\partial \phi}\right] f_{0,p}(\phi, \mathbf{u}) = 0,$$
$$f_{0,p}(\phi, \mathbf{u}) = f_{0,p}(\phi + \pi, \mathbf{u}). \tag{37}$$

As the eigenfunctions $\psi_{\mathbf{m}}(\mathbf{u})$ of the operator $L_{\mathbf{u}}$ form a complete function set [20], we can expand $f_{0,p}(\phi, \mathbf{u})$ as a series in terms of $\psi_{\mathbf{m}}(\mathbf{u})$, i.e.,

$$f_{0,p}(\phi, \mathbf{u}) = \sum_{m_1=0,\ldots,m_n=0}^{\infty} f_{0,p}^{(\mathbf{m})}(\phi) \psi_{\mathbf{m}}(\mathbf{u}). \tag{38}$$

The substitution of Eq. (38) into Eq. (37) leads to the fact that each coefficient $f_{0,p}^{(\mathbf{m})}(\phi)$ is the solution to

$$\left[\omega \frac{\partial}{\partial \phi} + \lambda_{\mathbf{m}}\right] f_{0,p}^{(\mathbf{m})}(\phi) = 0,$$
$$f_{0,p}^{(\mathbf{m})}(\phi) = f_{0,p}^{(\mathbf{m})}(\phi + \pi), \tag{39}$$

in which the only non-zero periodic solution is $f_{0,p}^{(0)}(\phi) = C$, which corresponds to the eigenvalue $\lambda_0 = 0$, and C is an integral constant. Thus, $f_{0,p}(\phi, \mathbf{u})$ can be expressed as

$$f_{0,p}(\phi, \mathbf{u}) = C\psi_0(\mathbf{u}). \tag{40}$$

Furthermore, we can select $\psi_0 = 1$ [7], and, without loss of generality, we let

$$f_{0,p}(\phi, \mathbf{u}) = 1. \tag{41}$$

Consider the equation of order ε in Eq. (35). The substitution of Eq. (41) into the second equation of Eq. (35) yields

$$L_0 f_{1,p}(\phi, \mathbf{u}) = g_{1,p} - pf(\mathbf{u})\rho_{l1}(\phi). \tag{42}$$

The solvability condition for this equation immediately leads to

$$g_{1,p} = \langle pf(\mathbf{u})\rho_{l1}(\phi), \psi_0^*(\mathbf{u})\rangle = p\int_0^{\pi} \rho_{l1}(\phi)\,\mathrm{d}\phi \int_D f(\mathbf{u})\psi_0^*(\mathbf{u})\,\mathrm{d}\mathbf{u} = 0, \tag{43}$$

where $\psi_0^*(\mathbf{u}) \in \mathrm{Ker}(L_0^*) = \{C\psi_0^*(\mathbf{u})\colon\ C \text{ is an arbitrary constant}\}$. Eq. (42) then reduces to

$$\left[L_{\mathbf{u}} + \omega \frac{\partial}{\partial \phi}\right] f_{1,p}(\phi, \mathbf{u}) = -pf(\mathbf{u})\rho_{l1}(\phi),$$
$$f_{1,p}(\phi + \pi, \mathbf{u}) = f_{1,p}(\phi, \mathbf{u}). \tag{44}$$

After expanding $f_{1,p}(\phi, \mathbf{u})$ and $f(\mathbf{u})$ along $\psi_{\mathbf{m}}(\mathbf{u})$, we obtain

$$f_{1,p}(\phi, \mathbf{u}) = \sum_{m_1,\dots,m_n=0}^{\infty} f_{1,p}^{(\mathbf{m})}(\phi)\psi_{\mathbf{m}}(\mathbf{u}),$$
$$f(\mathbf{u}) = \sum_{m_1,\dots,m_n=0}^{\infty} f^{(\mathbf{m})}\psi_{\mathbf{m}}(\mathbf{u}), \tag{45}$$

where

$$f^{(\mathbf{m})} = \langle f(\mathbf{u}), \psi_{\mathbf{m}}^{*}(\mathbf{u}) \rangle. \tag{46}$$

Substituting Eq. (45) into Eq. (44) and equating the coefficients of the same eigenfunctions, we obtain

$$\left[\lambda_{\mathbf{m}} + \omega\frac{\partial}{\partial\phi}\right] f_{1,p}^{(\mathbf{m})}(\phi) = -pf^{(\mathbf{m})}\rho_{l1}(\phi), \quad m \neq 0,$$
$$\left[\lambda_{\mathbf{m}} + \omega\frac{\partial}{\partial\phi}\right] f_{1,p}^{(\mathbf{m})}(\phi) = 0, \quad m = 0. \tag{47}$$

In Eq. (47), each $f_{1,p}^{(\mathbf{m})}(\phi)$ is a π-periodic function of the variable ϕ, which can be easily obtained via a direct integration, i.e.

$$f_{1,p}^{(\mathbf{m})}(\phi) = \begin{cases} f_{1,p}^{(0)}(\phi) = C_1, & m = 0, \\ -\dfrac{p}{2} f^{(\mathbf{m})}\left[\dfrac{(C_1^{(\mathbf{m})}\cos 2\phi + C_2^{(\mathbf{m})}\sin 2\phi)}{\lambda_{\mathbf{m}}^2 + 4\omega^2} + \dfrac{\sigma_2}{\lambda_{\mathbf{m}}}\right], & m \neq 0, \end{cases} \tag{48}$$

where

$$C_1^{(\mathbf{m})} = 2\omega\sigma_1 - \lambda_{\mathbf{m}}\sigma_2, \quad C_2^{(\mathbf{m})} = -(2\omega\sigma_2 + \lambda_{\mathbf{m}}\sigma_1). \tag{49}$$

In the first equation of Eq. (48), C_1 is an integral constant, and it is not difficult to verify that C_1 contributes nothing to the expression of the moment Lyapunov exponent. Finally, $f_{1,p}(\phi, \mathbf{u})$, which is the solution to Eq. (44), can be expressed as

$$f_{1,p}(\phi, \mathbf{u}) = C_1 + \sum_{\substack{m_1,m_2,\dots,m_n=0 \\ m\neq 0}}^{\infty} f_{1,p}^{(\mathbf{m})}(\phi)\psi_{\mathbf{m}}(\mathbf{u}) = \sum_{m_1,m_2,\dots,m_n=0}^{\infty} f_{1,p}^{(\mathbf{m})}(\phi)\psi_{\mathbf{m}}(\mathbf{u}). \tag{50}$$

4.3. Moment Lyapunov exponent and top Lyapunov exponent

Employing the above results, the third equation of Eq. (35) becomes

$$L_0 f_{2,p}(\phi, \mathbf{u}) = g_{2,p} - p\rho_{l2}(\phi) - f(\mathbf{u})\left[\phi_{l1}(\phi)\frac{\partial}{\partial\phi} + p\rho_{l1}(\phi)\right] f_{1,p}(\phi, \mathbf{u}),$$
$$f_{2,p}(\phi + \pi, \mathbf{u}) = f_{2,p}(\phi, \mathbf{u}), \tag{51}$$

in which the solvability condition leads to

$$\left\langle g_{2,p} - p\rho_{l2}(\phi) - f(\mathbf{u})\left[\phi_{l1}\frac{\partial}{\partial\phi} + p\rho_{l1}(\phi)\right]f_{1,p}(\phi,\mathbf{u}), \psi_0^*(\mathbf{u})\right\rangle = 0. \tag{52}$$

After the integration, we obtain

$$\begin{aligned}
g_{2,p} &= I_1 + I_2 + I_3,\\
I_1 &= \frac{1}{\pi}\langle p\rho_{l2}(\phi), \psi_0^*(\mathbf{u})\rangle = \frac{p}{\pi}\int_D \psi_0^*(\mathbf{u})\,\mathrm{d}\mathbf{u}\int_0^\pi \rho_{l2}(\phi)\,\mathrm{d}\phi = \frac{1}{2}p\beta,\\
I_2 &= \frac{1}{\pi}\sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\left\langle f(\mathbf{u})\phi_{l1}(\phi)\frac{\partial}{\partial\phi}f_{1,p}^{(\mathbf{m})}(\phi,\mathbf{u}), \psi_0^*(\mathbf{u})\right\rangle\\
&= \frac{p}{\pi}\sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\left\{\frac{f^{(\mathbf{m})}\hat{f}^{(\mathbf{m})}}{\lambda_{\mathbf{m}}^2+4\omega^2}\int_0^\pi[\phi_{l1}(\phi)(C_{\mathbf{m}}^{(1)}\sin 2\phi - C_{\mathbf{m}}^{(2)}\cos 2\phi)]\,\mathrm{d}\phi\right\}\\
&= -\frac{p}{4}\sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\left[\frac{\lambda_{\mathbf{m}}f^{(\mathbf{m})}\hat{f}^{(\mathbf{m})}}{\lambda_{\mathbf{m}}^2+4\omega^2}(\sigma_1^2+\sigma_2^2)\right]\\
&= \frac{1}{8}p(\sigma_1^2+\sigma_2^2)S_f(2\omega),\\
I_3 &= \frac{p}{\pi}\sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\langle f(\mathbf{u})\rho_{l1}(\phi)f_{1,p}^{(\mathbf{m})}(\phi,\mathbf{u}), \psi_0^*(\mathbf{u})\rangle\\
&= -\frac{p^2}{2\pi}\sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\left\{\frac{f^{(\mathbf{m})}\hat{f}^{(\mathbf{m})}}{\lambda_{\mathbf{m}}^2+4\omega^2}\int_0^\pi[\rho_{l1}(\phi)(C_{\mathbf{m}}^{(1)}\cos 2\phi + C_{\mathbf{m}}^{(2)}\sin 2\phi)]\,\mathrm{d}\phi\right.\\
&\qquad\left. + \frac{f^{(\mathbf{m})}\hat{f}^{(\mathbf{m})}}{\lambda_{\mathbf{m}}}\sigma_2\int_0^\pi \rho_{l1}(\phi)\,\mathrm{d}\phi\right\}\\
&= -\frac{p^2}{8}\sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\left[\frac{\lambda_{\mathbf{m}}f^{(\mathbf{m})}\hat{f}^{(\mathbf{m})}}{\lambda_{\mathbf{m}}^2+4\omega^2}(\sigma_1^2+\sigma_2^2) + \frac{2f^{(\mathbf{m})}\hat{f}^{(\mathbf{m})}}{\lambda_{\mathbf{m}}}\sigma_2^2\right]\\
&= \frac{1}{16}p^2(\sigma_1^2+\sigma_2^2)S_f(2\omega) + \frac{1}{8}p^2\sigma_2^2 S_f(0),
\end{aligned} \tag{53}$$

where

$$\hat{f}^{(\mathbf{m})} = \langle f(\mathbf{u})\psi_{\mathbf{m}}(\mathbf{u}), \psi_0^*(\mathbf{u})\rangle. \tag{54}$$

Furthermore, in Eq. (53) the following relationships are applied:

$$S_f(2\omega) = -\sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{2\lambda_{\mathbf{m}}f^{(\mathbf{m})}\hat{f}^{(\mathbf{m})}}{\lambda_{\mathbf{m}}^2+4\omega^2}, \quad S_f(0) = -\sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{2f^{(\mathbf{m})}\hat{f}^{(\mathbf{m})}}{\lambda_{\mathbf{m}}}. \tag{55}$$

Synthesizing the above results gives

$$g_{2,p} = \tfrac{1}{2}p\beta + \tfrac{1}{8}p(\sigma_1^2 + \sigma_2^2)S_f(2\omega) + \tfrac{1}{16}p^2(\sigma_1^2 + \sigma_2^2)S_f(2\omega) + \tfrac{1}{8}p^2\sigma_2^2 S_f(0). \tag{56}$$

Next, the solution to Eq. (51) is required. Before the derivation, the functions of $f_{2,p}(\phi, \mathbf{u})$ and $f(\mathbf{u})f_{1,p}(\phi, \mathbf{u})$ should be expanded along $\psi_{\mathbf{m}}(\mathbf{u})$, i.e.,

$$f_{2,p}(\phi, \mathbf{u}) = \sum_{m_1,\ldots,m_n=1}^{\infty} f_{2,p}^{(\mathbf{m})}(\phi)\psi_{\mathbf{m}}(\mathbf{u}),$$

$$f(\mathbf{u})f_{1,p}(\phi, \mathbf{u}) = \sum_{m_1,\ldots,m_n=0}^{\infty}\sum_{l_1,\ldots,l_n=0}^{\infty} [f_{1,p}^{(\mathbf{m})}(\phi)\tilde{f}^{(\mathbf{m},\mathbf{l})}]\psi_{\mathbf{l}}(\mathbf{u}), \tag{57}$$

where

$$\tilde{f}^{(\mathbf{m},\mathbf{l})} = \langle f(\mathbf{u})\psi_{\mathbf{m}}(\mathbf{u}), \psi_{\mathbf{l}}^*(\mathbf{u})\rangle. \tag{58}$$

Substituting Eq. (57) into Eq. (51) and solving the equation, we obtain

$$f_{2,p}^{(0)}(\phi) = \Gamma_1^{(0)}\cos(4\phi) + \Gamma_2^{(0)}\sin(4\phi) + \Gamma_3^{(0)}\cos(2\phi) + \Gamma_4^{(0)}\sin(2\phi),$$

$$f_{2,p}^{(\mathbf{l})}(\phi) = \sum_{\substack{m_1,m_2,\ldots,m_n=0\\ m\neq 0}}^{\infty} (\Gamma_{\mathbf{m},1}^{(\mathbf{l})}(\phi) + \Gamma_{\mathbf{m},2}^{(\mathbf{l})}(\phi) + \Gamma_{\mathbf{m},3}^{(\mathbf{l})}(\phi) + \Gamma_{\mathbf{m},4}^{(\mathbf{l})}(\phi) + \Gamma_{\mathbf{m},5}^{(\mathbf{l})}(\phi)), \quad l = \sum_{i=1}^{n} l_i \neq 0, \tag{59}$$

in which $f_{2,p}(\phi, \mathbf{u})$ is determined. The coefficients in Eq. (59) are given in Appendix A.

The fourth equation in Eq. (35) is now considered, i.e.,

$$L_0 f_{3,p} = g_{0,p}f_{3,p} + g_{1,p}f_{2,p} + g_{2,p}f_{1,p} + g_{3,p}f_{0,p} - (L_2 + p\rho_2)f_{1,p} - (L_1 + pf(\mathbf{u})\rho_1)f_{2,p}, \tag{60}$$

in which the solvability condition becomes

$$\langle g_{2,p}f_{1,p} + g_{3,p}f_{0,p} - (L_2 + p\rho_2)f_{1,p} - (L_1 + pf(\mathbf{u})\rho_1)f_{2,p}, \psi_0^*(\mathbf{u})\rangle = 0. \tag{61}$$

After some direct integration, we finally obtain

$$\begin{aligned} g_{3,p} &= \frac{1}{\pi}\langle -(L_1 + pf(\mathbf{u})\rho_1)f_{2,p}, \psi_0^*(\mathbf{u})\rangle_{\phi,\mathbf{u}} \\ &= \sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\sum_{l_1,l_2,\ldots,l_n=0}^{\infty}\Bigg[-\frac{1}{8}p(p+2)(\sigma_1^2+\sigma_2^2)[2\sigma_1\omega(\lambda_{\mathbf{m}}+\lambda_{\mathbf{l}}) \\ &\quad - p\sigma_2\frac{\lambda_{\mathbf{l}}}{\lambda_{\mathbf{m}}}(\lambda_{\mathbf{m}}^{\mathbf{2}} + 2\omega^2) + 2p\sigma_2\omega^2\Big]\widehat{\Lambda}_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 2\omega) \\ &\quad + \frac{1}{8}p^3\sigma_2^3\widehat{\Lambda}_{\mathbf{m}}^{(\mathbf{l})}(0,0) + \frac{1}{16}p^2(p+2)\sigma_2(\sigma_1^2+\sigma_2^2)\lambda_{\mathbf{m}}\widehat{\Lambda}_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 0)\Bigg], \end{aligned} \tag{62}$$

where

$$\overset{\frown}{\Lambda}{}^{(\mathbf{l})}_{\mathbf{m}}(2\omega,2\omega)=\frac{f^{(\mathbf{m})}\tilde{f}^{(\mathbf{m},\mathbf{l})}\hat{f}^{(\mathbf{l})}}{[\lambda_{\mathbf{m}}^2+4\omega^2][\lambda_{\mathbf{l}}^2+4\omega^2]},$$

$$\overset{\frown}{\Lambda}{}^{(\mathbf{l})}_{\mathbf{m}}(2\omega,0)=\frac{f^{(\mathbf{m})}\tilde{f}^{(\mathbf{m},\mathbf{l})}\hat{f}^{(\mathbf{l})}}{[\lambda_{\mathbf{m}}^2+4\omega^2]\lambda_{\mathbf{l}}},\quad \overset{\frown}{\Lambda}{}^{(\mathbf{l})}_{\mathbf{m}}(0,0)=\frac{f^{(\mathbf{m})}\tilde{f}^{(\mathbf{m},\mathbf{l})}\hat{f}^{(\mathbf{l})}}{\lambda_{\mathbf{m}}\lambda_{\mathbf{l}}}. \tag{63}$$

Substituting Eq. (43), Eq. (56), and Eq. (62) into the first expression of Eq. (34), we obtain the asymptotic expansions of the moment Lyapunov exponent and the top Lyapunov exponent

$$g_{\varepsilon,p}=\varepsilon^2 p\left[\frac{1}{2}\beta+\frac{1}{8}(\sigma_1^2+\sigma_2^2)S_f(2\omega)\right]+\frac{\varepsilon^2p^2}{16}[(\sigma_1^2+\sigma_2^2)S_f(2\omega)+2\sigma_2^2S_f(0)]+\varepsilon^3 g_{3,p}+o(\varepsilon^3),$$

$$\lambda=\left.\frac{\partial g_\varepsilon(p)}{\partial p}\right|_{p=0}=\varepsilon^2\left[\frac{1}{2}\beta+\frac{1}{8}(\sigma_1^2+\sigma_2^2)S_f(2\omega)\right]+o(\varepsilon^3). \tag{64}$$

This result matches the derivation that is described in Eq. (31) of Ref. [15], which is derived for the case of a small value of p. The top Lyapunov exponent matches the result that is described in Eq. (9.4.19) of Ref. [2].

5. Almost-sure stability and stability in probability for a nonlinear stochastic system

In this section, the stability properties of the nonlinear stochastic system that is described in Eq. (22), including the almost-sure stability and the stability in probability, are investigated. By using the same methods proposed in Section 4, we can derive a standard FPK equation that governs $p^{(0)}0(a,\tau)$ and the probability density function of the amplitude process $a(t)$ of order ε^0, which is identical to that which is derived from the stochastic averaging method for the case of a broadband noise excitation. Based on this FPK equation, we obtain the corresponding Ito stochastic differential equation that governs the process $a(t)$, from which the almost-sure stability condition and the stability in probability condition are examined.

5.1. Formulation

For the nonlinear stochastic system that is described in Eq. (22), it is well known that for the limit $\varepsilon\to 0$, the process $a(t)$ is clearly a slow variable, whereas the process $\phi(t)$ is a fast variable, and thus $a(t)$ will not change significantly over a time interval of the order $O(1)$. To investigate the response of a system over a time interval of order $O(\varepsilon^2)$, as in Refs. [19,21], the time variable is scaled as $t=\tau/\varepsilon^2$, where τ is a slow time scale. As the vector process $(a,\phi,\mathbf{u})$ forms a diffusive process of dimension $(n+2)$, the relevant probability density function $p_\varepsilon(a,\phi,\mathbf{u},\tau)$ satisfies the FPK equation as

$$\hat{L}_\varepsilon^* p_\varepsilon(a,\phi,\mathbf{u},\tau)=0, \tag{65}$$

where

$$\hat{L}_\varepsilon^* = \hat{L}_0^* + \varepsilon \hat{L}_1^* + \varepsilon^2 \hat{L}_2^*,$$

$$\hat{L}_0^* = L_{\mathbf{u}}^* - \omega \frac{\partial}{\partial \phi}, \quad \hat{L}_1^* = -f(\mathbf{u})\left[\frac{\partial}{\partial \phi}\phi_1 + \frac{\partial}{\partial a}a_1\right], \quad \hat{L}_2^* = -\frac{\partial}{\partial \phi}\phi_2 - \frac{\partial}{\partial a}a_2 - \frac{\partial}{\partial \tau}. \tag{66}$$

As in the case of the linear system, $p_\varepsilon(a, \phi, \mathbf{u}, \tau)$ is sought as an expansion in powers of ε

$$p_\varepsilon(a, \phi, \mathbf{u}, \tau) = p_0(a, \phi, \mathbf{u}, \tau) + \varepsilon p_1(a, \phi, \mathbf{u}, \tau) + \cdots + \varepsilon^n p_n(a, \phi, \mathbf{u}, \tau) + \cdots. \tag{67}$$

The substitution of the above expression into Eq. (65) leads to the following sequence of Poisson equations:

$$\hat{L}_0^* p_0 = 0, \quad \hat{L}_0^* p_1 = -\hat{L}_1^* p_0, \quad \hat{L}_0^* p_2 = -\hat{L}_1^* p_1 - \hat{L}_2^* p_0, \ldots \ . \tag{68}$$

In addition, it should be noted that each equation with the form $\hat{L}_0^* p(a, \phi, \mathbf{u}, \tau) = q(a, \phi, \mathbf{u}, \tau)$ satisfies the solvability condition as

$$\frac{1}{\pi}\int_0^\pi \mathrm{d}\phi \int_D \mathrm{d}\mathbf{u}\, q(a, \phi, \mathbf{u}, \tau) = 0. \tag{69}$$

We then proceed to investigate the solution to the first equation of Eq. (68), which is the leading term of asymptotical expansion in Eq. (67). For simplicity, the dependence upon the variable τ is omitted in the first two equations of Eq. (68). Being a function of the argument $\mathbf{u}$, $p_0(a, \phi, \mathbf{u})$ is expanded along the eigenfunctions of the operator $L_{\mathbf{u}}^*$, i.e.,

$$p_0(a, \phi, \mathbf{u}) = \sum_{m_1, m_2, \ldots, m_n = 0}^{\infty} p_{\mathbf{m}}^{(0)}(a, \phi)\psi_{\mathbf{m}}^*(\mathbf{u}) \tag{70}$$

and each term of $p_{\mathbf{m}}^{(0)}(a, \phi)$ satisfies

$$\left[-\omega \frac{\partial}{\partial \phi} + \lambda_{\mathbf{m}}\right] p_{\mathbf{m}}^{(0)}(a, \phi) = 0. \tag{71}$$

Among these $p_{\mathbf{m}}^{(0)}(a, \phi)$ $(m = 0, 1, \ldots)$, $p_0^{(0)}(a, \phi)$ is the only non-zero periodic solution that corresponds to the conditions of $m = 0$ and $\lambda_0 = 0$. With this result, the solution to the first equation of Eq. (68) becomes

$$p_0(a, \phi, \mathbf{u}) = p_0^{(0)}(a)\psi_0^*(\mathbf{u}), \tag{72}$$

where the coefficient of $p_0^{(0)}(a)$ remains to be determined from the subsequent solvability condition of the third equation in Eq. (68).

Substituting Eq. (72) into the right-hand side of the second equation in Eq. (68) results in

$$\begin{aligned}&\left[-\omega \frac{\partial}{\partial \phi} + L_{\mathbf{u}}^*\right] p_1(a, \phi, \mathbf{u}) \\ &\quad = \frac{1}{2} f(\mathbf{u})\psi_0^*(\mathbf{u})\left\{[\sigma_1 \sin(2\phi) + \sigma_2 \cos(2\phi)]\left[1 - a\frac{\partial}{\partial a}\right] + \sigma_2\left[1 + a\frac{\partial}{\partial a}\right]\right\} p_0^{(0)}(a).\end{aligned} \tag{73}$$

Substituting the expansion of $p_1(a,\phi,\mathbf{u})$

$$p_1(a,\phi,\mathbf{u}) = \sum_{m_1,m_2,\ldots,m_n=0}^{\infty} p_{\mathbf{m}}^{(1)}(a,\phi)\psi_{\mathbf{m}}^*(\mathbf{u}) \tag{74}$$

into Eq. (73) leads to

$$\left[-\omega\frac{\partial}{\partial\phi}+\lambda_{\mathbf{m}}\right]p_{\mathbf{m}}^{(1)}(a,\phi) = \frac{1}{2}\hat{f}^{(\mathbf{m})}\left\{[\sigma_1\sin(2\phi)+\sigma_2\cos(2\phi)]\left[1-a\frac{\partial}{\partial a}\right] + \sigma_2\left[1+a\frac{\partial}{\partial a}\right]\right\}p_0^{(0)}(a), \tag{75}$$

where $p_{\mathbf{m}}^{(1)}(a,\phi)$ is periodic with π and can be obtained through direct integration, i.e.,

$$p_{\mathbf{m}}^{(1)}(a,\phi) = \frac{1}{2}\left\{[\kappa_{\mathbf{m}}^{(1)}(2\omega)\cos 2\phi + \kappa_{\mathbf{m}}^{(2)}(2\omega)\sin 2\phi]\left[1-a\frac{\partial}{\partial a}\right] + \kappa_{\mathbf{m}}(0)\left[1+a\frac{\partial}{\partial a}\right]\right\}p_0^{(0)}(a), \tag{76}$$

where

$$\kappa_{\mathbf{m}}^{(1)}(2\omega) = \frac{\lambda_{\mathbf{m}}\sigma_2+2\omega\sigma_1}{\lambda_{\mathbf{m}}^2+4\omega^2}, \quad \kappa_{\mathbf{m}}^{(2)}(2\omega) = \frac{\lambda_{\mathbf{m}}\sigma_1-2\omega\sigma_2}{\lambda_{\mathbf{m}}^2+4\omega^2}, \quad \kappa_{\mathbf{m}}(0) = \frac{\sigma_2}{\lambda_{\mathbf{m}}}. \tag{77}$$

Thus, the function of $p_1(a,\phi,\mathbf{u})$ is obtained.

To determine $p_0^{(0)}(a,\tau)$ and then $p_0(a,\phi,\mathbf{u},\tau)$, the solvability condition for the third equation in Eq. (68) is considered, i.e.,

$$-\int_0^{2\pi}\mathrm{d}\phi\int_D[\hat{L}_1^* p_1(a,\phi,\mathbf{u},\tau)+\hat{L}_2^* p_0(a,\phi,\mathbf{u},\tau)]\,\mathrm{d}\mathbf{u} = 0. \tag{78}$$

Substituting Eq. (72) and Eq. (76) into the above equation, we obtain

$$\begin{aligned}
-\int_0^{2\pi}\mathrm{d}\phi\int_D \hat{L}_1^* p_1(a,\phi,\mathbf{u},\tau)\,\mathrm{d}\mathbf{u} &= \left[-\frac{3}{8}a\sigma_2^2 S_f(0) - \frac{1}{16}aS_f(2\omega)(\sigma_1^2+\sigma_2^2)\right]\frac{\partial}{\partial a}p_0^{(0)}(a,\tau) \\
&\quad + \left[-\frac{1}{8}a^2\sigma_2^2 S_f(0) - \frac{1}{16}a^2 S_f(2\omega)(\sigma_1^2+\sigma_2^2)\right]\frac{\partial^2}{\partial a^2}p_0^{(0)}(a,\tau) \\
&\quad + \left[-\frac{1}{8}\sigma_2^2 S_f(0) + \frac{1}{16}S_f(2\omega)(\sigma_1^2+\sigma_2^2)\right]p_0^{(0)}(a,\tau) \\
&\quad - \int_0^{2\pi}\mathrm{d}\phi\int_D \hat{L}_2^* p_0(a,\phi,\mathbf{u},\tau)\,\mathrm{d}\mathbf{u} \\
&= \frac{\partial}{\partial\tau}p_0^{(0)}(a,\tau) + \left[\frac{1}{2}\beta+\frac{3}{8}a^2\omega^2\delta+\frac{1}{8}a^2\gamma\right]a\frac{\partial}{\partial a}p_0^{(0)}(a,\tau) \\
&\quad + \left[\frac{1}{2}\beta+\frac{9}{8}a^2\omega^2+\frac{3}{8}a^2\right]p_0^{(0)}(a,\tau),
\end{aligned} \tag{79}$$

which leads to the following standard FPK equation for $p_0^{(0)}(a,\tau)$, i.e.,

$$\frac{\partial}{\partial\tau}p_0^{(0)}(a,\tau) = \frac{1}{2}\frac{\partial^2}{\partial a^2}[\psi^2(a)p_0^{(0)}(a,\tau)] - \frac{\partial}{\partial a}[\varphi(a)p_0^{(0)}(a,\tau)], \quad a,\tau\in[0,+\infty), \tag{80}$$

where

$$\psi^2(a) = \tfrac{1}{4}a^2[\sigma_2^2 S_f(0) + \tfrac{1}{2}S_f(2\omega)(\sigma_1^2 + \sigma_2^2)],$$

$$\varphi(a) = \tfrac{1}{8}a\left[\sigma_2^2 S_f(0) + \tfrac{3}{2}S_f(2\omega)(\sigma_1^2 + \sigma_2^2) + 4\beta\right] - \tfrac{1}{8}a^3[3\omega^2\delta + \gamma]. \tag{81}$$

This result is identical to that which is derived from the stochastic averaging method for the case of a broadband real noise excitation. Corresponding to this FPK equation, the Ito stochastic differential equation for the amplitude process $a(\tau)$ of order $O(\varepsilon^0)$ can be obtained:

$$\mathrm{d}a(\tau) = \varphi(a)\,\mathrm{d}\tau + \psi(a)\,\mathrm{d}W(\tau), \tag{82}$$

where $W(\tau)$ is the Wiener process of unit intensity, and the function $\psi(a)$ is defined as

$$\psi(a) = [\psi^2(a)]^{1/2} = \frac{a}{2}\left[\sigma_2^2 S_f(0) + \frac{1}{2}S_f(2\omega)(\sigma_1^2 + \sigma_2^2)\right]^{1/2}. \tag{83}$$

5.2. Almost-sure stability

In this subsection, the almost-sure stability, or the stability with probability 1, for the system that is described in Eq. (82) is examined.

The linearization of Eq. (82) becomes

$$\mathrm{d}a(\tau) = \varphi^*(a)\,\mathrm{d}\tau + \psi^*(a)\,\mathrm{d}W(\tau),$$

$$\psi^*(a) = \psi(a),$$

$$\varphi^*(a) = \tfrac{1}{8}a[\sigma_2^2 S_f(0) + \tfrac{3}{2}S_f(2\omega)(\sigma_1^2 + \sigma_2^2) + 4\beta], \tag{84}$$

in which the solution is

$$a(\tau) = a(\tau_0)\exp\left\{\left[\frac{\varphi^*(a)}{a} - \frac{1}{2}\left(\frac{\psi^*(a)}{a}\right)^2\right]\tau + \frac{\psi^*(a)}{a}W(\tau)\right\}. \tag{85}$$

It is well known that $W(\tau) \sim (\tau \log\log \tau)^{1/2}$ as $\tau \to \infty$ with probability 1 (w.p.1), and thus we can conclude that $a(\tau) \to 0$ as $\tau \to \infty$ w.p.1, provided that

$$\frac{\varphi^*(a)}{a} - \frac{1}{2}\left[\frac{\psi^*(a)}{a}\right]^2 < 0, \tag{86}$$

which means that

$$\tfrac{1}{2}\beta + \tfrac{1}{8}S_f(2\omega)(\sigma_1^2 + \sigma_2^2) < 0. \tag{87}$$

This result matches the stability criterion that is derived from the top Lyapunov exponent in Eq. (64).

5.3. Stability in probability

In this subsection, the stability in probability for Eq. (82) is examined. Before the analysis, some relevant definitions are first introduced.

In general, for a stochastic process $x(t;x_0,t_0)$, $x(t_0)=x_0$, and $t\geqslant t_0$, $\|x(t;x_0,t_0)\|$ is a suitable norm of $x(t;x_0,t_0)$. The stability in probability for $x(t;x_0,t_0)$ is defined in such a way that the trivial solution $x(t;x_0,t_0)=0$ is said to be stable in probability if, for every pair of $\varepsilon_1,\varepsilon_2>0$, there exists a $\delta(\varepsilon_1,\varepsilon_2,x_0,t_0)>0$ such that

$$\mathrm{Prob}[\|x(t;x_0,t_0)\|\geqslant\varepsilon_1]\leqslant\varepsilon_2,\quad t\geqslant t_0, \tag{88}$$

provided that $\|x_0\|\leqslant\delta$, where x_0 is assumed to be deterministic. In addition, the trivial solution is said to be asymptotically stable if and only if it is an exit or an attractive natural boundary and the other boundary is an entrance or a repulsive natural boundary (see p. 265 in Ref. [1]).

Almost-sure stability is also called the Lyapunov stability with probability 1, and ensures that the absolute maxima of almost all of the functions are bounded in the entire time interval $[t_0,t]$. In contrast, stability in probability is concerned with the convergence properties of sample functions at an arbitrary instant in time $t\geqslant t_0$, and is therefore not of the Lyapunov type and is generally less stringent. According to Lin and Cai [1], when applied to a linear system, the convergence of the solution at an arbitrary time instant $t\geqslant t_0$ guarantees the same for the entire time interval, and the two types of stability conditions become equivalent. In this subsection, we investigate the stability in probability for the nonlinear system that is described in Eq. (22), and verify whether it is equivalent to the relevant almost-sure stability condition.

The method used here originates from Ref. [1]. The principle is such that for the diffusion process $a(\tau)$, the two boundaries at $a=0$ and ∞ are both singular, and if one of the system parameters, such as β, changes, then the boundaries will change simultaneously, from which the stability condition will be determined.

As in Ref. [1], to check the types of singular boundaries for the diffusion process of $a(\tau)$, the drift and diffusion coefficients, which are defined in Eq. (81), should first be examined.

The result of $\psi^2(a)=0$ at $a=0$ leads us to the information that the boundary at $a=0$ is singular of the first kind. Furthermore, as $\varphi(0)=0$, the boundary at $a=0$ is in fact a trap. According to Ref. [1], for a diffusion process $a(\tau)$, the diffusion exponent, drift exponent, and characteristic value at $a=0$, which is a singular boundary of the first kind, are defined, respectively, as:

- the diffusion exponent α_l:

$$\psi^2(0)=O(|a-0|)^{\alpha_l}, \tag{89}$$

- the drift exponent β_l:

$$\varphi(0)=O(|a-0|)^{\beta_l}, \tag{90}$$

- the characteristic value c_l:

$$c_l = \lim_{a \to 0^+} \frac{2\varphi(a)(a-0)^{\alpha_l - \beta_l}}{\psi^2(a)}, \tag{91}$$

that at $a = 0$ lead to

$$\alpha_l = 2, \quad \beta_l = 1, \quad c_l = 1 + \frac{S_f(2\omega)(\sigma_1^2 + \sigma_2^2) + 4\beta}{\sigma_2^2 S_f(0) + \frac{1}{2} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)}. \tag{92}$$

After validating these results with the relevant terms in Table 4.5.2 in Ref. [1] (see p. 134 of Ref. [1]), we conclude that if $\frac{1}{2}\beta \geqslant -\frac{1}{8} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)$, then $c_l \geqslant \beta_l$, $a = 0$ is a repulsive natural boundary, and if $\frac{1}{2}\beta < -\frac{1}{8} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)$, then $c_l < \beta_l$ and $c_l < 1$, $a = 0$ is an attractive natural boundary.

For the boundary at $a = +\infty$ under the conditions of $\beta \neq -\frac{1}{4}[\sigma_2^2 S_f(0) + \frac{3}{2} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)]$ and $|\varphi(+\infty)| = +\infty$, from which we know that the boundary of $a = +\infty$ is singular of the second kind at infinity, and that its diffusion and drift exponents and characteristic value are defined, respectively, in Ref. [1],

- the diffusion exponent α_r:

$$\psi^2(0) = O(|a|)^{\alpha_r}, \tag{93}$$

- the drift exponent β_r:

$$\varphi(0) = O(|a|)^{\beta_r}, \tag{94}$$

- the characteristic value c_r:

$$c_r = \lim_{a \to 0^+} -\frac{2\varphi(a)|a|^{\alpha_r - \beta_r}}{\psi^2(a)}. \tag{95}$$

In view of the parameters δ and γ, the three quantities at the boundary $a = +\infty$ should be investigated in three separate cases.

Case 1: $|\delta| + |\gamma| \neq 0$ and $3\omega^2\delta + \gamma > 0$. Under this condition, it leads to

$$\alpha_r = 2, \quad \beta_r = 3, \quad c_r = \frac{3\omega^2\delta + \gamma}{\sigma_2^2 S_f(0) + \frac{1}{2} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)}, \quad \varphi(+\infty) < 0. \tag{96}$$

According to Table 4.54 in Ref. [1, p. 137], we know that the right boundary at $a = +\infty$ is an entrance. Thus, for this case, the stability conditions in probability for the nonlinear stochastic system that is described in Eq. (82) are given as:

- if $\frac{1}{2}\beta \geqslant -\frac{1}{8} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)$, then $a = 0$ is not asymptotically stable in probability;
- if $\frac{1}{2}\beta < -\frac{1}{8} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)$, then $a = 0$ is asymptotically stable in probability.

Case 2: $|\delta| + |\gamma| \neq 0$ and $3\omega^2\delta + \gamma < 0$. This condition leads to

$$\alpha_r = 2, \quad \beta_r = 3, \quad c_r = \frac{3\omega^2\delta + \gamma}{\sigma_2^2 S_f(0) + \frac{1}{2} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)}, \quad \varphi(+\infty) > 0, \tag{97}$$

which, according to Table 4.54 in Ref. [1], immediately results in the revelation that the right boundary at $a = +\infty$ is an exit. Thus, the trivial solution to the nonlinear stochastic system that is described in Eq. (82) is not asymptotically stable in probability.

Case 3: $\delta = \gamma = 0$, which means that the system that is described in Eq. (82) is linear, and then

$$\alpha_r = 2, \quad \beta_r = 1, \quad c_r = 0. \tag{98}$$

From Table 4.54 in Ref. [1], we know that the boundary at $a = +\infty$ is a repulsive natural boundary, and for this case the stability conditions in probability for the linear stochastic system that is described in Eq. (82) ($\delta = \gamma = 0$) are the same as for the nonlinear system, i.e.,

- if $\frac{1}{2}\beta \geqslant -\frac{1}{8} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)$, then $a = 0$ is not asymptotically stable in probability;
- if $\frac{1}{2}\beta < -\frac{1}{8} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)$, then $a = 0$ is asymptotically stable in probability.

In summarizing the foregoing result, we conclude that under the condition of $\beta \neq -\frac{1}{4}[\sigma_2^2 S_f(0) + \frac{3}{2} S_f(2\omega)(\sigma_1^2 + \sigma_2^2)]$, the asymptotic stability conditions in probability of the trivial solutions for both the linear and nonlinear ($|\delta| + |\gamma| \neq 0$ and $3\omega^2\delta + \gamma > 0$) stochastic systems match the condition of the almost-sure stability for the same linear stochastic system.

6. Conclusions

In this paper, we investigated the stability properties of a nonlinear Duffing-Van der Pol oscillator that is excited parametrically by a real noise, including the moment Lyapunov exponent, the maximal Lyapunov exponent, and the stability in probability. The real noise excitation is assumed to be an integrable function of the output of a linear filter system, which is an n-dimensional Ornstein–Uhlenbeck vector process. In this study, we removed the detailed balance condition and the strong mixing condition, which is the prerequisite for the stochastic averaging method. For the case of an arbitrary finite real number p, a perturbation method and the spectrum representation of the Fokker–Planck operator of the linear filter system are employed to derive the asymptotic expansion of the moment Lyapunov exponent $g(p, x_0)$ and the top Lyapunov exponent, which match the results of Arnold et al. [15] for the case of a small p. Furthermore, we also examine the stability properties of a nonlinear stochastic system. The standard FPK equation that governs the amplitude process $a(\tau)$ is obtained, and is identical to that which is derived from the stochastic averaging method for the case of a broadband noise excitation. It should be noted that the method that is used in this paper can also be applied to many other stochastic systems for which stochastic averaging methods are not available. Using the method that is proposed by Lin and Cai [1], the almost-sure stability and the stability in probability for the relevant nonlinear stochastic system are examined.

Appendix A. The coefficients in the expression of $f_{2,p}^{(\mathbf{m})}(\phi)$

The coefficients in the expression of $f_{2,p}^{(\mathbf{m})}(\phi)$ are

$$\Gamma_1^{(0)} = \frac{1}{32}\frac{p}{\omega}\{(\sigma_1^2 - \sigma_2^2)\Phi(2\omega) - \sigma_1\sigma_2 S(2\omega)\}(-p^2 + 2p),$$

$$\Gamma_2^{(0)} = \frac{1}{32}\frac{p}{\omega}\{(\sigma_1^2 - \sigma_2^2)\Phi(2\omega) - \sigma_1\sigma_2 S(2\omega)\}(-p^2 + 2p),$$

$$\Gamma_3^{(0)} = \frac{1}{16}\frac{p}{\omega}\{-\sigma_2 p(\sigma_1 S(2\omega) + 2\sigma_2\Phi(2\omega) + \sigma_1 S(0)) + 2\sigma_1(2\sigma_1\Phi(2\omega) - \sigma_2 S(2\omega))\},$$

$$\Gamma_4^{(0)} = \frac{1}{16}\frac{p}{\omega}\{\sigma_2 p((2\sigma_1 + \sigma_2)S(2\omega) + \sigma_2 S(0)) - 2\sigma_1(\sigma_1 S(2\omega) + 2\sigma_2\Phi(2\omega))\}, \tag{A.1}$$

and

$$f_{2,p}^{(\mathbf{l})}(\phi) = \sum_{\substack{m_1,m_2,\ldots,m_n=0 \\ m\neq 0}}^{\infty}\{\Gamma_{\mathbf{m},1}^{(\mathbf{l})}(\phi) + \Gamma_{\mathbf{m},2}^{(\mathbf{l})}(\phi) + \Gamma_{\mathbf{m},3}^{(\mathbf{l})}(\phi) + \Gamma_{\mathbf{m},4}^{(\mathbf{l})}(\phi) + \Gamma_{\mathbf{m},5}^{(\mathbf{l})}(\phi)\}, \quad l = \sum_{i=1}^{n} l_i \neq 0,$$

$$\Gamma_{\mathbf{m},1}^{(\mathbf{l})}(\phi) = \frac{p(p-2)}{4}\Lambda_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 4\omega)\{(\sigma_1^2 - \sigma_2^2)\omega - \sigma_1\sigma_2\lambda_{\mathbf{m}}\}(4\omega\cos(4\phi) - \lambda_{\mathbf{l}}\sin(4\phi)),$$

$$\Gamma_{\mathbf{m},2}^{(\mathbf{l})}(\phi) = -\frac{p(p-2)}{8}\Lambda_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 4\omega)\{(\sigma_1^2 - \sigma_2^2)\lambda_{\mathbf{m}} + 4\sigma_1\sigma_2\omega\}(\lambda_{\mathbf{l}}\cos(4\phi) + 4\omega\sin(4\phi)),$$

$$\Gamma_{\mathbf{m},3}^{(\mathbf{l})}(\phi) = \frac{p}{2}\frac{\Lambda_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 2\omega)}{\lambda_{\mathbf{m}}}(2\omega\cos(2\phi) - \lambda_{\mathbf{l}}\sin(2\phi)) \times\{p\sigma_2(\sigma_1\lambda_{\mathbf{m}}^2 + \sigma_2\lambda_{\mathbf{m}}\omega + 2\sigma_1\omega^2) + \lambda_{\mathbf{m}}\sigma_1(\sigma_2\lambda_{\mathbf{m}} - 2\sigma_1\omega)\},$$

$$\Gamma_{\mathbf{m},4}^{(\mathbf{l})}(\phi) = -\frac{p}{2}\frac{\Lambda_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 2\omega)}{\lambda_{\mathbf{m}}}(\lambda_{\mathbf{l}}\cos(2\phi) + 2\omega\sin(2\phi)) \times\{p\sigma_2(\sigma_2\lambda_{\mathbf{m}}^2 - \sigma_1\lambda_{\mathbf{m}}\omega + 2\sigma_2\omega^2) - \lambda_{\mathbf{m}}\sigma_1(\sigma_1\lambda_{\mathbf{m}} + 2\sigma_2\omega)\},$$

$$\Gamma_{\mathbf{m},5}^{(\mathbf{l})}(\phi) = \frac{p}{8}\frac{\Lambda_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 0)}{\lambda_{\mathbf{m}}}\{p(3\sigma_2^2\lambda_{\mathbf{m}}^2 + 8\sigma_2^2\omega^2 + \sigma_1^2\lambda_{\mathbf{m}}^2) + 2\lambda_{\mathbf{m}}^2(\sigma_1^2 + \sigma_2^2)\}, \tag{A.2}$$

where

$$\Lambda_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 4\omega) = \frac{f^{(\mathbf{m})}\tilde{f}^{(\mathbf{m},\mathbf{l})}}{(\lambda_{\mathbf{m}}^2 + 4\omega^2)(\lambda_{\mathbf{l}}^2 + 16\omega^2)},$$

$$\Lambda_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 2\omega) = \frac{f^{(\mathbf{m})}\tilde{f}^{(\mathbf{m},\mathbf{l})}}{(\lambda_{\mathbf{m}}^2 + 4\omega^2)(\lambda_{\mathbf{l}}^2 + 4\omega^2)},$$

$$\Lambda_{\mathbf{m}}^{(\mathbf{l})}(2\omega, 0) = \frac{f^{(\mathbf{m})}\tilde{f}^{(\mathbf{m},\mathbf{l})}}{(\lambda_{\mathbf{m}}^2 + 4\omega^2)\lambda_{\mathbf{l}}}, \quad \Lambda_{\mathbf{m}}^{(\mathbf{l})}(0, 0) = \frac{f^{(\mathbf{m})}\tilde{f}^{(\mathbf{m},\mathbf{l})}}{\lambda_{\mathbf{m}}\lambda_{\mathbf{l}}}. \tag{A.3}$$

References

[1] Y.K. Lin, G.Q. Cai, *Probabilistic is Structural Dynamics, Advanced Theory and Applications*, McGraw-Hill, New York, 1995.

[2] L. Arnold, *Random Dynamical Systems*, Springer, Berlin, 1998.

[3] R. Khasminskii, Necessary and sufficient conditions for the asymptotic stability of linear stochastic system, *Theory of Probability and its Applications* 12 (1967) 144–147.

[4] F. Kozin, S. Prodromou, Necessary and sufficient conditions for almost sure sample stability of linear Ito equations, *SIAM Journal of Applied Mathematics* 21 (1971) 413–424.

[5] K. Nishoka, On the stability of two-dimensional linear stochastic systems, *Kodai Mathematical Seminar Report* 27 (1976) 211–230.

[6] S.T. Ariaratnam, W.C. Xie, Lyapunov exponents and stochastic stability of coupled linear systems under real noise excitation, *ASME Journal of Applied Mechanics* 59 (1992) 664–673.

[7] L. Arnold, G. Papanicolaou, V. Wihstutz, Asymptotic analysis of the Lyapunov exponents and rotation numbers of the random oscillator and applications, *SIAM Journal of Applied Mathematics* 46 (1986) 427–450.

[8] N. Sri Namachchivayam, H.J. Van Roessel, Maximal Lyapunov exponent and rotation numbers for two coupled oscillators driven by real noise, *Journal of Statistical Physics* 71 (1993) 549–567.

[9] M.M. Doyle, N. Sri Namachchivaya, Almost-sure asymptotic stability of a general four-dimensional system driven by real noise, *Journal of Statistical Physics* 75 (1994) 525–552.

[10] X.B. Liu, K.M. Liew, Lyapunov exponents for two nonlinear systems driven by real noises, *Proceedings of the Royal Society of London* A (2002) 2705–2719.

[11] X.B. Liu, K.M. Liew, The Lyapunov exponent for a codimension two bifurcation system that is driven by a real noise, *International Journal of Non-linear Mechanics* 32 (2003) 1495–1511.

[12] X.B. Liu, K.M. Liew, On the almost-sure stability condition for a co-dimension two bifurcation system under the parametric excitation of a real noise, *Journal of Sound and Vibration* 272 (2004) 85–107.

[13] L.A. Arnold, A formula connecting sample and moment stability of linear stochastic systems, *SIAM Journal of Applied Mathematics* 44 (1984) 793–802.

[14] R. Khasminskii, N. Moshchuk, Moment Lyapunov exponent and stability index for linear conservative system with small random perturbation, *SIAM Journal of Applied Mathematics* 58 (1998) 245–256.

[15] L. Arnold, M.M. Doyle, N. Sri Namachchivaya, Small noise expansion of moment Lyapunov exponents for general two-dimensional systems, *Dynamics and Stability of Systems* 12 (1997) 187–211.

[16] N. Sri Namachchivaya, H.J. Van Roessel, M.M. Doyle, Moment Lyapunov exponent for two coupled oscillators driven by real noise, *SIAM Journal of Applied Mathematics* 56 (1996) 1400–1423.

[17] N. Sri Namachchivaya, H.J. Van Roessel, Moment Lyapunov exponent and stochastic stability of two coupled oscillators driven by real noise, *ASME Journal of Applied Mechanics* 68 (2001) 903–914.

[18] D. Liberzon, R.W. Brockett, Spectral analysis of Fokker–Planck and related operators arising from linear stochastic differential equations, *SIAM Journal on Control Optimization* 38 (2000) 1453–1467.
[19] R.V. Roy, Stochastic averaging of oscillators excited by colored Gaussian processes, *International Journal of Non-linear Mechanics* 29 (1994) 463–475.
[20] C.W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, second ed., Springer, Berlin, 1997.
[21] M.M. Klosek-Dygas, B.J. Matkowsky, Z. Schuss, Stochastic stability on nonlinear oscillators, *SIAM Journal of Applied Mathematics* 48 (1988) 1115–1127.