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Journal of Sound and Vibration 285 (2005) 501–509

JOURNAL OF SOUND AND VIBRATION

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Short Communication

Are the eigensolutions of a 1-d.o.f. system with viscoelastic damping oscillatory or not?

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Received 7 July 2003; received in revised form 14 September 2004; accepted 20 September 2004
Available online 22 December 2004

Abstract

In this note, one considers a 1 d.-o.-f. oscillator consisting of a mass and a viscoelastic spring, the rheology of which is represented by a so-called Zener 3-parameters model. Such a system has three eigenmotions: two of them are damped with or without oscillations (as in the case of viscous damping), the third is damped without oscillation. Introducing appropriate parameters, one has performed a detailed discussion of the nature of these eigenmotions.

1. Introduction

In this brief note, the following notations are used: $\mathbf{x}$ represents a column of n scalar components $[\mathbf{x}^{\mathrm{T}} = (x_1, x_2, ..., x_n)]$, and $\mathbf{M}$ an $n \times n$ matrix.

It is well known that in the case of an N-d.o.f. *viscously* damped system, the motions of which are governed by a system of equations in the form:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{A}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t), \tag{1}$$

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2004.09.007

where the matrices **M**, **A**, **K** are real, symmetric and positive definite and where the damping forces depend only on the current value $\dot{\mathbf{x}}(t)$ of the velocities, a search for eigensolutions in the form

$$\mathbf{x}(t) \cong e^{st} \tag{2}$$

leads to N pairs of values for s (which are real negative, or complex conjugate with a negative real part). The question of the oscillatory nature of the eigensolutions of such systems is of permanent interest (for instance, conditions which give rise to a non-oscillatory nature for *all* eigensolutions may be found in Ref. [1]).

If one considers now an N-d.o.f. system with *viscoelastic* damping, the motions of which are governed, in the time domain, by a system of equations of the form:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \int_{-\infty}^{t} \mathbf{a}(t-u)\dot{\mathbf{x}}(u)\,\mathrm{d}u + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \tag{3}$$

(in which the damping forces depend on the whole history of the velocities $\{\dot{\mathbf{x}}(u);\ -\infty < u < t\}$, with a memory functional $\mathbf{a}(t-u)$ weighted towards the recent past), it is worth noting that if such a system is submitted to a harmonic excitation $\mathbf{f}(t) = \mathbf{f}_0 e^{i\Omega t}$, the determination of its *forced response* $\mathbf{x}(t) = \mathbf{X}(\Omega)e^{i\Omega t}$ (and of all related quantities such as velocities, dissipation, damping…) does not require a knowledge of the eigenfrequencies and eigenmodes, because the complex amplitude $\mathbf{X}(\Omega)$ is governed, in the frequency domain, by the algebraic system:

$$[-\mathbf{M}\Omega^2 + \mathbf{K}^*(\Omega)]\,\mathbf{X}(\Omega) = \mathbf{f}_0, \tag{4}$$

the resolution of which is direct (see, for instance, Ref. [2]).

Concerning the free vibrations of such an N-d.o.f. system with viscoelastic damping, the question of its eigenfrequencies and eigenmodes has been clarified by Adhikari in a series of papers (see, for instance, Ref. [3]). A search for eigensolutions of Eq. (3) in the same form as Eq. (2) leads now to $m = 2N + p > 2N$ values for s, giving rise to solutions which are either damped without oscillations or damped with oscillations. The aim of this note is to discuss the nature of these roots in the case of a 1-d.o.f. system with *viscoelastic* damping, consisting of a mass and a "Zener-type" viscoelastic spring.

2. Viscous damping

One starts by recalling the elementary case of a classical viscously damped 1-d.o.f. system consisting of a mass and a "solid-like Kelvin–Voigt viscoelastic spring" (Fig. 1).

The motions of such a system are governed by the classical equation:

$$m\ddot{x}(t) + a\dot{x}(t) + kx(t) = f(t), \tag{5}$$

and it is well known that its two eigensolutions are given by $x(t) \cong e^{s_\pm t}$, where the two conjugate values $s_\pm$ may be written in the form of normalized expressions:

$$s_\pm = \omega(-\delta \pm i\sqrt{1-\delta^2}) \quad \text{where } \omega = \sqrt{\frac{k}{m}} > 0 \text{ and } \delta = \frac{a}{2\sqrt{km}} > 0, \tag{6}$$

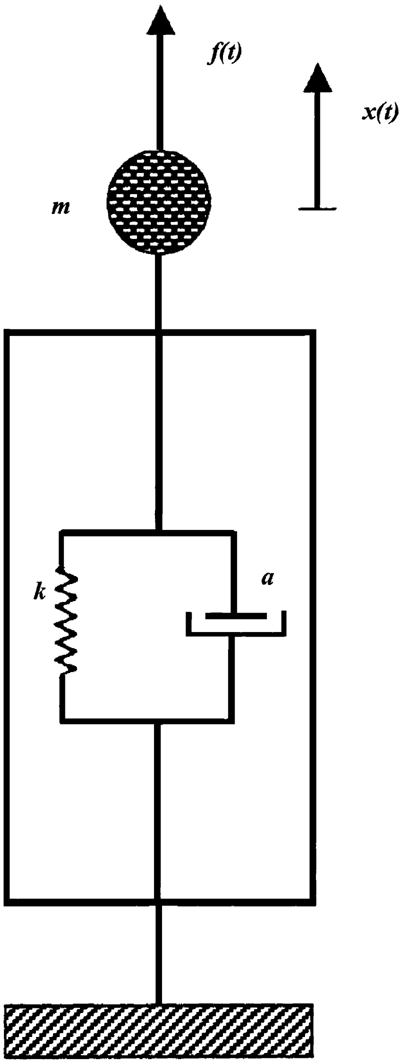

Fig. 1. One-d.o.f. oscillator with Kelvin–Voigt damping.

giving rise to solutions which are either damped without oscillations when the so-called *rate of critical damping* δ exceeds 100%, or damped with oscillations (with a *structural* decreasing time $\tau = m/2a$ depending on the viscous damping coefficient a—which is a purely rheological characteristic—*and on the mass* m) when δ is less than 100%.

3. Viscoelastic damping

One considers now the case of a viscoelastically damped 1-d.o.f. system consisting of a mass and a "solid-like Zener viscoelastic spring" with characteristics $m > 0$, $a > 0$, $k_1 > 0$ and long-term stiffness $k_\infty > 0$ (Fig. 2):

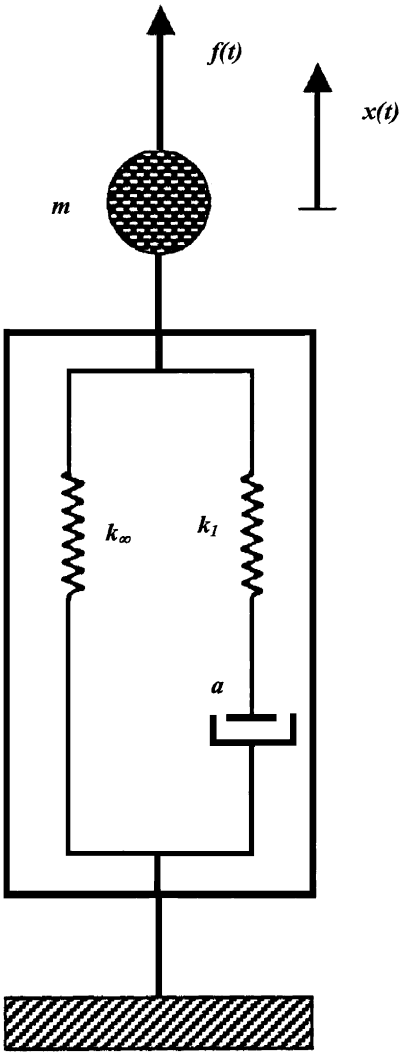

Fig. 2. One-d.o.f. oscillator with Zener damping.

It will be convenient to introduce:

- the *rheological* relaxation time τ' (independent of the mass m) given by $a = k_1\tau'$;
- the instantaneous stiffness $k_0 > k_\infty$ given by $k_1 = k_0 - k_\infty$;
- the *rheological* retardation time $\theta' > \tau'$ (independent of the mass m) given by $\theta' k_\infty = \tau' k_0$.

The equation governing the motions of this system may be written as

$$m\ddot{x}(t) + k_1 \int_{-\infty}^{t} \mathrm{e}^{-(t-u)/\tau'} \dot{x}(u)\,\mathrm{d}u + k_\infty x(t) = f(t). \tag{7}$$

The search of eigensolutions in the form $x(t) \cong e^{st}$ leads, by means of the change of variable:

$$\int_{-\infty}^{t} e^{-(t-u)/\tau'} e^{su}\,du = \int_{\infty}^{0} e^{-v/\tau'} e^{s(t-v)}(-dv) = e^{st}\int_{0}^{\infty} e^{-(s+1/\tau')v}\,dv = \frac{e^{st}}{s+1/\tau'} \tag{8}$$

and of the definitions of k_1 and of θ', to the following equation for s:

$$ms^2 + k_0\frac{s+1/\theta'}{s+1/\tau'} = 0, \tag{9}$$

or, assuming $s + 1/\tau' \neq 0$:

$$s^3 + \frac{1}{\tau'}s^2 + \omega'^2 s + \frac{1}{\theta'}\omega'^2 = 0 \quad \text{where } \omega'^2 = \frac{k_0}{m}. \tag{10}$$

Of course, Eq. (10) could be rewritten in the form

$$(s - s_0)(s^2 + 2\delta''\omega'' s + \omega''^2) = 0, \tag{11}$$

where ω'', δ'' and s_0 are to be determined by the identification of Eq. (10) with Eq. (9):

$$\begin{aligned} 2\delta''\omega'' - s_0 &= \frac{1}{\tau'},\\ \omega''^2 - 2\delta''\omega'' s_0 &= \omega'^2,\\ -s_0\omega''^2 &= \frac{1}{\theta'}\omega'^2. \end{aligned} \tag{12a−c}$$

The three roots of the cubic Eq. (10) may thus be written in the normalized form of:

- a real negative root given, from Eq. (12c), by $s_0 = -(1/\theta')(\omega'^2/\omega''^2)$, giving rise to a non-oscillating damped solution decreasing without oscillations with a *structural* decreasing time $\tau''' = -1/s_0 = \theta'(\omega''^2/\omega'^2)$;
- a pair of conjugate roots $s_\pm = \omega''(-\delta'' \pm i\sqrt{1-\delta''^2})$, where ω'' (which may be assumed to be >0) and δ'' (which may easily be demonstrated to be >0) are solutions of the nonlinear system obtained by substituting s_0 in Eqs. (12a) and (12b):

$$\begin{aligned} 2\delta''\omega'' + \frac{1}{\theta'}\frac{\omega'^2}{\omega''^2} &= \frac{1}{\tau'},\\ \omega''^2 + 2\delta''\frac{1}{\theta'}\frac{\omega'^2}{\omega''} &= \omega'^2, \end{aligned} \tag{13}$$

giving rise to damped eigensolutions, non-oscillating if $\delta'' > 1$ or oscillating (with a *structural* decreasing time $\tau'' = 1/\delta''\omega''$) if $\delta'' < 1$.

4. Nature of the solutions in the case of viscoelastic damping

To exhibit the nature (oscillating or not) of the two later solutions, it is of course possible to use the classical discussion of the roots of the cubic Eq. (10) resting on the sign of its discriminant D (see, for instance, Ref. [4]). It appears to be more convenient here to re-write Eq. (10). Introducing the two nondimensional positive parameters δ_τ and δ_θ defined by

$$\frac{1}{\tau'} = 2\delta_\tau\omega' \Leftrightarrow \delta_\tau = \frac{k_1\sqrt{m}}{2a\sqrt{k_0}}, \quad \frac{1}{\theta'} = 2\delta_\theta\omega' \Leftrightarrow \delta_\theta = \frac{k_\infty}{k_0}\delta_\tau < \delta_\tau, \tag{14}$$

and setting $\bar{s} = s/\omega'$, Eq. (10) takes the nondimensional form:

$$f(\bar{s}) = \bar{s}^3 + 2\delta_\tau\bar{s}^2 + \bar{s} + 2\delta_\theta = 0, \tag{15}$$

which shows that this discussion is managed by the two non-dimensional parameters δ_τ (which governs the first derivative $y = f'(x) = 3x^2 + 4\delta_\tau x + 1$ and the second derivative $y = f''(x) = 6x + 4\delta_\tau$ of the curve $y = f(x)$) and δ_θ (which governs the ordinate at the origin $y_0 = f(0) = 2\delta_\theta$ of the same curve).

Clearly, Eq. (15) has *three real negative roots* (giving rise to three damped non-oscillating eigensolutions) *when the two following conditions are satisfied*:

(1) the curve has two extremums, that is the discriminant of its derivative $y = f'(x)$ is positive:

$$\delta_\tau > \frac{3}{4}, \tag{16}$$

so that it has a relative maximum at the point:

$$a_-(\delta_\tau, \delta_\theta) = -\frac{2}{3}\left(\delta_\tau + \sqrt{\delta_\tau^2 - \delta_\theta^2}\right), \tag{17}$$

and a relative minimum at the point:

$$a_+(\delta_\tau, \delta_\theta) = -\frac{2}{3}\left(\delta_\tau - \sqrt{\delta_\tau^2 - \delta_\theta^2}\right); \tag{18}$$

(2) condition (16) being satisfied, the vertical positioning of the curve $y = f(x)$ must be such as shown in Fig. 3, between the two limiting cases of:
- Fig. 4, where $f(a_-) = 0$, i.e. $\delta_\theta = \delta_\theta^- = -\frac{1}{2}[a_-^3 + 2\delta_\tau a_-^2 + a_-]$, and
- Fig. 5, where $f(a_+) = 0$, i.e. $\delta_\theta = \delta_\theta^+ = -\frac{1}{2}[a_+^3 + 2\delta_\tau a_+^2 + a_+]$.

That is, δ_θ must satisfy the double inequality

$$\delta_\theta^- = -\frac{1}{2}[a_-^3 + 2\delta_\tau a_-^2 + a_-] < \delta_\theta < \delta_\theta^+ = -\frac{1}{2}[a_+^3 + 2\delta_\tau a_+^2 + a_+]. \tag{19}$$

Conversely, Eq. (15) has *one real negative root and two complex conjugate roots* (giving rise to one damped non-oscillating and two damped oscillating eigensolutions) *when the data* m, a, k_1 and k_∞ *are such that one of the there of the following sets of conditions are satisfied*:

(1) either:

$$\delta_\tau < \frac{3}{4}, \tag{20}$$

(2) or:

$$\delta_\tau > \frac{3}{4} \quad \text{and} \quad \delta_\theta < \delta_\theta^- = -\frac{1}{2}[a_-^3 + 2\delta_\tau a_-^2 + a_-], \tag{21}$$

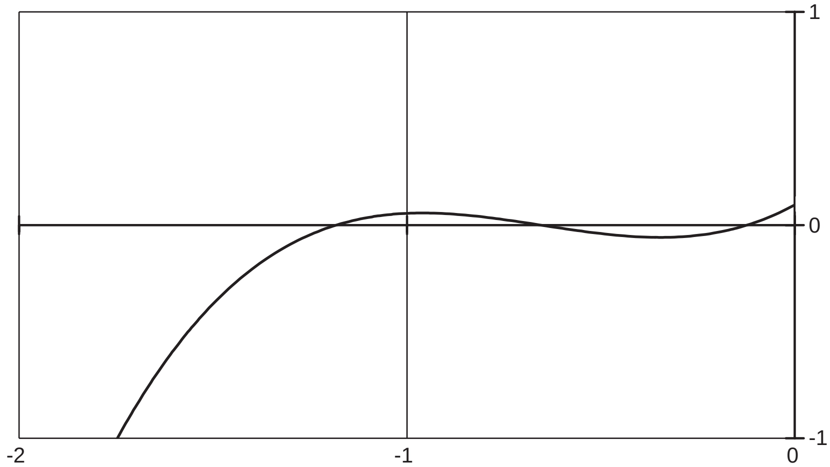

Fig. 3. Curve $y = f(x) = x^3 + 2\delta_\tau x^2 + x + 2\delta_\theta$. Vertical positioning when $\delta_\tau > \frac{3}{4}$ ($\omega' = 1,\ \delta_\tau = 0.98,\ \delta_\theta = 0.0475\ldots$).

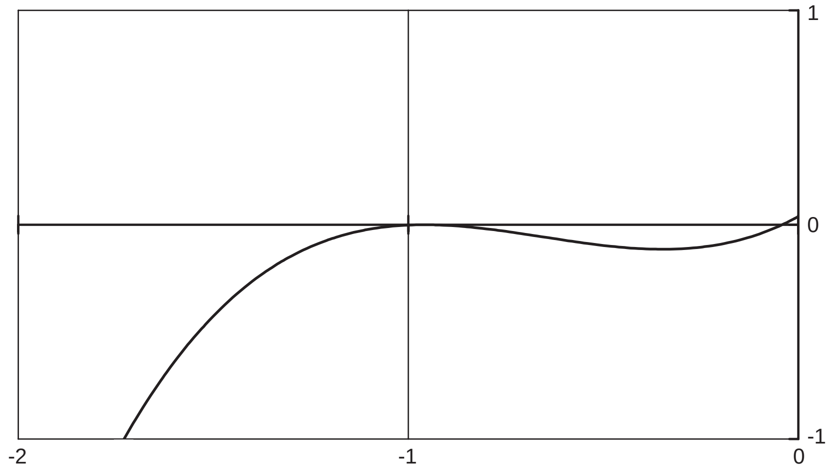

Fig. 4. Curve $y = f(x) = x^3 + 2\delta_\tau x^2 + x + 2\delta_\theta$. Case $f(a_-) = 0$ ($\omega' = 1,\ \delta_\tau = 0.98,\ \delta_\theta = 0.0192\ldots$).

(3) or:

$$\delta_\tau > \frac{3}{4} \quad \text{and} \quad \delta_\theta > \delta_\theta^+ = -\frac{1}{2}[a_+^3 + 2\delta_\tau a_+^2 + a_+]. \tag{22}$$

5. Viscous damping, a singular rheological limit of viscoelastic damping

The rheological behaviour of both springs relating the traction $T(t)$ on the spring to its extension $x(t)$ may be written as follows, using hereditary integrals with relaxation function $k(t)$ or

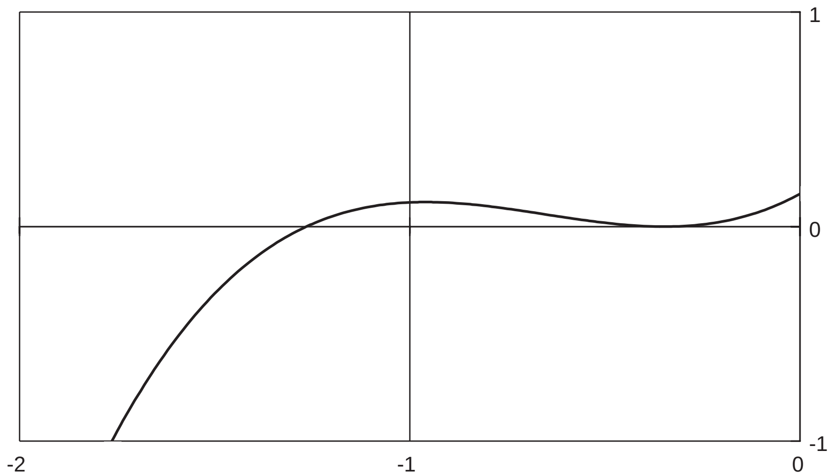

Fig. 5. Curve $y = f(x) = x^3 + 2\delta_\tau x^2 + x + 2\delta_\theta$. Case $f(a_+) = 0$ ($\omega' = 1,\ \delta_\tau = 0.98,\ \delta_\theta = 0.0764\ldots$).

Table 1
Relaxation and creep functions of Zener and Kelvin–Voigt models

	Relaxation function $k(t)$	Creep function $c(t)$
Zener model	$k_\infty + (k_0 - k_\infty)\mathrm{e}^{-t/\tau'}$	$\frac{1}{k_\infty} + \left(\frac{1}{k_0} - \frac{1}{k_\infty}\right)\mathrm{e}^{-t/\theta'}$
Kelvin–Voigt model	$k_\infty + a\delta(t) = k_0 + a\delta(t)$	$c(t) = \frac{1}{k_\infty}[1 - \mathrm{e}^{-t/\theta'}] = \frac{1}{k_0}[1 - \mathrm{e}^{-t/\theta'}]$

creep function $c(t)$:

$$T(t) = \int_{-\infty}^{t} k(t-u)\frac{\mathrm{d}x}{\mathrm{d}u}(u)\,\mathrm{d}u \Leftrightarrow x(t) = \int_{-\infty}^{t} c(t-u)\frac{\mathrm{d}T}{\mathrm{d}u}(u)\,\mathrm{d}u, \tag{23}$$

where, if there are discontinuities, the derivatives are to be taken in the sense of distributions.

The relaxation and creep functions are recalled in Table 1 (where $\delta(t)$ is Dirac's distribution). It is worth noting that the Kelvin–Voigt model seems to be obtained from the Zener model when, keeping k_∞ and a constant, $k_1 \to 0$ (hence $k_0 \to k_\infty$). As expected, the rheological relaxation time τ', given by $a = k_1\tau'$, disappears (it becomes infinite). But the Kelvin–Voigt model has a retardation time θ' given by $a = k_\infty\theta'$, and not by the limit of the relation $\theta' k_\infty = \tau' k_0$.

6. Conclusion

As mentioned earlier, the question of the eigenfrequencies and eigenmodes of an N-d.o.f. system with viscoelastic damping has been clarified by Adhikari. In the case of a 1-d.o.f. system with viscoelastic damping represented by a generalized rheological model with $2n + 1$ parameters, the characteristic equation generalizing (9) takes the form:

$$ms^2 + k_0 \frac{\left(s+\frac{1}{\theta_1}\right)\left(s+\frac{1}{\theta_2}\right)\ldots\left(s+\frac{1}{\theta_n}\right)}{\left(s+\frac{1}{\tau_1}\right)\left(s+\frac{1}{\tau_2}\right)\ldots\left(s+\frac{1}{\tau_n}\right)} = 0 \tag{24}$$

and it is easy to prove the result (more accurate than in the N-d.o.f. case) that, due to the basic properties of the relaxation modulus $k(t)$, the set of the $n+2$ solutions always split into:

- n solutions decreasing without oscillations with *structural* decreasing times $\tau'_1, \ldots, \tau'_n$;
- two solutions which decrease with or without oscillations (exactly as in the case of viscous damping).

In other words, in the case of a 1-d.o.f. system with viscoelastic damping, one cannot have more than two eigensolutions decreasing with oscillations.

By restricting in this note the area of the investigation to the simplest three parameter Zener model, a complete analytical representation of the solution has been obtained.

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