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Journal of Sound and Vibration 285 (2005) 615–635
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A variational basis for error analysis in finite element elastodynamic problems

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Received 1 July 2003; received in revised form 23 February 2004; accepted 26 August 2004
Available online 15 December 2004

Abstract

In this paper, the variational basis for finite element analysis of elastodynamic problems has been examined using the principle of virtual work. Using the principle of virtual work, two fundamental important theorems on errors in variationally correct formulations in computational elastodynamics have been discussed and illustrated with simple one-dimensional elements. A geometric interpretation of the behavior of these errors in approximate solutions from a variationally correct formulation has been presented using the frequency-error-hyperboloid. It has been shown that derivation of a complete and accurate mathematical description of the nature of errors in free vibration analysis involves a simultaneous consideration of errors in both displacement and strains. This is in sharp contrast to the error analysis in elastostatic problems where the variational basis involves only strains.

Furthermore, it has been observed that variationally correct and conforming formulations satisfy the projection theorems that result from the weak forms of elastodynamic problems by virtue of the virtual work principle. These formulations involve consistent mass matrices and yield eigenfrequencies that are always higher than the analytical values, independent of domain discretisation. This is not necessarily true for variationally incorrect lumped mass formulations in which no guarantee of the boundedness of the eigenvalues with respect to the exact ones can be given. With the help of sweep tests, it has been demonstrated that the computed eigenvalues with lumped mass formulations can be higher than, equal to, or lower than the exact values, depending on the finite element mesh.

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doi:10.1016/j.jsv.2004.08.039

1. Introduction

Finite element analysis can be regarded as a tool for obtaining approximate solutions to differential equations using piecewise assumed interpolation functions. Conventionally, the method involves the use of element equilibrium equations derived from the basic principles of variational calculus [1–5].

In the finite element analysis literature, it can be observed that while extensive research work has been reported on error analysis for elastostatic problems, definitive and conclusive work on error analysis for elastodynamic problems is relatively scanty. The best-fit paradigm of elastostatics, emerging out of the orthogonality condition that results from the Hu–Washizu's theorem, has been used successfully in interpreting the mechanism of computation and in predicting errors for finite element analysis of elastostatic problems [4,5]. In a mathematical rigorous fashion, the best-fit paradigm has been correlated to the projection theorems of function space algebra of elastostatics [1,4]. It has been shown how the projection theorems of function space algebra can be utilised to derive a priori error estimates and explain the various pathological problems of finite element analysis in elastostatics [6,7]. Recently, the virtual work principle has been extended [8], in a physically meaningful way, to examine the various variational statements of elastodynamic problems, derived earlier [1,4] using rigorous but abstract mathematics.

In error analysis of finite element analysis of elastodynamic problems, a posteriori approaches have been adopted by several investigators [9–14]. Mori [4] and Tong et al. [9] have shown that the lumped mass matrices result from using specific basis functions satisfying orthogonality conditions with respect to mass distribution in the element domain, without disturbing the displacement functions used for calculation of the strain energy. Strang and Fix [1] have branded such extra-variational techniques as "variational crimes". Tong et al. [9] have concluded that for *uniform* meshing, the convergence rate of lumped mass formulation is identical to that of the consistent mass formulation for second-order differential equations, without addressing the question of boundedness of the eigenvalues. Using only uniform meshing, Cook et al. [3,10,11] have observed that lumped mass methods yield lower natural frequencies than those of the consistent mass methods for uniform meshes, and have also recognised that no guarantee of bounds of the eigenvalues can be given for arbitrary meshing. Fried [12] conducted a posteriori error analysis using an error indicator that served as a measure of the relative change of an eigenvalue for the hierarchical finite element method. Fuenmayor et al. [13] have investigated discretisation errors and optimisation of *h*-adaptive process in finite element elastodynamics. They have argued that the eigenvalue error is primarily governed by the modal strain energy error compared to which the associated error in the modal kinetic energy is negligibly small, and therefore has been ignored in the analysis. Wiberg et al. [14] have presented an adaptive *h*-version finite element scheme to control the discretisation error in free vibration analysis. They have used the local error estimator that reflects the difference between the superconvergent patch recovery solution and the finite element solution with the original mesh. The strain energy distribution only has been used as a basis for a new mesh in the adaptive procedure.

Quite recently, attempts were made to derive error convergence rates and estimates for the finite element elastodynamics of one-dimensional elements like bar and Euler–Bernoulli beam [15] and Timoshenko beam [16]. Here, the qualities of stiffness and mass matrices were assessed independently using what were called the stress and momentum correspondence principles.

Underlying the approach is the tacit assumption that stresses (strains) and velocities (momentum) obtained through the finite element discretisation process are least square accurate approximations of the true stresses, etc. These are seen as consequences of projection theorems resulting from the virtual work principle.

Using first principles, the present work studies how the errors in eigenvalues in elastodynamic computations by finite element method can be interpreted. The Rayleigh quotient has been reviewed in the light of the principle of virtual work, and the equivalent projection theorem equation for elastodynamics has been derived. It has been shown that unlike the elastostatic case, the best-fit paradigm is violated at a global level, and a modified expression for the projections theorem results in elastodynamics. In elastodynamics, unlike elastostatics, the strain energy of the error differs from the error of the strain energy. The variational statement of Strang and Fix [1] has been re-derived using the virtual work principle. It has been shown how the various elastodynamic variational statements, valid for the variationally correct consistent mass formulations, are violated when lumped mass formulations are adopted. The consequences of employing lumped mass matrices are illustrated by the sweep tests. These show that lumped mass formulations lack the boundedness property of the eigenvalues. The computed eigenvalues with lumped mass formulations can be higher than, equal to, or lower than the exact values, depending on the finite element mesh. Accuracy of the consistent mass and lumped mass solutions is decided by the finite element meshing, and no general conclusions can be made. The simple one-dimensional linear two noded bar element has been used to illustrate the fundamental principles that guide finite element computations in conservative or self-adjoint problems.

The present paper also highlights the geometrical implications, in the abstract sense, of the various error statements in computational elastodynamics. The frequency-error-hyperboloid is shown to be a surface generated by the error norms of free vibration. The geometric implications of the Rayleigh quotient errors for variationally correct formulations have been presented in terms of the angles of projections for modal displacements and strains.

2. The Rayleigh quotient, the projection theorem and the energy-error rules for elastodynamics

2.1. Inner products

For the purpose of analysis, we first define two types of inner products and the norms they describe. These inner products are global in character, and are presented as summation over N^e elements of the complete domain of analysis.

2.1.1. Stiffness-inner product and norm

If $\{a\}$ and $\{b\}$ are vectors each of r-rows, and $[D]$ is a positive definite square rigidity matrix of size $r \times r$, then their *stiffness-inner product* is defined as

$$\langle a, b\rangle = \sum_{ele=1}^{N^e} \int_{ele} \{a\}^{\mathrm{T}}[D]\{b\}\,\mathrm{d}x \tag{1a}$$

and the *stiffness-norm squared* value of the vector $\{a\}$ is given as

$$\|a\|^2 = \langle a, a \rangle. \tag{1b}$$

2.1.2. Inertia-inner product and norm

If $\{c\}$ and $\{d\}$ are vectors each of s-rows, and $[\rho]$ is a positive definite square inertia density matrix of size $s \times s$, then their *inertia-inner product* is defined as

$$(c, d) = \sum_{ele=1}^{N^e} \int_{ele} \{c\}^{\mathrm{T}} [\rho] \{d\} \, \mathrm{d}x \tag{2a}$$

and the *inertia-norm squared* value of the vector $\{c\}$ is given as

$$|c|^2 = (c, c). \tag{2b}$$

2.2. Rayleigh quotient and the energy-error rules for elastodynamics

Free, simple harmonic vibration of a continuum in a particular normal mode with displacement modal function $u(x)$ and natural circular frequency ω can be expressed as a space (x) and time (t) dependent displacement function

$$\{U(x, t)\} = \{u(x)\} \mathrm{e}^{\mathrm{i}\omega t}. \tag{3}$$

If the approximate modal displacement function is denoted by some admissible vector $\{\overline{u}\}$ (satisfying the kinematic boundary conditions), and the resulting approximate modal strain vector is $\{\overline{\varepsilon}\}$, then the Rayleigh quotient is defined as

$$\overline{\omega}^2 = \frac{\|\overline{\varepsilon}\|^2}{|\overline{u}|^2}. \tag{4}$$

Here the parameter $\overline{\omega}$ represents the approximate value for the angular frequency (radians per sec) corresponding to the normal mode approximated by the admissible displacement function $\{\overline{u}\}$.

The weak form of the classical differential equation of free vibration readily yields the following expressions:

$$\omega^2 = \frac{\|\varepsilon\|^2}{|u|^2} \tag{5}$$

and

$$\omega^2 = \frac{\langle \overline{\varepsilon}, \varepsilon \rangle}{(\overline{u}, u)}. \tag{6}$$

Here $\{u\}$, $\{\varepsilon\}$ and ω are the analytical modal displacement vector, modal strain vector and the corresponding (exact) natural circular frequency, respectively. Note that Eqs. (4) and (5) can be obtained from the statement of the Rayleigh quotient, but Eq. (6) can be obtained only through the virtual work principle [1,5,8]. Combining Eqs. (4) and (6), one may obtain the following expression:

$$\langle \overline{\varepsilon}, \varepsilon - \overline{\varepsilon} \rangle = (\overline{u}, \omega^2 u - \overline{\omega}^2 \overline{u}). \tag{7}$$

Eq. (7) is a consequence of the virtual work principle, and can be interpreted as

Total virtual work done by error of stress on approximate strain
= Total virtual work done by error of inertia force on approximate displacement.

Combining Eqs. (4) and (5), we get another rule

$$\|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2 = \omega^2|u|^2 - \overline{\omega}^2|\overline{u}|^2 \tag{8}$$

or

Error of global strain energy = Error of global kinetic energy.

This can be interpreted as the elastodynamic energy-error rule, governing the error in energies due to discretisation.

It will be useful also to compute the energies of the errors in strain and displacements, due to discretisation process and to examine if a simple relationship exists between these quantities. The energy of the strain error can be expanded as follows:

$$\|\varepsilon - \overline{\varepsilon}\|^2 = \|\varepsilon\|^2 + \|\overline{\varepsilon}\|^2 - 2\langle\varepsilon, \overline{\varepsilon}\rangle = \|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2 - 2\langle\overline{\varepsilon}, \varepsilon - \overline{\varepsilon}\rangle. \tag{9}$$

Note that for elastostatics, it has been shown earlier [1,8] that the virtual work principle can be used to prove the following orthogonality condition:

$$\langle\overline{\varepsilon}, \varepsilon - \overline{\varepsilon}\rangle = 0. \tag{10a}$$

Thus from Eq. (8), one obtains the energy-error rule for elastostatics [5,8],

$$\|\varepsilon - \overline{\varepsilon}\|^2 = \|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2 \tag{10b}$$

i.e.

The strain energy of the error = error in the strain energy.

Using the energy-error rule (Eq. (8)) and the virtual work rule (Eq. (7)), one can have from Eq. (9)

$$\begin{aligned}\|\varepsilon - \overline{\varepsilon}\|^2 &= \omega^2|u|^2 - \overline{\omega}^2|\overline{u}|^2 - 2(\overline{u}, \omega^2 u - \overline{\omega}^2\overline{u}) = \omega^2|u|^2 - 2(\overline{u}, \omega^2 u) + \overline{\omega}^2|\overline{u}|^2 \\ &= \omega^2|u|^2 - 2(\overline{u}, \omega^2 u) + \omega^2|\overline{u}|^2 + [\overline{\omega}^2 - \omega^2]|\overline{u}|^2\end{aligned}$$

i.e.

$$\|\varepsilon - \overline{\varepsilon}\|^2 = \omega^2|u - \overline{u}|^2 + [\overline{\omega}^2 - \omega^2]|\overline{u}|^2. \tag{11}$$

The above equation has been presented earlier using the weak form by Strang and Fix [5], but with normalising the approximate displacement norm ($|\overline{u}|^2 = 1$).

The error in the eigenvalue is given by the expression derived from above as

$$\left[\frac{\overline{\omega}^2}{\omega^2} - 1\right] = \frac{\|\varepsilon - \overline{\varepsilon}\|^2 - \omega^2|u - \overline{u}|^2}{\omega^2|\overline{u}|^2}. \tag{12}$$

3. The frequency-error hyperboloid

From Eq. (11), one can derive the following equation:

$$\frac{|u-\overline{u}|^2}{|\overline{u}|^2}+\frac{\overline{\omega}^2}{\omega^2}-\frac{\|\varepsilon-\overline{\varepsilon}\|^2}{\omega^2|\overline{u}|^2}=1 \tag{13a}$$

or

$$\frac{X^2}{|\overline{u}|^2}+\frac{Y^2}{1^2}-\frac{Z^2}{\omega^2|\overline{u}|^2}=1, \tag{13b}$$

where $X=|u-\overline{u}|$, $Y=\overline{\omega}/\omega$ and $Z=\|\varepsilon-\overline{\varepsilon}\|$. With the approximate modal displacement vector $|\overline{u}|$ arbitrarily scaled to some chosen constant, ($|\overline{u}|^2=a^2$, say), and noting that all norms and frequencies are positive, Eq. (13b) can be interpreted to be the algebraic representation of the surface of the *first octant of a hyperboloid of one-sheet* (Fig. 1). This will be called as a frequency-error-hyperboloid.

It can be noted that this hyperboloid intersects the $Z=0$ plane (or the X–Y plane) in the quarter of an ellipse of semi-axes of magnitudes $|\overline{u}|$ and 1 along X and Y axes, respectively. The point E at the apex of this ellipse on the $Y=\overline{\omega}/\omega$ axis, of coordinates ($X=0$, $Y=\overline{\omega}/\omega=1, Z=0$) represents the analysis with approximate functions replaced by exact ones. A plane $Y=\overline{\omega}/\omega=1$ parallel to the X–Z plane (and tangential to this ellipse) through this point E intersects this octant of the hyperboloid along a *straight line EF*. The equation to this straight line EF on the $Y=\overline{\omega}/\omega=1$ plane is given by

$$Z=\omega X \tag{14a}$$

or

$$\|\varepsilon-\overline{\varepsilon}\|=\omega|u-\overline{u}|. \tag{14b}$$

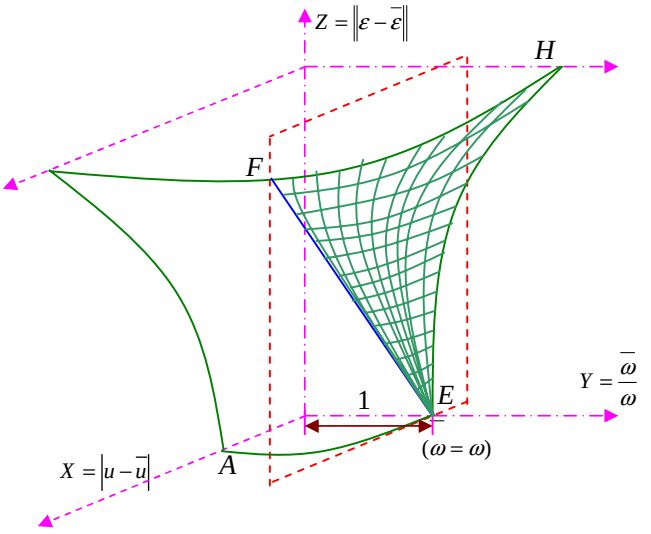

Fig. 1. Geometric interpretation of eigenvalue analysis of the variationally correct formulation using frequency-error hyperboloid. Approximate eigenvalues obtained from a variationally correct formulation lie in the shaded portion of hyperboloid.

For variationally correct formulations, the upper boundednesss ($\overline{\omega} \geqslant \omega$) of the Rayleigh quotient indicates that the only feasible surface that represents real computational results is that portion of the first octant of the hyperboloid that lies bounded by the straight line EF on one side on the $Y = \overline{\omega}/\omega = 1$ plane (given by Eq. (14)), and the hyperbola EH on the other side on the $X = 0$ plane, given by the following equation:

$$\frac{\overline{\omega}^2}{\omega^2} - \frac{Z^2}{\omega^2 |\overline{u}|^2} = 1. \tag{15}$$

This portion is shown as a shaded zone on the frequency-error-hyperboloid. It is now obvious that the computationally feasible surface of this hyperboloid is enveloped by the straight line EF (Eq. (14)) and the hyperbola EH (Eq. (15)), both originating from the critical point E. In this shaded surface, one observes that for a given modal displacement error norm $|u - \overline{u}|$ the strain error satisfies the condition

$$\|\varepsilon - \overline{\varepsilon}\| \geqslant \omega |u - \overline{u}|. \tag{16}$$

The equality sign is valid only when exact modal functions are used in the analysis, so that all errors vanish. Such a condition is represented by the point E which is also the limit of convergence of finite element computation by finer discretisation.

An approximate but variationally correct formulation for elastodynamics satisfies the following two conditions:

(a) The continuity of the derivatives as required by the weak form is satisfied within the element.
(b) The mass matrix is consistent, i.e. it is developed through the variation of the kinetic energy used in the weak form.

Variationally incorrect formulations violate at least one of conditions (a) and (b). For such formulations, Eqs. (6), (7) and (11) are not satisfied, and conditions (16) is not necessarily valid. Note that for all formulations, Eq. (8) is valid always since it springs from the conservation of energy only. Hence when consistent mass matrices are replaced by lumped ones, no guarantee of upper bound of the exact frequency can be given. In fact, the computed approximate frequency with lumped masses can be greater than, equal to or less than the exact frequency for the same mode, according to the distribution of the nodal points.

4. Order of convergence of approximate finite element eigenvalues

An important question raised in evaluating the quality of a finite element formulation is whether the convergence rate is uniformly optimal. One method to evaluate this is to plot the convergence to zero of a suitable error norm as meshes are refined and to verify if these lines (curves) have optimal slope [17]. It is desirable that the order of convergence be derived a priori from first principles, as was done for example using the correspondence principles [15,16]. In what follows, we now attempt to introduce an approach using the function space arguments and the Rayleigh quotient to predict the optimal slope of uniform convergence for finite element elastodynamical model.

The energy-error rule for elastodynamics discussed by Eq. (8) clearly shows that the error in the approximate finite element strain energy is exactly same as the error in the approximate finite element kinetic energy. In other words the order of convergence is same for both quantities. In this section, we shall make use of this theory to investigate the rate of convergence of the approximate eigenvalue of free vibration problems.

Before we derive the order of convergence estimate for elastodynamics, let us review the case of finite element elastostatics. In a finite element (i.e. the sub-domain region), the exact displacement, strain and stress fields ($u, \varepsilon,$ and σ) are replaced by finite element solutions ($\overline{u}$, $\overline{\varepsilon}$, and $\overline{\sigma}$). It is known from the projection theorem [1] or alternatively from the orthogonality condition arising from the Hu–Washizu theorem [5,15], that

$$\int_e \delta\overline{\varepsilon}^{\mathrm{T}}(\sigma - \overline{\sigma})\,\mathrm{d}x = 0. \tag{17}$$

From this, one can proceed to demonstrate that if displacement fields $\overline{u}$ are chosen complete to order x^n (for simplicity a one-dimensional problem with x, or non-dimensional ξ is chosen as the coordinate variable), so that strain (stress) fields are complete to the order x^{n-1}, then the finite element can model actual strain (stress) field of order x^n in a best-fit manner. Let

$$\sigma = a_0 + a_1 h L_1(\xi) + a_2 h^2 L_2(\xi) + \cdots + a_n h^n L_n(\xi), \tag{18}$$

where $L_n(\xi)$ are suitably normalised Legendre polynomials and h is the element length. This form allows us to exploit the orthogonality condition given by Eq. (17). From Eq. (17) one can show that

$$\overline{\sigma} = a_0 + a_1 h L_1(\xi) + a_2 h^2 L_2(\xi) + \cdots + a_{n-1} h^{n-1} L_{n-1}(\xi). \tag{19}$$

Thus a finite element computation produces approximate strains (stresses) which are accurate to $O(h^n)$. From this it is simple to show that the error of the energy (= energy of the error) is of the $O(h^{2n})$ [15].

To extend this theory to elastodynamics, one must carefully examine the energy error rule for elastodynamics (Eq. (8)). Let us introduce the idea of generalised mass in the same equation, where $|u|^2 = |\overline{u}|^2 = 1$; then we have from Eq. (8)

$$-(\|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2) = (\overline{\omega}^2 - \omega^2). \tag{20}$$

Eq. (20) clearly explains that the error in the approximate eigenvalue is still governed by the error in the strain energy. Therefore, the order of convergence discussed earlier for the elastostatic case is directly applicable to the approximate eigenvalue obtained from a variationally correct formulation. In other words, the finite element eigenvalues obtained from consistent mass formulation, $\overline{\omega}^2$ should have an order of convergence of the error in the strain energy, and therefore of $O(h^{2n})$. For the linear bar element, this will mean that a consistent mass finite element model will have a convergence of $O(h^2)$. This will be illustrated with numerical examples in Section 8.4.

5. Geometric interpretation of the Rayleigh quotient error for variationally correct formulations

In this section, we show how Rayleigh quotient error for variationally correct formulations can be interpreted geometrically using the function space approach. Using Cauchy–Schwarz inequality theorem, one can define θ_u as the included angle between the displacement vectors $\{u\}$ and $\{\overline{u}\}$ and θ_ε as the included angle between the corresponding strain vectors $\{\varepsilon\}$ and $\{\overline{\varepsilon}\}$, so that

$$\cos(\theta_u) = \frac{(u,\overline{u})}{|u||\overline{u}|}, \quad \cos(\theta_\varepsilon) = \frac{\langle \varepsilon, \overline{\varepsilon}\rangle}{\|\varepsilon\|\,\|\overline{\varepsilon}\|}. \tag{21a,b}$$

Thus

$$\frac{\cos\,\theta_u}{\cos\,\theta_\varepsilon} = \frac{(u,\overline{u})}{|u||\overline{u}|}\,\frac{\|\varepsilon\|\,\|\overline{\varepsilon}\|}{\langle \varepsilon, \overline{\varepsilon}\rangle} = \frac{1}{\omega^2}\,\frac{\|\varepsilon\|\,\|\overline{\varepsilon}\|}{|u||\overline{u}|} \quad \left(\text{because } \frac{\langle \varepsilon, \overline{\varepsilon}\rangle}{(u,\overline{u})} = \omega^2\right).$$

Squaring both sides of the above equation, one obtains,

$$\frac{\cos^2\theta_u}{\cos^2\theta_\varepsilon} = \frac{1}{\omega^4}\,\frac{\|\varepsilon\|^2\|\overline{\varepsilon}\|^2}{|u|^2|\overline{u}|^2} = \frac{1}{\omega^4}\,\omega^2\,\overline{\omega}^2 = \frac{\overline{\omega}^2}{\omega^2}. \tag{22}$$

Eq. (22) shows that the approximate and exact Rayleigh quotients are in the same ratio as the squares of the cosines of the included angles θ_u and θ_ε. Thus the error in the Rayleigh quotient (eigenvalue) is determined solely by the included angles θ_u and θ_ε which are given by Eqs. (21a,b), and is independent of the actual magnitudes (norms) of the modal displacement or strain vectors, which can be arbitrarily normalised.

For a variationally correct formulation (consistent mass), the ratio $\overline{\omega}/\omega$ exceeds unity for arbitrary meshing. This immediately implies that θ_u is *less* than θ_ε. Geometrically, one can interpret this fact in the following way. The *deviation* of the approximate modal displacement vector from the exact modal displacement vector is *less* than that of the approximate strain vector from the exact strain vector. This is demonstrated in Fig. 2.

6. Numerical experiments to illustrate the elastodynamic error rules

In this section we shall illustrate the elastodynamic energy-error rules presented in the previous section with some examples.

6.1. Analysis of floating bar (both ends free)

Consider a bar with both ends free (Fig. 3). The exact expressions for the modal displacement u and natural circular frequency ω for the fundamental antisymmetric mode are given by

$$u = -a\,\sin\left(\frac{\pi x}{L}\right), \quad \omega = \sqrt{\frac{\pi^2 E}{\rho L^2}},$$

where L is the total length of bar, E is the Young's modulus and ρ is the density of the material of the bar. Here x is measured with the bar center as the origin. The same problem has been analyzed

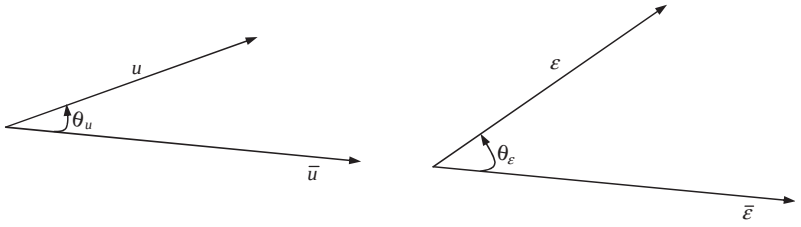

Fig. 2. Projection of vectors.

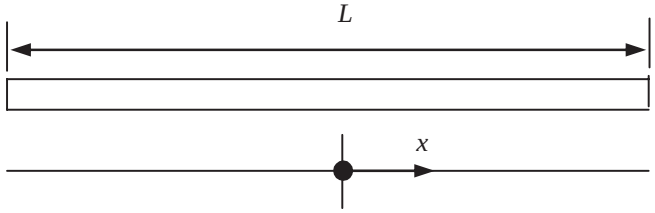

Fig. 3. Free–free bar.

using Rayleigh quotient with linear and cubic displacement functions representing approximately the fundamental (antisymmetric) mode. The approximate displacement functions chosen should satisfy geometric boundary condition for the antisymmetric mode,

$$\bar{u}(x=0)=0.$$

The admissible linear and cubic displacement functions representing the first mode are taken as $\bar{u}=b(-2x/L)$ and $\bar{u}=b(-2x/L)+c((x/L)-4x^3/L^3)$, respectively, so that the linear function has a single generalised degree of freedom b and the cubic function has two generalised degrees of freedom b and c. The approximate functions given above satisfy the necessary geometric boundary conditions for the antisymmetric mode. For the cubic function, the fundamental natural (antisymmetric) mode of the free–free bar corresponds to a specific ratio of the two generalised degrees of freedom, $(c/b=-1.076)$, so that the admissible cubic function representing the fundamental mode is given by

$$\bar{u}=b\left[\frac{-2x}{L}-1.076\left(\frac{x}{L}-\frac{4x^3}{L^3}\right)\right]. \tag{23}$$

The results of the analysis are presented in Table 1. It can be seen that the solutions satisfy the elastodynamic projection theorem and energy-error rule given by Eqs. (7) and (8).

6.2. *Analysis of a fixed–free bar*

A fixed–free bar shown in Fig. 4 has been analyzed using the exact method and using the Rayleigh quotient method using a linear modal function. Here the geometric boundary condition is $\bar{u}(x=0)=0$, where x is measured with the left end of the bar as origin. The results for the fundamental mode are tabulated in Table 2. It can be seen that the solutions satisfy the elastodynamic projection theorem and energy-error rule presented in Eqs. (7) and (8).

Table 1
Analysis results of the free–free bar using the exact and the approximate methods for the fundamental mode

	Exact solutions	Approximate solutions	
		Linear function	Cubic function
Modal displacement function	$u = -a\sin\left(\frac{\pi x}{L}\right)$	$\overline{u} = \frac{-2bx}{L}$	$\overline{u} = b\left\{\frac{-2x}{L} - 1.076\left(\frac{x}{L} - \frac{4x^3}{L^3}\right)\right\}$
Modal strain	$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{\pi}{L}a\cos\left(\frac{\pi x}{L}\right)$	$\overline{\varepsilon} = \frac{\mathrm{d}\overline{u}}{\mathrm{d}x} = \frac{-2b}{L}$	$\overline{\varepsilon} = \frac{\mathrm{d}\overline{u}}{\mathrm{d}x} = b\left\{\frac{-2}{L} - 1.076\left(\frac{1}{L} - \frac{12x^2}{L^3}\right)\right\}$
(Strain energy) × 2	$\|\varepsilon\|^2 = 4.9348\frac{AE}{L}a^2$	$\|\overline{\varepsilon}\|^2 = 4\frac{AE}{L}b^2$	$\|\overline{\varepsilon}\|^2 = 4.926\frac{AE}{L}b^2$
Inertia norm	$\|u\|^2 = \frac{4.9348}{\pi^2}(\rho AL)a^2$	$\|\overline{u}\|^2 = \frac{1}{3}(\rho AL)b^2$	$\|u\|^2 = \frac{4.926}{9.875}(\rho AL)b^2$
Eigenvalue	$\omega^2 = \pi^2\frac{E}{\rho L^2}$	$\overline{\omega}^2 = 12\frac{E}{\rho L^2}$	$\overline{\omega}^2 = 9.875\frac{E}{\rho L^2}$
(Kinetic energy) × 2	$\omega^2\|u\|^2 = 4.9348\frac{AE}{L}a^2$	$\overline{\omega}^2\|\overline{u}\|^2 = 4.0\frac{AE}{L}b^2$	$\overline{\omega}^2\|\overline{u}\|^2 = 4.926\frac{AE}{L}b^2$
(Error in strain energy)×2 $\|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2$		$\frac{AE}{L}(4.9348a^2 - 4.0b^2)$	$\frac{AE}{L}(4.9348a^2 - 4.926b^2)$
(Error in kinetic energy)×2 $\omega^2\|u\|^2 - \overline{\omega}^2\|\overline{u}\|^2$		$\frac{AE}{L}(4.9348a^2 - 4.0b^2)$	$\frac{AE}{L}(4.9348a^2 - 4.926b^2)$
$\langle\overline{\varepsilon}, \varepsilon - \overline{\varepsilon}\rangle$		$4\frac{AE}{L}(ab - b^2)$	$4.926\frac{AE}{L}(ab - b^2)$
$(\overline{u}, \omega^2 u - \overline{\omega}^2\overline{u})$		$4\frac{AE}{L}(ab - b^2)$	$4.926\frac{AE}{L}(ab - b^2)$
Elastodynamic projection theorem $\langle\overline{\varepsilon}, \varepsilon - \overline{\varepsilon}\rangle - (\overline{u}, \omega^2 u - \overline{\omega}^2\overline{u})$ (Eq. (7))		0	0
Elastodynamic energy-error rule $[\|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2] - [\omega^2\|u\|^2 - \overline{\omega}^2\|\overline{u}\|^2]$ (Eq. (8))		0	0

6.3. Analysis of a simply supported beam

Consider a simply supported Euler beam as shown in Fig. 5. The exact expressions for the modal transverse displacement w and natural circular frequency ω for the fundamental transverse mode are given by

$$w = a\sin\left(\frac{\pi x}{L}\right), \quad \omega = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}},$$

where L is the length of beam, A and I are the sectional area and sectional moment of inertia of the beam, respectively. Here the coordinate x is measured with left end of the beam as origin. The same problem has been analyzed using Rayleigh quotient with a quadratic displacement function representing approximately the fundamental transverse mode. The approximate displacement functions should satisfy geometric boundary condition for the fundamental

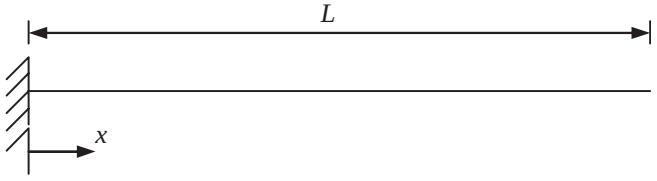

Fig. 4. Fixed–free bar.

Table 2
Analysis results of the fixed–free bar using the exact and the approximate methods for the fundamental mode

	Exact solutions	Approximate solutions
Modal displacement function	$u = a\sin\left(\frac{\pi x}{2L}\right)$	$\overline{u} = \frac{bx}{L}$
Modal strain	$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\pi}{2L}\, a\cos\left(\frac{\pi x}{2L}\right)$	$\overline{\varepsilon} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{b}{L}$
(Strain energy) × 2	$\|\varepsilon\|^2 = \frac{\pi^2}{8}\frac{AE}{L}a^2$	$\|\overline{\varepsilon}\|^2 = \frac{AE}{L}b^2$
Inertia norm	$\|u\|^2 = \frac{1}{2}(\rho AL)a^2$	$\|\overline{u}\|^2 = \frac{1}{3}(\rho AL)b^2$
Eigenvalue	$\omega^2 = \frac{\pi^2}{4}\frac{E}{\rho L^2}$	$\overline{\omega}^2 = 3\frac{E}{\rho L^2}$
(Kinetic energy) × 2	$\omega^2\|u\|^2 = \frac{\pi^2}{8}\frac{AE}{L}a^2$	$\overline{\omega}^2\|\overline{u}\|^2 = \frac{AE}{L}b^2$
(Error in strain energy) × 2 $\|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2$		$\frac{AE}{L}\left\{\left(\frac{\pi^2}{8}\right)a^2 - b^2\right\}$
(Error in kinetic energy) × 2 $\omega^2\|u\|^2 - \overline{\omega}^2\|\overline{u}\|^2$		$\frac{AE}{L}\left\{\left(\frac{\pi^2}{8}\right)a^2 - b^2\right\}$
$\langle\overline{\varepsilon}, \varepsilon - \overline{\varepsilon}\rangle$		$\frac{AE}{L}(ab - b^2)$
$(\overline{u}, \omega^2 u - \overline{\omega}^2\overline{u})$		$\frac{AE}{L}(ab - b^2)$
Elastodynamic projection theorem $\langle\overline{\varepsilon}, \varepsilon - \overline{\varepsilon}\rangle - (\overline{u}, \omega^2 u - \overline{\omega}^2\overline{u})$ (Eq. (7))		0
Elastodynamic energy-error rule $[\|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2] - [\omega^2\|u\|^2 - \overline{\omega}^2\|\overline{u}\|^2]$ (Eq. (8))		0

transverse mode,

$$\overline{w}(x = 0) = 0 \quad \text{and} \quad \overline{w}(x = L) = 0.$$

Let us choose the approximate quadratic displacement function representing the fundamental transverse mode as $\overline{w} = b(x/L)(1 - x/L)$, which satisfies the geometric boundary conditions. It is observed that the analysis results satisfy the elastodynamic error rules presented in Eqs. (7) and (8). The results are tabulated in Table 3.

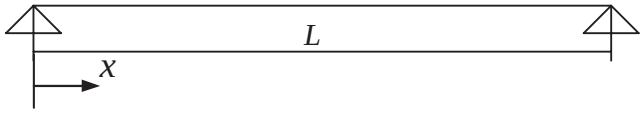

Fig. 5. Simply supported beam.

Table 3
Analysis results of the simply supported Euler beam using the exact and the approximate methods for the fundamental mode

	Exact solutions	Approximate solution
Modal displacement function	$w = a\sin\left(\frac{\pi x}{L}\right)$	$\overline{w} = b\left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right)$
Modal strain	$\varepsilon = \left(-\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) = a\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right)$	$\overline{\varepsilon} = \left(-\frac{\mathrm{d}^2 \overline{w}}{\mathrm{d}x^2}\right) = \frac{2b}{L^2}$
(Strain energy) × 2	$\|\varepsilon\|^2 = \left(\frac{\pi^4}{2}\right)\frac{EI}{L^3}a^2$	$\|\overline{\varepsilon}\|^2 = \frac{4EI}{L^3}b^2$
Inertia norm	$\|u\|^2 = \frac{1}{2}(\rho AL)a^2$	$\|\overline{u}\|^2 = \frac{1}{30}(\rho AL)b^2$
Eigenvalue	$\omega^2 = \pi^4\frac{EI}{\rho AL^4}$	$\overline{\omega}^2 = 120\frac{EI}{\rho AL^4}$
(Kinetic energy) × 2	$\omega^2\|u\|^2 = \left(\frac{\pi^4}{2}\right)\frac{EI}{L^3}a^2$	$\overline{\omega}^2\|\overline{u}\|^2 = \frac{4EI}{L^3}b^2$
(Error in strain energy) × 2 $\|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2$		$\frac{EI}{L^3}\left\{\left(\frac{\pi^4}{2}\right)a^2 - 4b^2\right\}$
(Error in kinetic energy) × 2 $\omega^2\|u\|^2 - \overline{\omega}^2\|\overline{u}\|^2$		$\frac{EI}{L^3}\left\{\left(\frac{\pi^4}{2}\right)a^2 - 4b^2\right\}$
$\langle\overline{\varepsilon}, \varepsilon - \overline{\varepsilon}\rangle$		$4\frac{EI}{L^3}(\pi ab - b^2)$
$(\overline{u}, \omega^2 u - \overline{\omega}^2\overline{u})$		$4\frac{EI}{L^3}(\pi ab - b^2)$
Elasodynamic projection theorem $\langle\overline{\varepsilon}, \varepsilon - \overline{\varepsilon}\rangle - (\overline{u}, \omega^2 u - \overline{\omega}^2\overline{u})$ (Eq. (7))		0
Elastodynamic energy-error rule $[\|\varepsilon\|^2 - \|\overline{\varepsilon}\|^2] - [\omega^2\|u\|^2 - \overline{\omega}^2\|\overline{u}\|^2]$ (Eq. (8))		0

7. Consistent mass versus lumped mass in finite element analysis for elastodynamic problems

For computational efficiency, engineers often employ the lumped mass technique, which effectively replaces the consistent (non-diagonal) mass matrices by lumped (diagonal) mass matrices. The present section examines the variational correctness of finite element results for elastodynamic analysis using these different methodologies.

The weak form of the elastodynamic differential equation yields a mass matrix $[M^{ce}]$ from the inner product with the approximate modal function in the following manner:

$$\begin{aligned}(\overline{u},\overline{u}) &= \sum_e \int_e \{\overline{u}\}^{\mathrm{T}}\rho\{\overline{u}\}\,\mathrm{d}V \\ &= \sum_e \{\delta^e\}^{\mathrm{T}} \int_e [N]^{\mathrm{T}}\rho[N]\,\mathrm{d}V\{\delta^e\} \\ &= \sum_e \{\delta^e\}^{\mathrm{T}}[M]^{ce}\{\delta^e\},\end{aligned} \tag{24}$$

where $[N]$ is the shape function matrix for the approximate modal displacement function in an element e and $\{\overline{u}\} = [N]\{\delta^e\}$, where $\{\delta^e\}$ is the nodal displacement vector for the element. The *consistent* mass matrix is given by

$$[M^{ce}] = \int_e [N]^{\mathrm{T}}\rho[N]\,\mathrm{d}V. \tag{25}$$

If the mass matrix is computed according to Eq. (24), then the elastodynamic error Eqs. (12) and (13a,b) are satisfied, since it is consistent with the weak form of the elastodynamic differential equation.

However, for computational convenience, engineers often use the *lumped* mass matrix instead of the consistent one. In a lumped mass matrix, all the off-diagonal elements are set equal to zero, and the masses are lumped only in the diagonal elements of the matrix. With the lumped mass matrix, the inner product of Eq. (23) is replaced by the expression

$$\sum_e \{\delta^{e*}\}^{\mathrm{T}}[M^{le}]\{\delta^{e*}\} = \sum_e \sum_i m_i^e(\delta_i^{e*})^2, \tag{26}$$

where m_i^e is the mass associated for the ith diagonal term for the lumped mass matrix $[M^{le}]$ for the element e. The term δ_i^{e*} denotes the ith displacement component of the displacement vector $\{\delta_i^{e*}\}$ for the lumped mass case. Using the lumped mass formulation effectively replaces Eq. (4) by

$$(\overline{\varepsilon}^*,\overline{\varepsilon}^*) = (\overline{\omega}^*)^2.\sum_e \sum_i m_i^e(\delta_i^{e*})^2, \tag{27}$$

where the approximate modal displacement function for the lumped mass in an element is given by $\{\overline{u}^*\} = [N]\{\delta^{e*}\}$ and the corresponding eigenvalue is $(\overline{\omega}^*)^2$. If $\{\overline{u}\}$ and $(\overline{\omega})^2$ are replaced by $\{\overline{u}^*\}$ and $(\overline{\omega}^*)^2$ then Eqs. (7), (8) and (11) are violated. In other words, *elastodynamic results of finite element analysis with the lumped mass matrices are variationally incorrect*. However, from the principle of conservation of energy, a modified energy error rule, given below is satisfied by the lumped mass formulations.

$$\|\varepsilon\|^2 - \|\overline{\varepsilon}^*\|^2 = \omega^2|u|^2 - (\overline{\omega}^*)^2\sum_e \sum_i m_i^e(\delta_i^{e*})^2. \tag{28}$$

Note that the inner product $|\overline{u}^*|^2$ of Eq. (8) is replaced by Eq. (26).

A variationally correct finite element formulation for elastodynamics with consistent mass matrix *always* yields eigenvalues (natural frequencies) higher than those obtained by analytical methods for arbitrary meshing. This is not necessarily true if lumped mass formulation is

employed. In fact, lumped mass analysis can yield eigenvalues which are either lower than, or higher than, or equal to the exact eigenvalue according to the position of the nodes. This is illustrated with numerical examples in the next section.

8. Numerical experiments with the linear bar element

8.1. Element displacement and stiffness

Consider the linear two noded bar element shown in Fig. 6. The approximate modal displacement $\overline{u}$ is given by linear interpolation function as

$$\overline{u} = N_1 u_1 + N_2 u_2 \quad \text{where } N_1 = \left(1 - \frac{x}{l}\right) \quad \text{and} \quad N_2 = \left(\frac{x}{l}\right). \tag{29}$$

Here l is the element length and x is measured with the left end of the element as origin, and u_i is the nodal displacement vector at node i. The modal strain–displacement relation is given as

$$\{\overline{\varepsilon}\} = \left\{\frac{\mathrm{d}\overline{u}}{\mathrm{d}x}\right\} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [B]\{\delta^e\},$$

where $[B]$ is the modal-strain displacement matrix and $\{\delta^e\} = [u_1 \ u_2]^{\mathrm{T}}$ is the element nodal displacement vector. The element rigidity matrix is $[D] = EA$ where A is the sectional area and E is the Young's modulus of the material. The element stiffness matrix is given by

$$[k^e] = \int_{x=0}^{l} [B]^{\mathrm{T}}[D][B]\,\mathrm{d}x = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{30}$$

8.2. Consistent mass formulation

When the mass matrix is constructed so that it is consistent with the weak form and the displacement approximation, it is referred to as consistent mass matrix. The consistent mass matrix for the linear bar element it is given by Eq. (24) as

$$[M^{ce}] = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{31}$$

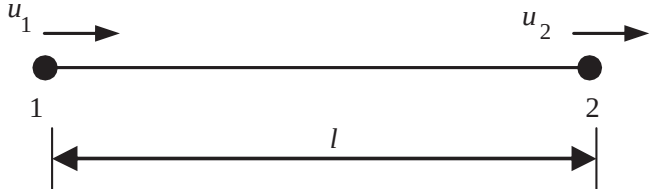

Fig. 6. Two noded linear bar element.

8.3. Lumped mass formulation

The lumped mass matrix is formulated by lumping of mass at the element nodes. This produces a diagonal mass matrix. For the linear bar element the lumped mass matrix is given by

$$[M^{le}] = \frac{\rho A l}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{32}$$

Mass lumping greatly simplifies matrix calculation involving the extraction of eigenvalues.

8.4. Frequency analysis

A fixed–fixed bar has been analyzed with different types of discretisations (with uniform and varying element lengths) using consistent and lumped mass matrices. Figs. 7 and 8 show the discretisation schemes using two and three bar elements.

The results of the analysis are graphically presented in Figs. 9 and 10. It can be observed that the approximate eigenvalues for a given mode using the consistent mass matrix are always higher than the exact eigenvalue of the respective mode for all kinds of meshing. In other words, the exact eigenvalue forms the lower bound of the sequence of approximate eigenvalues using consistent mass formulation. This is a fact which is independent of the discretisation schemes. This confirms what we had projected from the frequency-error-hyperboloid earlier.

However, with lumped mass formulation, the discretisation decides whether the approximate eigenvalues will be lower than or higher than the exact eigenvalue. In fact, for some critical discretisation, the approximate eigenvalue can even be equal to the exact one, as indicated by the intersecting points A and B in Figs. 9 and 10.

Thus when the lumped mass matrix is used, the guarantee of obtaining upper bounds to the exact eigenfrequencies of the structure is lost; in some particular cases, it is possible to show that the eigenfrequencies will be underestimated, but no proof of it can be given in general.

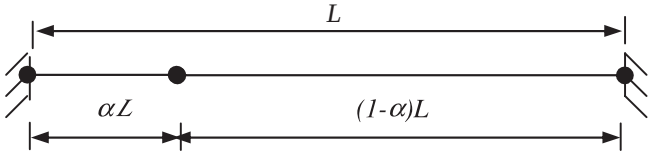

Fig. 7. Element discretisation details (2-element case).

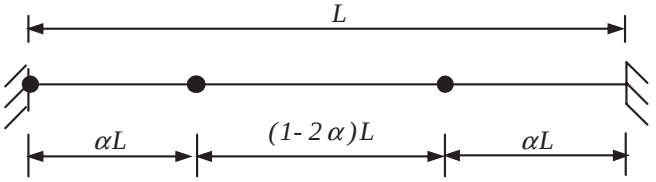

Fig. 8. Element discretisation details (3-element case).

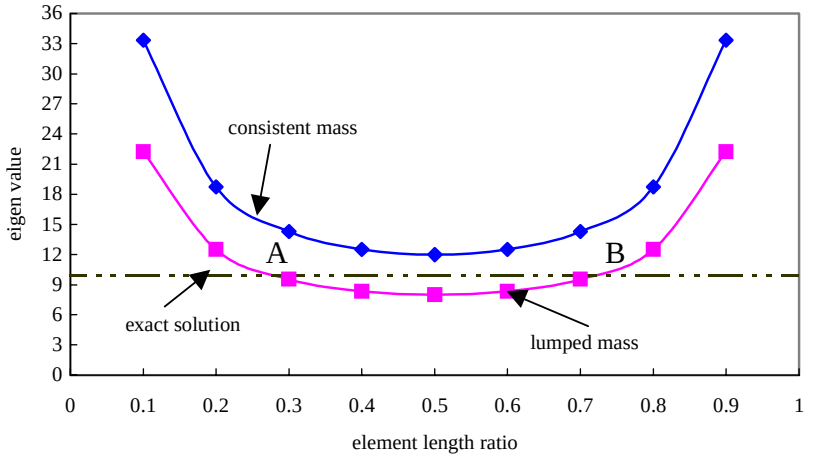

Fig. 9. Variation of eigenvalue with change in element length (2-element case) for the fundamental mode using both consistent and lumped mass.

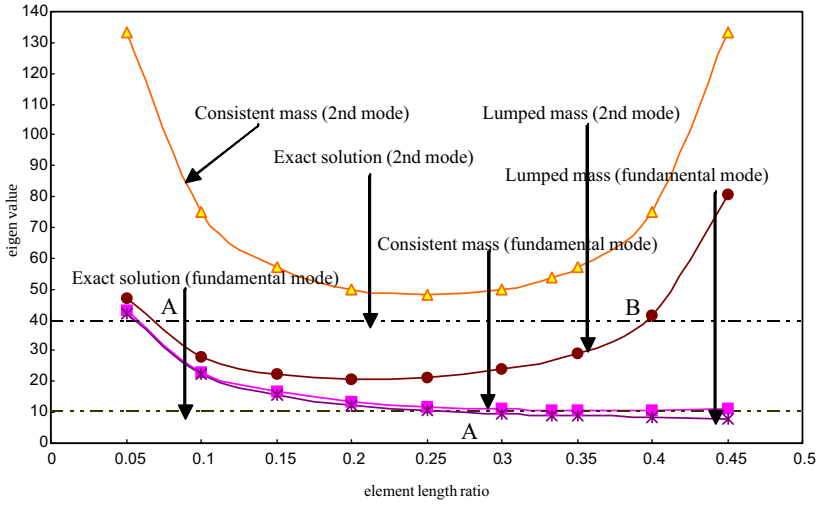

Fig. 10. Variation of eigenvalue with change in element length (3-element case) for the fundamental and 2nd mode using both consistent and lumped mass.

While it is generally true that consistent mass formulations yield eigenfrequencies higher than the exact value, it has been observed that for the present problem, analysis with the lumped mass formulation employing equal length elements, yields approximate eigenfrequencies which are lower than the exact value. The convergence trend of the errors in the eigenvalues with increasing number of equal length elements N is shown in Fig. 11. The error in the eigenvalue is defined as, $e = ((\overline{\omega}^2/\omega^2) - 1)$, where ω^2 is the exact eigenvalue and $\overline{\omega}^2$ is the approximate finite element eigenvalue. For consistent mass analysis the error e always turns out to be a small positive number but for lumped mass analysis it can be positive, negative or zero. Therefore, for the purpose of

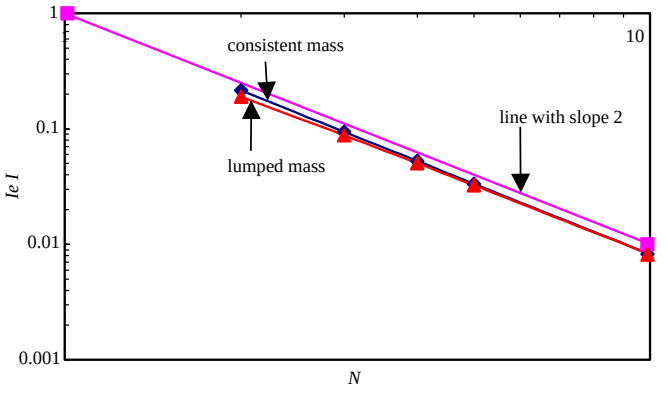

Fig. 11. Convergence of eigenvalue with increase in number of equal length elements.

obtaining the convergence graph we used only the modulus of the error e. It is seen that, as discussed in Section 4, the error has an order of convergence of $O(h^2)$, where $h = L/N$ is the element length. Here L is the total length of bar and N is the number of elements.

We see from Fig. 11 that the errors for an equal length element mesh from the consistent and lumped mass formulations are nearly exactly equal in magnitude but opposite in sign. One can envisage a physical picture where the lumped mass case is equivalent to a formulation where the discretisation of the mass (or inertia) properties leads to a heavier configuration than the consistent mass case. Thus while both cases show a second-order convergence rate, it is possible to manipulate the mass matrix such that the errors of the order of h^2 canceled out, giving a fourth-order (i.e. $O(h^4)$) accuracy. This is in fact the basis of higher-order mass of Goudreau [18], where the new mass matrix is obtained as the average of the lumped and consistent mass matrices, i.e. $[M^{he}] = \frac{1}{2}\{[M^{ce}] + [M^{le}]\}$.

8.5. Illustration of the energy-error rule and projection theorem

It is interesting to examine whether the lumped mass formulations conserve the relationships described by the elastodynamic energy-error rules and projection theorems. Table 4 tabulates these quantities for the equal length two and three element solutions for consistent mass and Table 5 repeats these for lumped mass. It is very clearly seen that due to the extra variational nature of the lumped mass formulation, the mass became "heavier" (compare the third rows (inertia norm) in Tables 4 and 5). Even then, from the principle of conservation of energy the lumped mass formulations satisfy the modified elastodynamic energy-error rule (Eq. (27)). But it will be meaningless to talk of a projection theorem for the variationally incorrect lumped mass formulations because, the inertia-inner product is now mixed with continuous (exact solution) and discrete (lumped mass solution) functions. These arguments are confirmed with the algebraic expressions presented in Tables 4 and 5.

The parameters a and b present in the expressions of Tables 4 and 5 are the amplitudes of the exact and approximate finite element solutions, respectively.

Table 4
Analysis results for the fixed–fixed bar with exact solution and with two and three equal length bar elements using the consistent mass formulation for the fundamental mode

	Exact solution	Approximate finite element solutions	
		2-element solution	3-element solution
(Strain energy)×2	$\|\varepsilon\|^2 = 4.9348\,\frac{AE}{L}\,a^2$	$\|\bar{\varepsilon}\|^2 = 4\,\frac{AE}{L}\,b^2$	$\|\bar{\varepsilon}\|^2 = 4.5\,\frac{AE}{L}\,b^2$
Inertia norm	$\|u\|^2 = \frac{4.9348}{\pi^2}(\rho AL)a^2$	$\|\bar{u}\|^2 = \frac{1}{3}(\rho AL)b^2$	$\|\bar{u}\|^2 = \frac{4.5}{10.8}(\rho AL)b^2$
Eigenvalue	$\omega^2 = \pi^2\,\frac{E}{\rho L^2}$	$\bar{\omega}^2 = 12\,\frac{E}{\rho L^2}$	$\bar{\omega}^2 = 10.8\,\frac{E}{\rho L^2}$
(Kinetic energy)×2	$\omega^2\|u\|^2 = 4.9348\,\frac{AE}{L}\,a^2$	$\bar{\omega}^2\|\bar{u}\|^2 = 4\,\frac{AE}{L}\,b^2$	$\bar{\omega}^2\|\bar{u}\|^2 = 4.5\,\frac{AE}{L}\,b^2$
(Error in strain energy)×2 $\|\varepsilon\|^2 - \|\bar{\varepsilon}\|^2$		$\frac{AE}{L}\,(4.9348a^2 - 4b^2)$	$\frac{AE}{L}\,(4.9348a^2 - 4.5b^2)$
(Error in kinetic energy)×2 $\omega^2\|u\|^2 - \bar{\omega}^2\|\bar{u}\|^2$		$\frac{AE}{L}\,(4.9348a^2 - 4b^2)$	$\frac{AE}{L}\,(4.9348a^2 - 4.5b^2)$
$\langle\bar{\varepsilon}, \varepsilon - \bar{\varepsilon}\rangle$		$4\,\frac{AE}{L}\,(ab - b^2)$	$4.5\,\frac{AE}{L}(ab - b^2)$
$(\bar{u}, \omega^2 u - \bar{\omega}^2\bar{u})$		$4\,\frac{AE}{L}(ab - b^2)$	$4.5\,\frac{AE}{L}\,(ab - b^2)$
Elastodynamic projection theorem $\langle\bar{\varepsilon}, \varepsilon - \bar{\varepsilon}\rangle - (\bar{u}, \omega^2 u - \bar{\omega}^2\bar{u})$ (Eq. (7))		0	0
Elastodynamic energy-error rule $[\|\varepsilon\|^2 - \|\bar{\varepsilon}\|^2] - [\omega^2\|u\|^2 - \bar{\omega}^2\|\bar{u}\|^2]$ (Eq. (8))		0	0

9. Conclusion

An energy-error rule and the projection theorems have been derived using the variational (weak form) approach of Strang and Fix [5] for the elastodynamic problems. It has been demonstrated with numerical examples that a variationally correct formulation (consistent mass) always satisfies these theorems but any extra-variational formulation (lumped mass) violates these. The effect of replacing consistent mass by lumped mass has been critically examined.

Using the frequency-error hyperboloid, an attempt has been made to present a geometric interpretation for the errors associated in the computations of eigenfrequencies of structural mechanics problems with variationally correct formulations. For variationally incorrect formulations, such a rule is violated. Furthermore, the geometric significance of the upper bound of the Rayleigh quotient for variationally correct formulations has been shown in terms of included angles between exact and approximate modal solutions. The guarantee of upper bound is lost when a lumped mass is used; this fact has been illustrated with numerical examples using the two noded linear bar element.

Table 5
Analysis results for the fixed–fixed bar with exact solution and with two and three equal length bar elements using the lumped mass formulation for the fundamental mode

	Exact solution	Approximate finite element solutions	
		2-element solution	3-element solution
(Strain energy)×2	$\|\varepsilon\|^2 = 4.9348\,\frac{AE}{L}\,a^2$	$\|\bar{\varepsilon}^*\|^2 = 4\,\frac{AE}{L}\,b^2$	$\|\bar{\varepsilon}^*\|^2 = 4.5\,\frac{AE}{L}\,b^2$
Inertia norm	$\|u\|^2 = \frac{4.9348}{\pi^2}(\rho AL)a^2$	$\sum_e \sum_i m_i^e(\delta_e^{i*})^2 = \frac{1}{2}(\rho AL)b^2$	$\sum_e \sum_i m_i^e(\delta_e^{i*})^2 = \frac{1}{2}(\rho AL)b^2$
Inertia norm as per Eq. (2a)		$\|\bar{u}\|^2 = \frac{1}{3}(\rho AL)b^2$	$\|\bar{u}\|^2 = \frac{5}{12}(\rho AL)b^2$
Eigenvalue	$\omega^2 = \pi^2\frac{E}{\rho L^2}$	$(\bar{\omega}^*)^2 = 8\,\frac{E}{\rho L^2}$	$(\bar{\omega}^*)^2 = 9\,\frac{E}{\rho L^2}$
(Kinetic energy)×2	$\omega^2\|u\|^2 = 4.9348\,\frac{AE}{L}\,a^2$	$(\bar{\omega}^*)^2 \sum_e \sum_i m_i^e(\delta_i^{e*})^2 = 4\,\frac{AE}{L}\,b^2$	$(\bar{\omega}^*)^2 \sum_e \sum_i m_i^e(\delta_i^{e*})^2 = 4.5\,\frac{AE}{L}\,b^2$
(Error in strain energy)×2 $\|\varepsilon\|^2 - \|\bar{\varepsilon}^*\|^2$		$\frac{AE}{L}(4.9348a^2 - 4.0b^2)$	$\frac{AE}{L}(4.9348a^2 - 4.5b^2)$
(Error in kinetic energy)×2 $\omega^2\|u\|^2 - (\bar{\omega}^*)^2 \sum_e \sum_i m_i^e(\delta_i^{e*})^2$		$\frac{AE}{L}(4.9348a^2 - 4.0b^2)$	$\frac{AE}{L}(4.9348a^2 - 4.5b^2)$
$\langle \bar{\varepsilon}^*, \varepsilon - \bar{\varepsilon}^* \rangle$		$4\,\frac{AE}{L}\,(ab - b^2)$	$4.5\,\frac{AE}{L}\,(ab - b^2)$
$\|\varepsilon\|^2 - \|\bar{\varepsilon}^*\|^2 - [\omega^2\|u\|^2 - (\bar{\omega}^*)^2 \sum_e \sum_i m_i^e(\delta_i^{e*})^2]$ (Eq. (28))		0	0

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