Available online at www.sciencedirect.com
SCIENCE DIRECT

ELSEVIER

Journal of Sound and Vibration 285 (2005) 759–765

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Short Communication

Vibration of a rotating shaft with randomly varying internal damping

M.F. Dimentberg*

Mechanical Engineering Department, Worcester Polytechnic Institute, 100 Institute Road, Worcester, MA 01609, USA

Received 14 October 2004; accepted 4 November 2004
Available online 20 January 2005

Abstract

A simple Jeffcott rotor is considered with both external and internal damping. Coefficient of internal damping is subject to temporal random variations which may occasionally bring the rotor into the domain of dynamic instability. The corresponding sporadic outbreaks in the rotor's vibrational response (whirl) are studied by applying the Krylov–Bogoliubov averaging method to the complex equation of motion and using parabolic approximation for the random coefficient of the internal damping. This results in an explicit analytical solution for the radius of whirl which may be used for predicting reliability of the rotor. Furthermore, a convenient procedure is described for interpreting measured on-line test data for the rotor. Namely, the mean value of the coefficient of internal damping as well as its standard deviation and mean frequency of temporal variations may be estimated directly from the trace of whirl radius which exhibits spontaneous random outbreaks in response.

1. Introduction

Internal or "rotating" damping is a well-known source of potential dynamic instability of shafts operating at supercritical speeds [1]. This kind of destabilizing damping may be present due to energy dissipation in the shaft's material or rubbing between rotating components. Similar effect in some cases may also be the result of fluid flow in labyrinth seals, journal bearings, etc., with a

*Tel.: +1 508 831 5368; fax: +1 508 831 5680.
E-mail address: diment@wpi.edu (M.F. Dimentberg).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2004.11.025

model of internal damping providing at least qualitatively an adequate description for flow-induced dynamic instabilities [2]. In such cases, certain random temporal variations in the flow conditions may be expected sometimes. The shaft may then be occasionally brought into the instability domain for brief periods of time even if it is nominally stable, i.e. if the mean or expected value of the internal damping coefficient corresponds to a stable rotation of the shaft. Sporadic outbreaks of the shaft's vibration would then be observed due to these short-term excursions of the internal damping coefficient into the instability domain, with a small level of response between the relatively rare outbreaks. This kind of response may be observed if the shaft operates close to the instability threshold. The name "intermittency" may be used for this behaviour by analogy with fluid mechanics where it is used for the transitional state between laminar and turbulent flows. For vibrating systems with random variations of parameters this name may be applied to the transitional state between responses within domains of stability and instability [3]. As long as the system's reliability is not hundred per cent guaranteed in case of operation within such a transitional state the response analysis may be of importance both for predicting reliability at the design stage of a shaft system and for interpreting results of its tests where intermittent behaviour is observed.

This kind of analysis has been made in Ref. [3] for a single dof system with a randomly varying damping coefficient. A similar analysis is presented in this paper for a simple single-mass/two dof Jeffcott rotor which is prone to dynamic instability of the forward-whirl type. The analysis is based on the Krylov–Bogoliubov (KB) averaging method [4] and on an asymptotic parabolic approximation for the randomly varying internal damping coefficient within the instability domain [5,6]. It results in an explicit relation between peak values of a radius of the shaft's whirl and of the damping coefficient during its brief excursion into the instability domain. In this way the reliability analysis for the system as based on the solution to the first-passage problem for the response or on evaluating probability density function (p.d.f.) of response peaks is reduced to the corresponding problems for the internal damping coefficient. Furthermore, the analytical solution is used also to derive a simple identification procedure for the system from its measured (on-line) response. Namely, the mean value of the damping coefficient can be estimated as well as its standard deviation and mean frequency of its temporal variations.

2. Analysis of transient response

Consider a simple Jeffcott rotor with weightless shaft of a stiffness K rotating with angular velocity v. The horizontal shaft carries a disk of mass m at its midspan and possesses both external or "nonrotating" damping and internal or "rotating" damping with corresponding damping coefficients c_n and c_r, respectively. The latter of these coefficients experience temporal variations which are assumed to be slow compared with the shaft's natural frequency or critical speed. Let $x(t)$ and $y(t)$ be lateral horizontal and vertical displacements, respectively, of the disk's centre in the inertial frame with origin at the undeformed shaft's axis. Then, neglecting gravity force for sufficiently high rotation speeds and introducing complex displacement $z = x + \mathrm{i}y$, $\mathrm{i} = \sqrt{-1}$ one can write the following single complex equation of motion [1,2]:

$$\ddot{z} + 2(\alpha + \beta(t))\dot{z} + \Omega^2 z - 2\mathrm{i}\beta(t)vz = 0, \tag{1}$$

where $\Omega^2 = K/m$, $\alpha = c_n/2m$, $\beta = c_r/2m$. Let $\beta(t)$ be a stationary random process with mean value β_0 so that $\beta(t) = \beta_0 + q(t)$, where $q(t)$ is its zero-mean part with power spectral density (PSD) $\Phi_{qq}(\omega)$. We also denote by σ_q and λ standard deviation and mean frequency, respectively, of $q(t)$, where

$$\sigma_q^2 = \int_{-\infty}^{\infty} \Phi_{qq}(\omega)\,\mathrm{d}\omega \quad \text{and} \quad \lambda^2 = \sigma_q^{-2} \int_{-\infty}^{\infty} \omega^2 \Phi_{qq}(\omega)\,\mathrm{d}\omega. \tag{2}$$

The complex Eq. (1) can be solved analytically for the transient response of a lightly damped shaft by the asymptotic Krylov–Bogoliubov method of averaging over the response period [4]. Thus, assume that both damping ratios are small, namely $\alpha/\Omega \ll 1$, $\beta/\Omega \ll 1$ and also $\lambda \ll \Omega$, the last condition being that of slow temporal variations in the internal damping. The approximate solution may be represented then in the same form as that for the corresponding undamped system, namely

$$z = A_+ \exp(\mathrm{i}\Omega t) + A_- \exp(-\mathrm{i}\Omega t), \quad \dot{z} = \mathrm{i}\Omega[A_+ \exp(\mathrm{i}\Omega t) - A_- \exp(-\mathrm{i}\Omega t)]. \tag{3}$$

Here, A_+ and A_- are amplitudes (complex in general) of the forward and backward whirl, respectively. In the present case of transient response of a damped system they are functions of time which are found to be slowly varying under the above assumptions. Therefore, the KB-averaging over the response period can be applied after the basic equation is reduced to a form with small parameter at the RHS. Resolving the equations (3) for A_+ and A_- and differentiating provides this reduction, so that

$$\begin{aligned} \dot{A}_+ &= (1/2)\frac{\mathrm{d}}{\mathrm{d}t}[(z + \dot{z}/\mathrm{i}\Omega)\exp(-\mathrm{i}\Omega t)] \\ &= (1/2\mathrm{i}\Omega)(\ddot{z} + \Omega^2 z)\exp(-\mathrm{i}\Omega t) \\ &= (1/2\mathrm{i}\Omega)[-2(\alpha+\beta)\dot{z} + 2\mathrm{i}v\beta z]\exp(-\mathrm{i}\Omega t) \\ &= -(\alpha+\beta)(A_+ \exp(\mathrm{i}\Omega t) - A_- \exp(-\mathrm{i}\Omega t))\exp(-\mathrm{i}\Omega t) \\ &\quad + (\beta v/\Omega)(A_+ \exp(\mathrm{i}\Omega t) + A_- \exp(-\mathrm{i}\Omega t))\exp(-\mathrm{i}\Omega t) \\ &\cong (-\alpha - \beta + \beta v/\Omega)A_+ \end{aligned} \tag{4}$$

and similarly,

$$\dot{A}_- \cong -(\alpha + \beta + \beta v/\Omega)A_-. \tag{5}$$

The last, approximate equality in Eq. (4) is actually obtained by applying the KB-averaging [4], which resulted just in neglecting terms with the complex exponent $\exp(-2\mathrm{i}\Omega t)$; function $\beta(t)$ which is assumed to be slowly varying compared with $\exp(\mathrm{i}\Omega t)$; $\exp(-\mathrm{i}\Omega t)$ is regarded as constant in the "fast" time as long as its variations within response period $2\pi/\Omega$ should be small. Thus, the KB-averaging resulted in uncoupling equations for the forward and backward whirl.

It can be seen from Eq. (4) that in case of constant β the shaft is stable if $v/\Omega < 1 + \alpha/\beta$ and unstable otherwise; the instability is possible only in case of supercritical operation speed, that is $v/\Omega > 1$, whereas the mode of instability is clearly seen to be the forward whirl. Whilst these results are clearly the same as those obtained from exact solution to Eq. (1) with $\beta = \text{const}$ [1], reduction to the single first-order ODE (1) provides also the possibility for analytical study of transient vibrations. As for the backward whirl, it always decays as can be seen from Eq. (5). Therefore $A_-(t) \equiv 0$ as long as $A_-(0) = 0$. This initial condition will be adopted here.

The following analysis proceeds along the same lines as that for a single dof system with random temporal variations of damping [3]. Denoting

$$\gamma = \alpha/(v/\Omega - 1) - \beta_0, \quad \text{where } \gamma > 0,\ g(t) = q(t)/\sigma_q,\ \tau = \lambda t,\ u = \gamma/\sigma_q, \tag{6}$$

Eq. (4) is reduced to

$$\mathrm{d}A_+/\mathrm{d}\tau = (v/\Omega - 1)(\sigma_q/\lambda)[-u + g(\tau)]A_+. \tag{7}$$

Assumption $\gamma > 0$ implies that the "nominal" shaft, i.e. one without variations in the internal damping, is stable. However, it becomes "temporary unstable" whenever the (slow) random process $g(\tau)$ crosses level u with positive derivative and remains temporarily above this level. To predict the response amplitude during its corresponding outbreak one can use the following asymptotic parabolic approximation for $g(\tau)$ after this upcrossing [5] (also called Slepian model [6]):

$$\begin{aligned} &g_u(t/u) \cong u + (1/u)(\varsigma t - \lambda^2 t^2/2), \text{ so that} \\ &g_u(t) \cong u + \varsigma t - (u/2)(\lambda t)^2 \text{ and } \max_t g_u(t) = g_u(\varsigma/\lambda^2 u) = g_p = u + \varsigma^2/2\lambda^2 u. \end{aligned} \tag{8}$$

Here, subscript "u" for $g(t)$ implies that the process is considered immediately after upcrossing the level u; this subscript shall be dropped in the following whereas subscript "p" will be used for peak values of $g(t)$ and of the response variables. A local origin of the nondimensional time $\tau = 0$ is introduced here at the instant of upcrossing whereas ς is a random slope of $g(\tau)$ at this instant; it is a random variable with standard deviation λ and in case of a Gaussian $g(t)$ it has the Raleigh p.d.f. [5,6]. Subscript "p" for g denotes peak value of this process within domain $g > u$. Substituting now expression (8) into the ODE (7) and integrating yields

$$A_+(\tau) = A_+(0)\exp\{\kappa[(\varsigma/\lambda)(\tau^2/2) - u\tau^3/6]\}, \quad \text{where } \kappa = (v/\Omega - 1)(\sigma_q/\lambda). \tag{9}$$

Whilst the resulting amplitude $A_+(\tau)$ will be complex as long as its initial value at $\tau = 0$ is, a similar expression can be obtained for an essentially real quantity with a clear physical meaning—radius of whirl $r(t)$. Thus relation (3) with $A_-(t) \cong 0$ and solution (9) yield

$$r(\tau) = \sqrt{x^2 + y^2} = \sqrt{zz^*} = \sqrt{A_+(0)A_+^*(0)} = r(0)\exp\{\kappa[(\varsigma/\lambda)(\tau^2/2) - u\tau^3/6]\}. \tag{10}$$

Here, star superscript denotes complex conjugate quantity whereas $r(0) = \sqrt{A_+(0)A_+^*(0)}$ is the initial radius of whirl which should not be zero to obtain a nonzero response; its value for design analyses should be estimated by evaluating the response level during operation within the stability domain. For example, the vibration of a stable shaft may be excited by its unbalance due to offset c of the disk's mass centre from the geometrical centre; in this case, the initial radius of whirl may be estimated as $r(0) = c/[1 - (\Omega/v)^2]$ if $v - \Omega \gg \alpha$, that is, in case of steady operation far beyond the critical rotation speed [1]. It should be added also that in real-life applications the initial conditions would not necessarily be the same as adopted in this paper; the actual reason for absence of a backward whirl within the instability domain is high magnification of the forward whirl during excursions into this domain which makes this mode of response dominant over the whole response background of the stable shaft.

According to the solution (10), the peak value of $r(t)$ is attained at the final instant $\tau_f = 2\varsigma/\lambda u$ of the excursion into the instability domain—when $g(t)$ becomes equal to u for the second time—and this peak value is found to be

$$r_p = r(\tau_f) = r(0)\exp[(2\kappa/3u^2)(\varsigma/\lambda)^3]. \tag{11}$$

Thus, the peak value of the whirl radius attained due to short-term excursion into the domain of dynamic instability has been obtained in terms of a nondimensional slope ς/λ of internal damping coefficient variation at the instant of the excursion. The latter may be excluded using the expression (8), thereby providing a direct relation between $\bar{r}_p = r_p/r(0)$ and g_p—that is between peak values of radii ratio and of $g(t)$. Denote this relation as $\bar{r}_p = h(g_p)$ for $g_p \geqslant u$ and its inverse by h^{-1}, then

$$g_p = u + (\varsigma/\lambda)^2(1/2u) = h^{-1}(\bar{r}_p) = u + (\sqrt[3]{u}/2)[(3/2\kappa)\ln \bar{r}_p]^{2/3}. \tag{12}$$

These relations open the way to reliability predictions for the shaft system based on relevant statistics of $g(t)$. Thus, the first-passage problem for $r(t)$ with barrier $r*$ is reduced to that for $g(t)$ with barrier $g* = h^{-1}(\bar{r}*)$ as evaluated by relation (12). Furthermore, p.d.f. of $g(t)$ can be used to obtain p.d.f. of $\bar{r}_p$; the latter may be of importance for evaluating low-cycle fatigue life for a system subject to sporadic short-term dynamic instability. To this end the p.d.f. $p_g(g_p)$ of *peaks* of $g(t)$ is obtained first from that of the $g(t)$ itself as described in Refs. [5,6]; then the basic relation for p.d.f. of a function of a random variable is applied [5]:

$$p(\bar{r}_p) = p_g(h^{-1}(\bar{r}_p))|\mathrm{d}h^{-1}/\mathrm{d}\bar{r}_p|. \tag{13}$$

It should be just kept in mind that the p.d.f. (13) is nonzero for $\bar{r}_p \geqslant 1$ rather than for $\bar{r}_p \geqslant 0$ as long as the radius of whirl is not zero in a stable state where $g(t) < u$. It goes without saying that the *unconditional* p.d.f. $p(\bar{r}_p)$ is normalized not to unity but to $\text{Prob}(g(t) > u)$, that is to total probability for dynamic instability.

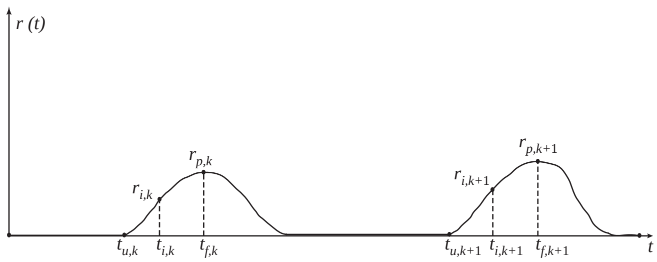

Fig. 1. Typical sketch of shaft vibration level illustrating intermittency due to short-term dynamic instability—radius of whirl $r(t)$. Time instants $t_f = t_u + 2\varsigma/\lambda^2$ and $t_i = t_u + \varsigma/\lambda^2$ are identified for any k from the corresponding radii $r_p = r(t_f)$ and $r_i = r(t_i)$ which are peak value of $r(t)$ and value of $r(t)$ at the inflexion point of curve $\ln r(t)$, respectively. Here, t_u is starting at the instant of the response outbreak, that is, instant of upcrossing level u by $g(t)$; it can be identified in this figure as $t_u = t_f - 2(t_f - t_i) = 2t_i - t_f$.

3. Identification of damping from the observed response

The explicit analytical solution (10) is particularly convenient for evaluating properties of the shaft system from its measured on-line response which exhibits sporadic nonoverlapping outbreaks in whirl radius as illustrated in Fig. 1. To this end one can use peak radius r_p which is attained at the instant $\tau_f = 2\varsigma/\lambda u$ for each outbreak, together with the corresponding radius r_i at the inflexion point of the curve ln $r(\tau)$. As can be seen from the solution (10), this inflexion point corresponds to the peak of $g(t)$ which is attained at local time instant $\tau_i = \varsigma/\lambda u = \tau_f/2$.

Therefore,

$$r_i = r(\tau_i) = r(0)\exp[(\kappa/3u^2)(\varsigma/\lambda)^3], \quad \text{so that } r_p/r_i = \exp[(\kappa/3u^2)(\varsigma/\lambda)^3]. \tag{14}$$

Thus, for each one of the observed response outbreaks (two such outbreaks are shown in Fig. 1) one can identify in a global time frame the instants $t_f = t_u + \tau_f/\lambda$ and $t_i = t_u + \tau_i/\lambda$ which correspond to the peak radius r_p and inflexion-point radius r_i, respectively; subscript "u" denotes the instants of upcrossings which can also be identified as long as $t_u = t_f - 2(t_f - t_i) = 2t_i - t_f$. We may now apply averaging for the time difference $t_f - t_i$ and for the amplitude ratio over all observed outbreaks of response (which may be properly numbered by using additional subscripts; thus, subscripts "k" and "$k+1$" are assigned for outbreaks shown in Fig. 1). This averaging which is equivalent to probabilistic averaging for an ergodic $g(t)$ will be denoted by angular brackets.

Let process $g(t)$ be Gaussian, so that ς the Raleigh p.d.f. [5,6], namely

$$p(\varsigma) = (\varsigma/\lambda^2)\exp(-\varsigma^2/2\lambda^2). \tag{15}$$

Then,

$$\langle t_f - t_i\rangle = \langle \tau_i\rangle/\lambda = (1/\lambda^2 u)\int_0^\infty \varsigma p(\varsigma)\,\mathrm{d}\varsigma = \sqrt{\pi/2}/\lambda u \tag{16}$$

and from the relation (14),

$$\langle \ln(r_p/r_i)\rangle = (\kappa/3u^2)\int_0^\infty (\varsigma/\lambda)^3 p(\varsigma)\,\mathrm{d}\varsigma = (\kappa/u^2)\sqrt{\pi/2}. \tag{17}$$

Now one may apply the formula for mean number of upcrossings n_u of the level u per unit time by the Gaussian process $g(t)$ [5,6], namely

$$n_u = (\lambda/2\pi)\exp(-u^2/2). \tag{18}$$

Its reciprocal is clearly seen to be the mean or expected time interval between two consecutive upcrossings; one such interval namely $t_{u,k+1} - t_{u,k}$ can be identified in Fig. 1. Therefore frequency λ and scaled total apparent mean damping factor u can be identified from the expressions (16) and (18). Finally, formula (17) can be used to calculate κ and thus σ_q as long as the quantity in its LHS is estimated by averaging over all observed response outbreaks. Thus, the above procedure provides on-line estimates both for the mean apparent damping coefficient—which may be regarded as a nominal stability margin—and for standard deviation and mean frequency of its random temporal variations.

4. Conclusions

The Jeffcott rotor with temporal random variations of its coefficient of internal or "rotating" damping has been considered. This rotor may exhibit spontaneous transient outbreaks in its dynamic response (forward whirl) due to brief excursions of the above coefficient into the domain of dynamic instability. Analysis of this transient response has been made using parabolic approximation for random temporal variations of the internal damping coefficient during the excursions together with the asymptotic Krylov–Bogoliubov method of averaging over the response period. It resulted, in particular, in explicit relation for peak value of the radius of whirl in terms of peak value of the negative apparent total damping coefficient during the corresponding excursion. In this way, reliability analysis for the shaft system as based on the solution to the first-passage problem for the response or on evaluating probability density of response peaks is reduced to the corresponding problems for the randomly varying coefficient of internal damping. The above analytical solution has also been used to derive a simple identification procedure for the rotor from its measured vibrational response. Namely, the mean value of the damping coefficient can be estimated on-line as well as its standard deviation and mean frequency of its temporal variations.

References

[1] G. Genta, *Vibrations of Structures and Machines*, Springer, New York, 1995.

[2] A. Tondl, *Some Problems of Self-excited Vibrations of Rotors*, Monographs and Memoranda Series, # 17, SVUSS, Bechovice, Czechoslovakia, 1974.

[3] M.F. Dimentberg, A. Naess, Short-term dynamic instability of a system with randomly varying damping, *Journal of Vibration and Control*, in press.

[4] N.N. Bogoliubov, Yu.A. Mitropol'sky, *Asymptotic Method in the Theory of Nonlinear Oscillations*, Gordon and Breach, New York, 1961.

[5] R.L. Stratonovich, *Topics in the Theory of Random Noise*, vol. 2, Gordon and Breach, New York, 1967.

[6] M.R. Leadbdetter, G. Lundgren, H. Rootzen, *Extremes and Related Properties of Random Sequences and Processes*, Springer, New York, 1983.