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Journal of Sound and Vibration 285 (2005) 1171–1179

JOURNAL OF SOUND AND VIBRATION

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Short Communication

The homotopy-perturbation method applied for solving complex-valued differential equations with strong cubic nonlinearity

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Received 2 April 2004; received in revised form 5 October 2004; accepted 12 October 2004
Available online 21 December 2004

Abstract

In this paper the homotopy perturbation method is adopted for solving a complex-valued second-order strongly nonlinear differential equation. Homotopy with an imbedding parameter $p \in [0, 1]$ is constructed. The perturbation procedure with parameter p transforms the strongly nonlinear differential equation into a system of linear complex-valued differential equations whose solutions give the approximate solution of the initial differential equation. To illustrate the effectiveness and convenience of the suggested procedure, a Duffing equation with strong cubic nonlinearity is considered. The periodic solution in the first approximation is obtained. The solution is compared with the exact one and shows good agreement.

1. Introduction

Many analytical procedures have been developed for solving complex-valued differential equations with small nonlinearity. The multiple-scales method [1], the Bogolubov–Mitropolski method [2], the Krylov–Bogolubov method i.e., the method of slow varying amplitude and phase and also the generalized averaging method are used for solving these differential equations (see Refs. [3–5]). All of them represent perturbation methods based on the exact closed-form analytic

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2004.10.026

solution of the corresponding linear differential equation as a generating equation. The main assumption is that the solution of the differential equation with small nonlinearity is close to the solution of the linear differential equation. Based on the mentioned methods the elliptic-Krylov–Bogolubov method, the harmonic balance method with elliptic functions, elliptic perturbation method, the Bogolubov–Mitropolski method with Jacobian elliptic functions, etc. are developed, which are applied for solving complex-valued strong nonlinear differential equations (see Refs. [6–9]).

The main disadvantage of the methods mentioned is that they require the exact analytic solution of the strong nonlinear differential equation to be known before the perturbation method is applied. To express the closed-form solution in analytical form is usually impossible. It exists only for a few differential equations with certain strong nonlinearities and for certain initial conditions. Besides, the traditional perturbation techniques are based on the small parameter. Unfortunately, this requirement is too over strict and most nonlinear equations have no small parameter at all.

Very often for applications it is not necessary to obtain the exact solution of the complex-valued strongly nonlinear differential equation, and an approximate analytical solution is quite satisfactory. In this paper an approximate analytic procedure for solving strong nonlinear complex-valued differential equation is developed. For solving such differential equations the homotopy perturbation method is adopted. The homotopy method is known in topology and Liao [10] was the first to apply it for solving the first- and second-order strong nonlinear differential equations. The homotopy is constructed with the imbedding parameter $p \in [0, 1]$ and due to homotopy description for the boundary values 0 and 1 the differential equation is linear and strong nonlinear, respectively. For the so obtained differential equation He [11–13] applied the perturbation procedure where the imbedding parameter p is considered as a small parameter.

In this paper the homotopy perturbation method is adopted for solving a complex-valued second-order strong nonlinear differential equation

$$\ddot{z} + f(z, \dot{z}, \mathrm{cc}) = 0 \tag{1}$$

with initial conditions

$$z(0) = z_0^*, \qquad \dot{z}(0) = \dot{z}_0^*, \tag{2}$$

where z is a complex function, cc is the complex conjugate function of z and z_0^* and $\dot{z}_0^*$ are constant complex values. The differential equation describes the oscillatory motion of a one-mass system with two degrees of freedom (dof). To illustrate the method an example of a Duffing equation with strong cubic nonlinearity is considered. The periodic solution in the first approximation is obtained.

2. The homotopy perturbation method

On introducing an operator F, the differential equation (1) takes the simple form

$$F(z) = 0, \quad z \in \mathbb{C}. \tag{3}$$

The operator F is divided into two parts: L is linear and N is nonlinear and Eq. (3) is

$$L(z) + N(z) = 0. \tag{4}$$

By homotopy technique proposed by Liao [13], the homotopy of Eq. (4) $\mathbb{C} \times [0,1] \to \mathbb{C}$ is constructed, which satisfies

$$\Im(Z,p) = pF(Z) + (1-p)(L(Z) - L(z_0)) = 0, \quad z \in \mathbb{C}, \; p \in [0,1] \tag{5}$$

with the initial conditions

$$Z(0,p) = z_0^*, \quad p \in [0,1],$$
$$\dot{Z}(0,p) = \dot{z}_0^*, \quad p \in [0,1], \tag{6}$$

where p is the imbedding parameter and z_0 is the initial approximation of Eq. (3), which satisfies the initial conditions (2). Different values of p correspond to different differential equations and their solutions. For $p = 0$ it is $\Im(Z,0) = L(Z) - L(z_0) = 0$ and the corresponding solution of the linear case is $Z(t,0) = z_0(t)$. For $p = 1$ it is $\Im(Z,1) = F(Z) = 0$ and $Z(t,1) = z(t)$ represents the solution of the original differential equation. So, $Z(t,p) : z_0(t) \simeq z(t)$ are homotopies. The solution $z(t)$ is a two frequency function as it corresponds to complex-valued second-order differential equation (1). The frequencies of the solution $z(t)$ are $\alpha\omega$ and $-\omega\beta$ where α and β depend on the nonlinearity. For the linear case, it is $\alpha(0) = 1$ and $\beta(0) = 1$ and the frequencies of the linear solution $z_0(t)$ are ω and $-\omega$.

Assuming that the imbedding parameter p is a "small parameter", the power series solution in p of Eq. (5) is

$$Z(t,p) = z_0 + pZ_1(t) + p^2 Z_2(t) + \cdots. \tag{7}$$

Setting $p = 1$ results in the approximate solution of Eq. (1)

$$z = \lim_{p \to 1} Z(t,p) = z_0 + Z_1(t) + Z_2(t) + \cdots. \tag{8}$$

Substituting Eq. (7) into Eqs. (5) and (6) and separating the terms with p^0 and p^1 following two differential equations are obtained:

$$p^0: \quad L(Z_0) - L(z_0) = 0, \quad Z_0(0) = z_0(0) = z_0^*, \quad \dot{Z}_0(0) = \dot{z}_0^*, \tag{9}$$

$$p^1: \quad L(Z_1) + F(Z_0) = 0, \quad Z_1(0) = 0, \quad \dot{Z}(0) = 0. \tag{10}$$

The solution of the differential equation (9) is $Z_0 = z_0$. The solution has periodical properties as the differential equation (9) describes the vibration motion of a one-mass system with two dof. Substituting the solution Z_0 into Eq. (10) and separating the secular terms which cause the resonant case, the correction of frequencies due to nonlinearity in the first approximation α and β are obtained. Solving the differential equation (10) and using the obtained values for α and β the solution in the first approximation is

$$z(t) = z_0 + Z_1(t). \tag{11}$$

3. Example 1

The complex-valued differential equation of Duffing type is

$$\ddot{z} + \omega^2 z + bz(z\bar{z}) = 0, \tag{12}$$

where ω and b are known constant values and the initial conditions are Eq. (2). This mathematical model describes the vibrations of a rotor with strong cubic nonlinearity [14]. To obtain the approximate solution of Eq. (12) the homotopy perturbation is applied.

A homotopy which satisfies $\mathbb{C} \times [0, 1] \to \mathbb{C}$ is constructed

$$L(Z) - L(z_0) + pL(z_0) + pZ(Z\bar{Z})b = 0, \tag{13}$$

where $L(Z) = \ddot{Z} + \omega^2 Z$. The initial approximation of Eq. (13) is assumed in the form

$$z_0 = A\exp(\mathrm{i}\alpha\omega t) + B\exp(-\mathrm{i}\beta\omega t), \tag{14}$$

where A and B are complex functions that satisfy the initial conditions (2)

$$A + B = z_0^*, \quad \alpha A - \beta B = -\frac{\mathrm{i}\dot{z}_0^*}{\omega}. \tag{15}$$

The α and β are unknown functions of nonlinearity. Using the suggested perturbation procedure and the approximate solution (14) the differential equation (10) is

$$\begin{aligned} L(Z_1) = & -\{A[\omega^2(1-\alpha^2) + b(A\bar{A} + 2B\bar{B})]\exp(\mathrm{i}\alpha\omega t) \\ & + B[\omega^2(1-\beta^2) + b(2A\bar{A} + B\bar{B})]\exp(\mathrm{i}\beta\omega t) \\ & - bA^2\bar{B}\exp[\mathrm{i}(2\alpha+\beta)\omega t] - b\bar{A}B^2\exp[-\mathrm{i}(\alpha+2\beta)\omega t], \end{aligned} \tag{16}$$

where $L(Z_1) = \ddot{Z}_1 + \omega^2 Z_1$, and $\bar{A}$ and $\bar{B}$ are the complex conjugate functions of A and B. Let us eliminate the secular terms in the differential equation. Two algebraic equations are obtained whose solutions are

$$\alpha^2 = 1 + \frac{b(A\bar{A} + 2B\bar{B})}{\omega^2}, \quad \beta^2 = 1 + \frac{b(2A\bar{A} + B\bar{B})}{\omega^2}. \tag{17}$$

It is evident that for the linear case when $b = 0$, it is $\alpha(0) = \beta(0) = 1$. The reduced differential equation is

$$\ddot{Z}_1 + \omega^2 Z_1 = -bA^2\bar{B}\exp[\mathrm{i}(2\alpha+\beta)\omega t] - b\bar{A}B^2\exp[-\mathrm{i}(\alpha+2\beta)]. \tag{18}$$

The differential equation (18) is a non-homogenous linear one. The solution of Eq. (18) is

$$\begin{aligned} Z_1 = & C_1\exp(\mathrm{i}\alpha\omega t) + C_2\exp(-\beta\mathrm{i}\omega t) \\ & + \frac{bA^2\bar{B}}{\omega^2[(2\alpha+\beta)^2 - 1]}\exp[\mathrm{i}(2\alpha+\beta)\omega t] \\ & + \frac{b\bar{A}B^2}{\omega^2[(\alpha+2\beta)^2 - 1]}\exp[-\mathrm{i}(\alpha+2\beta)\omega t], \end{aligned} \tag{19}$$

where according to the initial conditions $Z_1(0) = 0$, $\dot{Z}_1(0) = 0$ the complex constants C_1 and C_2 are

$$C_1 = \frac{b\bar{A}B^2}{2\omega^2(\alpha + 2\beta + 1)} - \frac{bA^2\bar{B}}{2\omega^2(2\alpha + \beta - 1)}, \tag{20}$$

$$C_2 = \frac{bA^2\bar{B}}{2\omega^2(2\alpha + \beta + 1)} - \frac{b\bar{A}B^2}{2\omega^2(\alpha + 2\beta - 1)}. \tag{21}$$

Using results (19)–(21) the first-order approximative solution of Eq. (12) is

$$\begin{aligned} z = {} & A\exp(\mathrm{i}\alpha\omega t) + B\exp(-\mathrm{i}\beta\omega t) \\ & + \left[\frac{b\bar{A}B^2}{2\omega^2(\alpha + 2\beta + 1)} - \frac{bA^2\bar{B}}{2\omega^2(2\alpha + \beta - 1)}\right]\exp(\mathrm{i}\omega t) \\ & + \left[\frac{bA^2\bar{B}}{2\omega^2(2\alpha + \beta + 1)} - \frac{b\bar{A}B^2}{2\omega^2(\alpha + 2\beta - 1)}\right]\exp(-\mathrm{i}\omega t) \\ & + \frac{bA^2\bar{B}}{\omega^2[(2\alpha + \beta)^2 - 1]}\exp[\mathrm{i}(2\alpha + \beta)\omega t] \\ & + \frac{b\bar{A}B^2}{\omega^2[(\alpha + 2\beta)^2 - 1]}\exp[-\mathrm{i}(\alpha + 2\beta)\omega t], \end{aligned} \tag{22}$$

where α and β are given with Eq. (17) and A and B with Eq. (15).

The system has two eigenfrequencies

$$\omega_1 = \omega\sqrt{1 + \frac{b}{\omega^2}(A\bar{A} + 2B\bar{B})}, \quad \omega_2 = \omega\sqrt{1 + \frac{b}{\omega^2}(2A\bar{A} + B\bar{B})} \tag{23}$$

and the corresponding periods of vibration are

$$T_1 = \frac{2\pi}{\omega\sqrt{1 + (b/\omega^2)(A\bar{A} + 2B\bar{B})}}, \quad T_2 = \frac{2\pi}{\omega\sqrt{1 + (b/\omega^2)(2A\bar{A} + B\bar{B})}}. \tag{24}$$

It is evident that both the frequencies and also both the periods depend on the initial values (2).

If it is assumed that the parameter b is small, expanding functions (23) into a Maclaurin series it is

$$\omega_1 = \omega\left(1 + \frac{b}{2\omega^2}(A\bar{A} + 2B\bar{B})\right), \quad \omega_2 = \omega\left(1 + \frac{b}{2\omega^2}(2A\bar{A} + B\bar{B})\right). \tag{25}$$

Comparing solutions (25) with those obtained in the paper [4] for the differential equation (12) with small parameter $b \ll 1$ using the method of slowly varying amplitude and phase it is obvious that they are the same.

4. Example 2

Let us consider the differential equation (12) with the following initial conditions:

$$z(0) = z_0^*, \qquad \dot{z}(0) = 0. \tag{26}$$

For these initial conditions the exact closed-form analytic solution of Eq. (12) is

$$z = 2A\,\mathrm{cn}(\Omega t, m), \tag{27}$$

where $\mathrm{cn}(\Omega t, m)$ is the Jacobi elliptic function [15], Ωt is the argument and m is the modulus. Introducing Eq. (27) and its second time derivative

$$\ddot{z} = -2\Omega^2\,\mathrm{cn}(\Omega t, m)[1 - 2m + 2m\,\mathrm{cn}^2(\Omega t, m)], \tag{28}$$

into Eq. (12) and separating the terms with the first and the third order of the elliptic function $\mathrm{cn}(\Omega t, m)$ the following system of two algebraic equations is obtained:

$$\omega^2 - \Omega^2(1 - 2m) = 0, \quad 2m\omega^2 - 4bA\bar{A} = 0. \tag{29}$$

Solving Eqs. (29) it is

$$\Omega^2 = \omega^2 + 4bA\bar{A}, \quad m = \frac{2bA\bar{A}}{\omega^2 + 4bA\bar{A}}. \tag{30}$$

Solution (27) is periodical with period

$$T = \frac{4K(m)}{\Omega}, \tag{31}$$

where $K(m)$ is the complete elliptic integral of the first kind [15].

Applying the homotopy perturbation method the first-order approximation solution of Eq. (12) is obtained. The approximate solution that corresponds to the linear differential equation ($b = 0$) has the form

$$z_0 = 2A\,\cos(\alpha\omega t), \tag{32}$$

where A is a complex function which is according to the initial conditions (26) $A = z_0^*/2$. Relation (32) represents a one-frequency solution. Using the suggested procedure the value of α in the first approximation is obtained and it is

$$\alpha = \sqrt{1 + \frac{3b}{\omega^2}A\bar{A}}. \tag{33}$$

For the initial conditions (26) the solution of Eq. (12) in the first approximation is

$$z = 2A\,\cos(\alpha\omega t) + \frac{2bA^2\bar{A}}{\omega^2(9\alpha^2 - 1)}[\cos(3\alpha\omega t) - \cos(\alpha\omega t)]. \tag{34}$$

The period of solution (34) is

$$T = \frac{2\pi}{\omega\sqrt{1 + (3b/\omega^2)A\bar{A}}}. \tag{35}$$

To illustrate the correctness of the obtained solution (34) let us compare the approximate and the exact analytic solutions. For $\omega^2 = 1$, $b = 1$ and initial condition $z_0^* = 0.2 + 0.6\mathrm{i}$ it is $\alpha = 1.14$, and the approximate solution (34) is

$$z = (0.2 + 0.6\mathrm{i})\left\{\cos(1.14t) + \tfrac{1}{107}[\cos(3.42t) - \cos(1.14t)]\right\}, \tag{36}$$

i.e.,

$$\begin{aligned} x_A &= 0.2\left\{\cos(1.14t) + \tfrac{1}{107}[\cos(3.42t) - \cos(1.14t)]\right\}, \\ y_A &= 0.6\left\{\cos(1.14t) + \tfrac{1}{107}[\cos(3.42t) - \cos(1.14t)]\right\} \end{aligned} \tag{37}$$

as $z = x_A + \mathrm{i}y_A$ where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, x_A and y_A are real functions. The period of solution is

$$T_A = \frac{2\pi}{1.14} = 5.509. \tag{38}$$

For the suggested coefficients and initial conditions the exact closed-form solution is according to Refs. (27) and (30)

$$z = (0.2 + 0.6\mathrm{i})\ \mathrm{cn}(1.183t, 0, 143), \tag{39}$$

i.e.,

$$\begin{aligned} x_N &= 0.2\ \mathrm{cn}(1.183t, 0, 143), \\ y_N &= 0.6\ \mathrm{cn}(1.183t, 0, 143), \end{aligned} \tag{40}$$

where $z = x_N + \mathrm{i}y_N$. The corresponding period (30) is

$$T = \frac{4K(0.143)}{1.183} = 5.510. \tag{41}$$

Comparing periods (38) and (41) it is concluded that the difference is negligible. In Fig. 1 the approximate x_A and y_A (37) and exact x_N and y_N (40) solutions as functions of time t are plotted. The absolute error between approximate x_A and y_A and exact solution x_N and y_N, respectively, is also shown in Fig. 1. The solutions are in good agreement, the absolute error is negligibly small for sufficient long time period (the error is less than 2% for $t = 50$). The error has a tendency to increase in time.

5. Conclusion

It can be concluded:

1. The perturbation procedure with the imbedding parameter $p \in [0, 1]$ applied for the complex-valued strong nonlinear differential equation, obtained using the homotopy method, transforms it into a system of linear complex-valued differential equations whose solutions give the approximate solution of the initial differential equation.

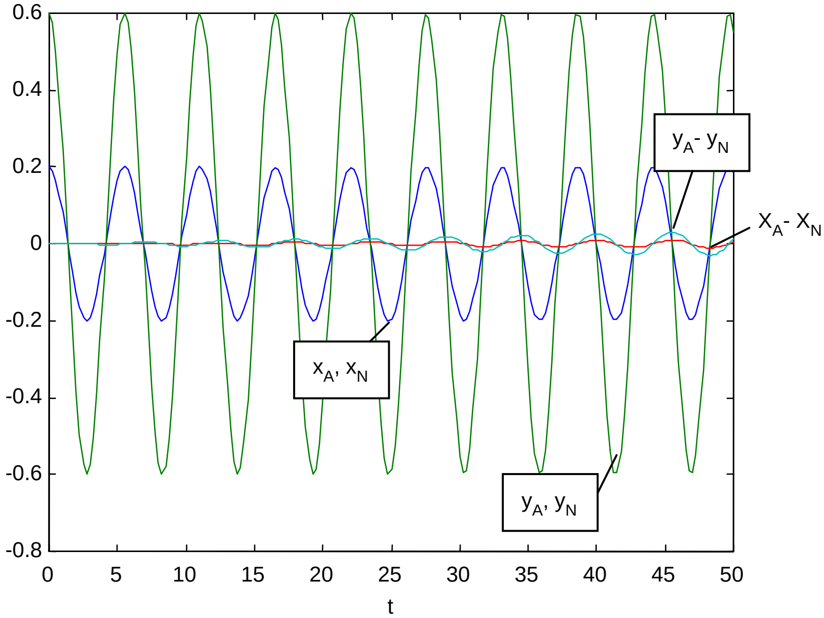

Fig. 1. Time histories of the approximate solution (x_A,, y_A), exact analytical solution (x_N, y_N) and absolute difference between approximate and exact solution ($x_A - x_N$, $y_A - y_N$).

2. The homotopy perturbation method has an advantage in comparison to the traditional perturbation methods (elliptic-Krylov–Bogolubov, elliptic method, etc.) as it is based on the linear complex-valued differential equations whose solutions are usually known.
3. The first-order approximative analytic solution obtained applying the homotopy perturbation method for solving strong nonlinear complex-valued second-order differential equation is in very good agreement with the exact closed form analytic solution.
4. The suggested procedure gives both the approximate frequencies and both the periods of the solution as a function of nonlinearity and initial conditions. The periods obtained applying the homotopy perturbation method are in good agreement with exact values.

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