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Short Communication

On the analytical approximations to the periodic solutions of the $x^{(2n+2)/(2n+1)}$ potential

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Recently, Cooper and Mickens [1] have studied the dynamics of a system modeled by the equation of motion [2]:

$$\ddot{x} + x^{1/(2n+1)} = 0, \tag{1}$$

$$x(0) = x_0, \dot{x}(0) = 0, \tag{2}$$

where n is a positive integer and over dots denote differentiation with respect to time, t. A higher-order harmonic balance method, combined with a numerical solution for one particular value of the initial conditions, has been used to construct an analytical approximation to a system modeled by a $x^{4/3}$ potential (i.e., $n = 1$, in Eq. (1)). A functional form of the generalized harmonic balance [3] used, to construct the analytical approximation to Eq. (1) for $n = 1$, is

$$x(t) \approx \frac{A\cos(\omega t)}{1 + B\cos(2\omega t)}. \tag{3}$$

Here A, B and the angular frequency (ω) are to be determined as functions of the initial conditions expressed in Eq. (2). They have indicated the possibility of generalizing the suggested procedure to any positive integer n. However, the numerically obtained frequency value for the case, $n = 1$, is found to be less compared to its exact value. In fact, there is no necessity to obtain numerical solution of Eq. (1) for the determination of A and B in the above analytical approximation (3). The unknown A and B values in Eq. (3) can be obtained directly from the energy relation of Eq. (1) and the initial conditions (2).

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Motivated by work of Cooper and Mickens [1], this article presents an exact frequency expression for the equation of motion (1), and the constants A and B in Eq. (3) applicable to any positive integer n.

Multiplying Eq. (1) by $2\dot{x}$ and using the initial conditions (2), the energy relation after integration can be obtained as:

$$(\dot{x})^2 = I(x_0) - I(x) \tag{4}$$

where $I(x) = (1/p)\,x^{2p}$ and $p = (n+1)/(2n+1)$.

The restoring force function in the equation of motion (1) is an odd function. The behavior of oscillations is the same for both negative and positive amplitudes. From Eqs. (2) and (4), one obtains

$$\int_{x=0}^{x=x_0} \frac{\mathrm{d}x}{\sqrt{I(x_0) - I(x)}} = \int_{t=0}^{t=T/4} \mathrm{d}t = \frac{T}{4} = \frac{\pi}{2\omega} \tag{5}$$

which implies that

$$\omega x_0^q = \pi\sqrt{p} \Big/ \beta\left(\frac{1}{2p}, \frac{1}{2}\right). \tag{6}$$

Here, $q = n/(2n+1)$, T is the period and $\beta(a,b) = \int^1 \xi^{a-1}(1-\xi)^{b-1}\,\mathrm{d}\xi$, is a beta function, whose value can be found accurately using MATLAB. For the case, $x_0 = 1$ and $n = 1$, the value of angular frequency from Eq. (6) is found to be 1.0705, whereas the numerically integrated value reported by Cooper and Mickens [1] is 1.054.

Regarding the analytical approximation to Eq. (1), Eq. (3) satisfies one of the initial conditions (i.e., $\dot{x}(0) = 0$) in Eq. (2). To satisfy the other initial condition (i.e., $x(0) = x_0$), one has to express

$$A = (1+B)x_0. \tag{7}$$

At the quarter period (i.e., $t = T/4 = \pi/(2\omega)$):

$$x\left(\frac{\pi}{2\omega}\right) = 0, \;\; \dot{x}\left(\frac{\pi}{2\omega}\right) = -\sqrt{I(x_0)}. \tag{8}$$

The second condition in Eq. (8) is obtained from the energy relation (4). Using the conditions in Eq. (7) and (8) in Eq. (3), one obtains

$$B = \frac{1-b}{1+b}, \tag{9}$$

where $b = \sqrt{p}\omega x_0^q = \pi p/\beta\big(1/(2p), 1/2\big)$. It is interesting to note that the constant B of Eq. (3) is found to be independent of x_0 and it is only a function of the positive integer n. Fig. 1 shows the phase-plane diagram of the equation of motion (1) for $x_0 = 1$ and $n = 1$. The constants in Eq. (3) obtained are: $A = 1.0672$, $B = 0.0672$ and $\omega = 1.0705$. The phase-plane diagram from the analytical function (3) with the above parameters matches well with the actual one generated using Eq. (4). The values of ωx_0^q for $n = 10$, 100, 1000, obtained are: 1.1047, 1.1101, and 1.1107, respectively. The corresponding values of B in Eq. (3) are: 0.1114, 0.1193, and 0.1201, respectively. The present exact solution will be useful to validate the approximate periodic solutions for the equation of motion (1).

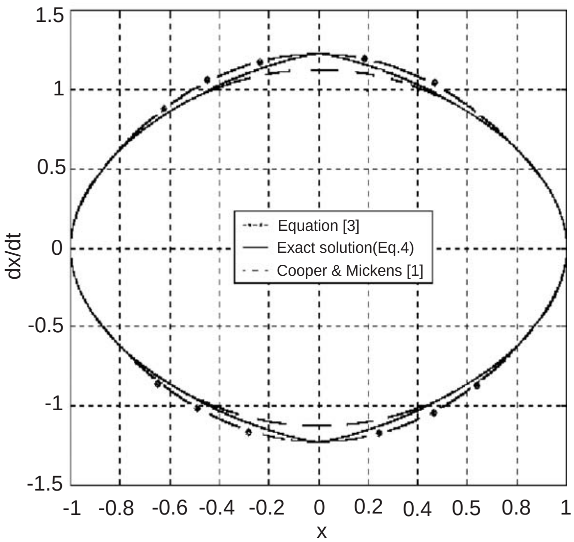

Fig. 1. Phase-plane diagram of the equation of motion (1) for the case, $n = 1$.

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