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Short Communication

Inverse vibration problem for inhomogeneous circular plate with translational spring

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Abstract

The free vibrations of uniform and homogeneous circular plates with translational springs have been studied in the literature for some time; although exact solutions have been found, no closed-form solution has been reported yet.

In this study, using the semi-inverse method we derive a closed-form solution for the natural frequency via postulating the vibration mode of the plate as a polynomial of the radial coordinate.

1. Introduction

The free vibrations of circular plates with the translational springs was studied by Leissa [1]. He derived the transcendental equations yielding the natural frequency of both axisymmetric and non-symmetric vibrations for the plate with both translational and rotational springs. The uniform plate was considered, with attendant transverse displacement expressed in terms of the Bessel functions. The Bubnov–Galerkin method to this problem was applied by Laura et al. [2].

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They approximated the transverse displacement as

$$W(r) \approx W_{N_1}(r) = \sum_{j=0}^{N_1} A_j(\alpha_j r^4 + \beta_j r^2 + 1) r^{2j}, \tag{1}$$

where N_1 denotes the number of terms retained, α_j and β_j are constants chosen so as to satisfy the boundary conditions. Note that the related paper on vibrations of circular plates with supports along the circumference is by Laura et al. [3]. To the best of our knowledge, there is no closed-form solutions reported to this problem. It may even seem at the first glance that there would be no closed-form solutions. This ambitious goal is addressed in this study.

We consider an inhomogeneous plate, with inhomogeneity in the form of the variable flexural rigidity. We pose and solve an inverse problem: we postulate the mode shape to be fourth-order polynomial, corresponding to the zero value of N_1 in Eq. (1). Then we find the flexural rigidity's variation along the radial coordinate, so as the inhomogeneous plate to have a postulated closed-form solution.

2. Basic equations

The differential equation governing free small axisymmetric vibrations of circular plates reads

$$D(r)r^3\nabla^2\nabla^2 W + \frac{\mathrm{d}D}{\mathrm{d}r}\left(2r^3\frac{\mathrm{d}^3 W}{\mathrm{d}r^3} + r^2(2+v)\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} - r\frac{\mathrm{d}W}{\mathrm{d}r}\right) + \frac{\mathrm{d}^2 D}{\mathrm{d}r^2}\left(r^3\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + vr^2\frac{\mathrm{d}W}{\mathrm{d}r}\right) - \rho h\omega^2 r^3 W = 0, \tag{2}$$

where h is the thickness of the plate, ρ the material density, v the coefficient of Poisson, r the radial coordinate, D the flexural rigidity, W the mode shape and ∇^2 the Laplace operator in polar coordinates,

$$\nabla^2 = \frac{\mathrm{d}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}. \tag{3}$$

The transverse displacement W is postulated to be in the form

$$W(r) = \alpha_0 + \alpha_2 r^2 + r^4. \tag{4}$$

We set

$$\rho h = \delta(r) \tag{5}$$

that we suppose to vary along the radial coordinate r as

$$\delta(r) = \sum_{i=0}^{m} a_i r^i. \tag{6}$$

Since W is a fourth-order polynomial expression in terms of r, in view of Eq. (6), the last term in the differential equation (2) is a polynomial expression of degree $m+7$. Moreover, the operator $\nabla^2\nabla^2$ in Eq. (2) involves the four-fold differentiation with respect to r. In order for the highest degree of the first term's polynomial expression in $Dr^3\nabla^2\nabla^2 W$ to be of order $m+7$, it is necessary and sufficient for the flexural rigidity to be represented as a polynomial of degree $m+4$. Thus, the sought flexural rigidity can be put in the form

$$D(r)=\sum_{i=0}^{m+4} b_i(r-R)^i. \tag{7}$$

3. Boundary conditions

The boundary conditions at the outer boundary $r=R$ consist of the bending moment M_r acting along the circumference sections to vanish, and the shearing force per unit length to be proportional to the deflection of the plate:

$$M_r(R)=0, \quad Q_r(R)+k_W W(R)=0, \tag{8,9}$$

where

$$M_r(r)=-D(r)\left(\frac{\mathrm{d}^2W}{\mathrm{d}r^2}+\frac{v}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right), \tag{10}$$

and k_W is the stiffness per unit of length of the translational spring. The shearing force per unit of length $Q_r(r)$ is obtained from the equilibrium equations (see Ref. [4, Eq. (53)]):

$$Q_r(r)=\frac{1}{r}\left[M_t(r)-M_r(r)-r\frac{\mathrm{d}M_r(r)}{\mathrm{d}r}\right], \tag{11}$$

where M_t denotes the bending moment per unit of length acting along the diametrical section rz of the plate

$$M_t(r)=-D(r)\left(\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}+v\frac{\mathrm{d}^2W}{\mathrm{d}r^2}\right). \tag{12}$$

The problem is posed as follows: *Determine the variation of the flexural rigidity $D(r)$ so that a plate with such $D(r)$ will possess the vibration mode defined in* Eq. (4).

4. Method of solution

The application of the boundary conditions given in Eqs. (8) and (9) permits the determination of the coefficients α_0 and α_2 of the mode shape polynomial expression defined in Eq. (4). Indeed, Eqs. (8) and (9) read

$$(12+4v)R^2+2\alpha_2(1+v)=0, \tag{13}$$

$$k_W R^4+k\alpha_2R^2-32b_0k_W\alpha_0R=0. \tag{14}$$

From Eqs. (13) and (14) we get

$$\alpha_0=-32\frac{Rb_0}{k_W}+\frac{5+v}{1+v}R^4, \tag{15}$$

$$\alpha_2 = -2\frac{(3+v)}{(1+v)}R^2. \tag{16}$$

So that the shape mode is written as

$$W(r) = -32\frac{Rb_0}{k_W} + \frac{5+v}{1+v}R^4 - 2\frac{(3+v)}{(1+v)}R^2r^2 + r^4. \tag{17}$$

Here it must be noted that the mode shape depends both upon the coefficient b_0 of the flexural rigidity and the stiffness of the translational spring k_W. Yet, it can be argued that it ought be anticipated that the *closed-form solution* would only be attainable for specific values and combinations of the system parameters, and for specific relationships between the mode shape and the system's characteristics.

Further steps involve the substitution of Eqs. (6), (7) and (17) into the governing differential equation (2) and demanding the so-obtained polynomial expression to vanish. This implies that all the coefficients in front of power r^i must be zero. This requirement is leading, in turn, to a set of algebraic equations inn terms of b_i, and ω^2. We consider various case for the inertial term $\delta(r)$ in Eq. (6).

5. Constant inertial term ($m = 0$)

As seen from Eq. (7), in this particular case, the flexural rigidity is sought as a fourth-order polynomial

$$D(r) = b_0 + b_1(r-R) + b_2(r-R)^2 + b_3(r-R)^3 + b_4(r-R)^4. \tag{18}$$

The differential equation (2) becomes

$$\sum_{i=0}^{7} c_i r^i = 0, \tag{19}$$

where

$$\begin{aligned}
c_0 &= c_1 = 0,\\
c_2 &= -4(3+v)(R^2b_1 - 2R^3b_2 + 3R^4b_3 - 4R^5b_4),\\
c_3 &= 64b_0 - 64Rb_1 + 16(1-v)R^2b_2 + 16(5+3v)R^3b_3\\
&\quad - 32(7+3v)R^4b_4 - \frac{5+v}{1+v}a_0R^4\omega^2 + 32\frac{a_0R}{k_W}b_0\omega^2,\\
c_4 &= 12(11+v)b_1 - 24(11+v)b_2 + 288R^2b_3 - 96(1-v)R^3b_4,\\
c_5 &= 32(7+v)b_2 - 96(7+v)b_3 + 128(9+v)R^2b_4 + 2\frac{3+v}{1+v}a_0R^2\omega^2,\\
c_6 &= 20(17+3v)b_3 - 80(17+3v)Rb_4,\\
c_7 &= 96(5+v)b_4 - a_0\omega^2.
\end{aligned} \tag{20}$$

Since the left-hand side of the differential equation (18) must vanish for any r within $[0; R]$, we demand that all the coefficients c_i to be zero. This leads to a homogeneous set of six non-linear algebraic equations for six unknowns. From the requirement $c_7 = 0$, the natural frequency squared is obtained as

$$\omega^2 = \frac{96 b_4 (5 + v)}{a_0} \tag{21}$$

in the desired closed-form solution. Upon substitution of Eq. (20) into Eq. (19), the remaining equations yield the coefficient in the flexural rigidity

$$\begin{aligned} b_0 &= \frac{b_4 R^4 k_W (5 + v)(7 + v)}{2(1 + v)[48 R b_4 (5 + v) + k]}, \\ b_1 &= -4 b_4 \frac{5 + v}{1 + v} R^3, \quad b_2 = -2 b_4 \frac{3 - v}{1 + v} R^2, \quad b_3 = 4 b_4 R. \end{aligned} \tag{22}$$

Hence, the flexural rigidity reads

$$D(r) = \left(\frac{R^4 k_W (5 + v)(7 + v)}{2(1 + v)[48 R b_4 (5 + v) + k_W]} - 4 \frac{5 + v}{1 + v} R^3 r - 2 \frac{3 - v}{1 + v} R^2 r^2 + 4 R r^3 + r^4 \right) b_4. \tag{23}$$

It must be stressed that the determined flexural rigidity of the plate depends on the stiffness k_W of the translational spring in a nonlinear manner. Only when there is a relation between $D(r)$ and k_W is the closed-form polynomial solution for the mode shape possible.

Substituting the expression for b_0 from Eq. (21) into the mode shape given by Eq. (17), the latter becomes

$$W(r) = \frac{5 + v}{1 + v} \left(1 - \frac{7 + v}{3(5 + v) + k_W / b_4 R} \right) R^4 - 2 \frac{(3 + v)}{(1 + v)} R^2 r^2 + r^4. \tag{24}$$

We introduce the non-dimensional constant

$$\beta = k_W / b_4 R. \tag{25}$$

The flexural rigidity and the mode shape are then expressed as

$$\begin{aligned} D(r) = \Bigg[&\frac{R^4 \beta (5 + v)(7 + v)}{2(1 + v)[48(5 + v) + \beta]} - 4 \frac{5 + v}{1 + v} R^3 (r - R) \\ &- 2 \frac{3 - v}{1 + v} R^2 (r - R)^2 + 4 R (r - R)^3 + (r - R)^4 \Bigg] b_4, \end{aligned} \tag{26}$$

$$W(r) = \frac{5 + v}{1 + v} \left(1 - \frac{7 + v}{3(5 + v) + \beta} \right) R^4 - 2 \frac{(3 + v)}{(1 + v)} R^2 r^2 + r^4. \tag{27}$$

Let

$$\rho = \frac{r}{R}. \tag{28}$$

We have

$$\frac{D(\rho)}{R^4 b_4} = \frac{(57 + 18v + v^2)\beta + 96(5 + v)(11 + 3v)}{2(1 + v)[48(5 + v) + \beta]} - 4\frac{3 + v}{1 + v}\rho^2 + \rho^4, \tag{29}$$

$$\frac{W(\rho)}{R^4} = \frac{5 + v}{1 + v}\frac{4(2 + v) + \beta}{3(5 + v) + \beta} - 2\frac{3 + v}{1 + v}\rho^2 + \rho^4. \tag{30}$$

As is seen, the coefficient b_4 can be chosen arbitrarily. Thus there are infinite amounts of closed-form solutions. Once one specifies b_{47}, a specific solution is obtained. The Figs. 1 and 2 represent the graph of $D(\rho)$ and $W(\rho)$ for different values of β with the coefficient of Poison v fixed to $v = 0.3$ and $b_4 = 1$.

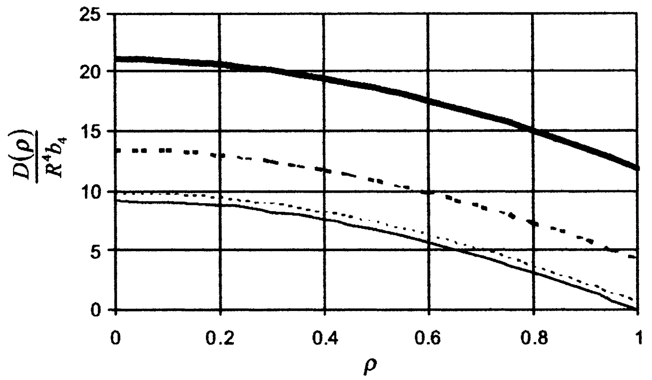

Fig. 1. Variation of the flexural rigidity of an inhomogeneous plate with translational spring on the border, a constant inertial term and a coefficient of Poison fixed to $v = 0.3$, when the non-dimensional coefficient β varies: —, $\beta = 0$; - - - -, $\beta = 10$; - - -, $\beta = 100$; ▬▬, $\beta = 1000$.

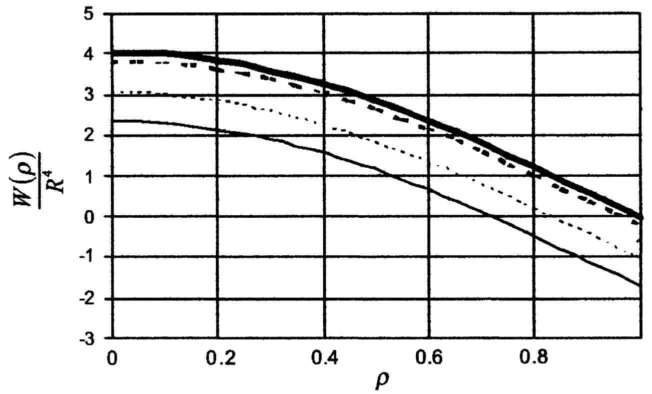

Fig. 2. Mode shape of an inhomogeneous plate with translational spring on the border, a constant inertial term and a coefficient of Poison fixed to $v = 0.3$ when the non-dimensional coefficient β varies: —, $\beta = 0$; - - - -, $\beta = 10$; - - -, $\beta = 100$; ▬▬, $\beta = 1000$.

6. Linearly varying inertial term ($m = 1$)

In this case, the inertial term is expressed as

$$\delta(r) = a_0 + a_1 r. \tag{31}$$

Let us introduce the non-dimensional coefficient γ defined such that

$$\gamma = \frac{a_0}{a_1 R}, \quad a_1 \neq 0. \tag{32}$$

Hence, the inertial term can be expressed with the reduced coordinate as

$$\delta(\rho) = a_0\left(1 + \frac{\rho}{\gamma}\right), \tag{33}$$

where ρ is defined in Eq. (28).

Instead of the set (20), we get here seven algebraic expressions for the coefficients of the flexural rigidity polynomial form defined in Eq. (7)

$$\begin{aligned}
c_0 &= c_1 = 0,\\
c_2 &= -4(3+v)(R^2 b_1 - 2R^3 b_2 + 3R^4 b_3 - 4R^5 b_4 + 5b_5),\\
c_3 &= 64b_0 - 64Rb_1 + 16(1-v)R^2 b_2 + 16(5+3v)R^3 b_3\\
&\quad - 32(7+3v)R^4 b_4 + 32(13+5v)R^5 b_5 - \frac{5+v}{1+v} a_0 R^4 \omega^2 + 32\frac{a_0 R}{k_W} b_0 \omega^2,\\
c_4 &= 12(11+v)b_1 - 24(11+v)b_2 + 288R^2 b_3 - 96(1-v)R^3 b_4\\
&\quad - 60(7+5v)R^4 b_5 + 32\frac{a_1 R}{k_W} b_0 \omega^2,\\
c_5 &= 32(7+v)b_2 - 96(7+v)b_3 + 128(9+v)R^2 b_4 - 1280R^3 b_5 + 2\frac{3+v}{1+v} a_0 R^2 \omega^2,\\
c_6 &= 20(17+3v)b_3 - 80(17+3v)Rb_4 + 100(31+5v)R^2 b_5 + 2\frac{3+v}{1+v} a_1 R^2 \omega^2,\\
c_7 &= 96(5+v)b_4 - 480(5+v)Rb_5 - a_0 \omega^2,\\
c_8 &= 28(23+5v)Rb_5 - a_1 \omega^2.
\end{aligned} \tag{34}$$

Since the polynomial expression of the differential equation must vanish for every positive r not greater than R, all the coefficients c_i must be equal to zero. From $c_8 = 0$, the natural frequency squared is obtained as

$$\omega^2 = \frac{28b_5(23+5v)}{a_1}. \tag{35}$$

Upon substitution of Eq. (35) into $c_5 = 0$, $c_6 = 0$, $c_7 = 0$ of Eqs. (34), the remaining equations yield the coefficient b_2, b_3, b_4 of the flexural rigidity:

$$
\begin{aligned}
b_2 &= \frac{-35(23+5v)(3-v)(17+3v)a_0 + 12(5+v)(15v^2-296v+1823)a_1 R}{60(5+v)(17+3v)(1+v)a_1} R^2 b_5,\\
b_3 &= \frac{35(23+5v)(17+3v)(1+v)a_0 + 6(5+v)(105v^2+568v-41)a_1 R}{30a_1(5+v)(17+3v)(1+v)} R b_5,\\
b_4 &= \frac{120(5+v)a_1 R + 7(23+5v)a_0}{24a_1(5+v)} b_5. \qquad (36)
\end{aligned}
$$

Let us consider now the set of equations $c_3 = 0$ and $c_4 = 0$ of Eqs. (34), substituting the values of ω^2, b_2, b_3, b_4 obtained in Eqs. (35) and (36). We obtain for b_0 and b_1 the following solution:

$$
\begin{aligned}
b_0 = {}& 32\frac{(7+v)(11+v)(17+3v)(23+5v)a_0 + (15v^3+463v^2+3789v+8885)a_1 R}{(1+v)(17+3v)[240(11+v)a_1 k_W + 3360(11+v)(23+5v)a_0 b_5 + 17920(23+5v)a_1 b_5]}\\
& \times R^4 k_W b_5,\\
b_1 = {}& -\{105(11+v)(23+5v)(17+3v)a_0 a_1 k_W + 6[18375a_0^2 b_5 + (60a_1^2 k_W + 475300a_0^2 b_5)v^3\\
& + (1456a_1^2 k_W + 4351690a_0^2 b_5)v^2 + (12060a_1^2 k_W + 17017700a_0^2 b_5)v\\
& + 29816a_1^2 k_W + 24236135a_0^2 b_5]R b_5\\
& + 28(23+5v)(705v^3+17633v^2+130715v+294723)R^2 a_0 a_1 b_5\\
& + 2688(23+5v)(15v^2+232v+721)a_1^2 R^3 b_5\}R^3 b_5/(17+3v)(1+v)[90(11+v)a_1^2 k_W\\
& + 1260(11+v)(23+5v)a_1 a_0 b_5 + 6720(23+5v)a_1^2 b_5]. \qquad (37)
\end{aligned}
$$

Taking into account the previous results (35)–(37), equation $c_2 = 0$ from Eq. (34) must now be satisfied. Two solutions for b_5 are obtained

$$b_5 = 0 \qquad (38)$$

or

$$b_5 = \frac{12(15v^3+364v^2+2619v+5546)a_1 k_W}{7(23+5v)(165a_0 v^3 + (329a_0 - 1440a_1 R)v^2 - (11441a_0 + 15936a_1 R)v - 37309a_0 - 38688a_1 R)R}. \qquad (39)$$

The first solution $b_5 = 0$ described in Eq. (38) must be dismissed since it leads to the trivial case with flexural rigidity that is identically zero all over the plate.

Eqs. (7), (17), (35)–(37) and (39) lead to a determinate solution in the case of a linearly varying inertial term. Indeed, the natural frequency squared is obtained by substituting the expression of b_5 into Eq. (35)

$$\omega^2 = \frac{48k_W(15v^3+364v^2+2619v+5546)}{a_1 R^2[(165v^3+329v^2-11441v-37309)\gamma - (1440v^2+15936v+38688)]}. \qquad (40)$$

The flexural rigidity and mode shape are expressed as follows, respectively

$$D(\rho) = \left[\frac{\gamma}{240}\frac{(165v^2 + 3314v + 12725)(3+v)^2}{(5+v)(17+v)(1+v)} - \frac{7\gamma}{6}\frac{(3+v)(23+5v)}{(1+v)(1+5v)}\rho^2 - \frac{9}{5}\frac{(33+5v)(3+v)}{(17+3v)(1+v)}\rho^3 + \frac{7\gamma}{24}\frac{(23+5v)}{(5+v)}\rho^4 + \rho^5\right] R^5 b_5, \quad (41)$$

$$\frac{W(\rho)}{R^4} = \frac{81}{35}\frac{(33+5v)(3+v)^2}{(1+v)(17+3v)(23+5v)} - 2\frac{3+v}{1+v}\rho^2 + \rho^4. \quad (42)$$

The physical realizability demands that the expression for the natural frequency ω^2 be positive. Analogously, $D(r)$ must take positive values in the interval $[0, R]$. The investigation of ω^2 shows that a solution exists in either of two cases

$$\gamma < \alpha \quad \text{for } a_1 > 0, \quad (43)$$

$$\gamma > \alpha \quad \text{for } a_1 < 0, \quad (44)$$

where α is defined by the expression

$$\alpha = \frac{1440v^2 + 15936v + 38688}{165v^3 + 329v^2 - 11441v - 37309}. \quad (45)$$

Indeed, since the numerator in Eq. (40) is positive, so should be the numerator. This implies that a_1 and the expression in the square parentheses should have the same sign. This requirement leads to conditions in Eqs. (43) and (44).

In the following the value of v is fixed at $v = 0.3$. The flexural rigidity D in Eq. (41) is a function of γ and ρ: $D(\gamma, \rho)$. We demand for D not to vanish in the interval $\rho \in [0, 1]$. Moreover, it should not change the sign in that interval. Considering ρ as a parameter, the value of γ that makes D

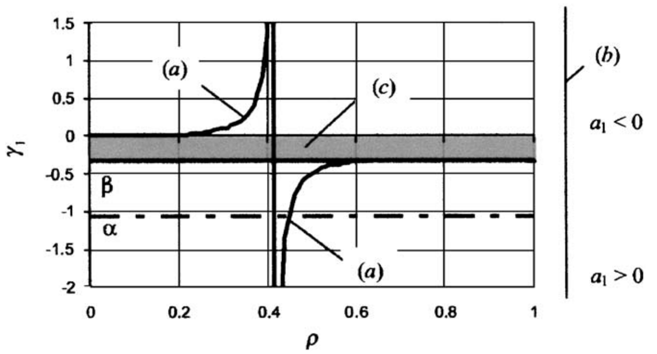

Fig. 3. Values of γ that make the flexural rigidity vanish are given by solid lines (a); (b) represents the relationship between the coefficient a_1 and the coefficient γ that allows a physically acceptable solution for the natural frequency squared ω^2; the shaded area (c) represents the range of values for γ that allow the flexural rigidity not to vanish.

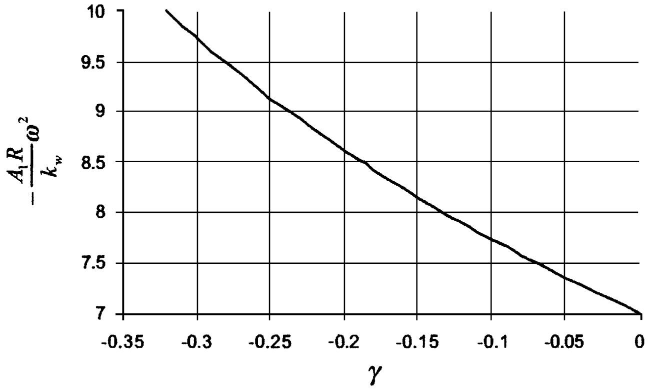

Fig. 4. Variation of the natural frequency squared versus $\gamma \in]\beta, 0[$; $v = 0.3$.

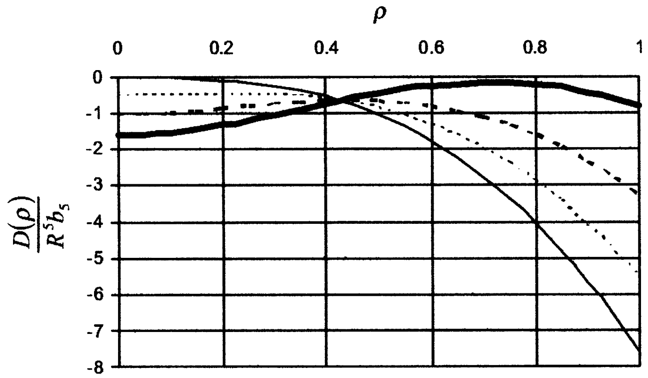

Fig. 5. Variation of the flexural rigidity of an inhomogeneous plate with translational spring on the boundary with a linearly decreasing inertial term along the radial coordinate and a coefficient of Poisson fixed to $v = 0.3$, for different value of $\gamma \in]\beta, 0[$: —, $\gamma = -0.01$; - - - -, $\gamma = -0.1$; - - -, $\gamma = -0.2$; ▬▬, $\gamma = -0.31$.

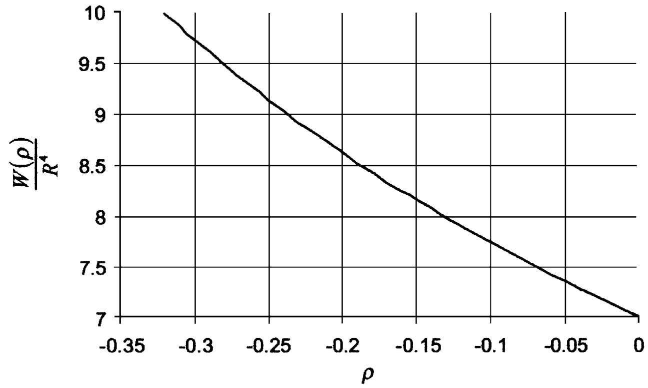

Fig. 6. Mode shape of an inhomogeneous plate with translational spring on the boundary with a linearly decreasing inertial term along the radial coordinate and a coefficient of Poisson fixed to $v = 0.3$.

vanish is

$$\gamma_1 = -\frac{25440\rho^3(-20493 + 2327\rho^2)}{299127609 - 1718133648\rho^2 + 29816100\rho^4}. \tag{46}$$

Fig. 3 represents the variations of γ_1 with ρ. The interpretation of this graph leads us to conclude that $a_1 > 0$ cannot be accepted since for all value of γ such as $\gamma < \alpha$, D vanishes in the interval [0,1]. Let us examine the case $a_1 < 0$. The shaded area that represents the admissible values for γ is defined by

$$\beta < \gamma < 0, \tag{47}$$

where $\beta \approx -0.3206643660$ is the maximum value of γ on the right of the vertical asymptote when $v = 0.3$. For such values of γ, the first term represented by the square bracket in Eq. (41) gets negative values. The condition for D to be positive depends then only upon the sign of b_5 as defined in Eq. (39). We can easily observe that

$$b_5 < 0 \quad \text{for } \gamma > \alpha. \tag{48}$$

Finally, the solution obtained in Eqs. (40)–(42) has a physical explication when

$$a_1 < 0 \quad \text{for } \beta < \gamma < 0. \tag{49}$$

Figs. 4–6 portray ω^2, $D(\rho)$ and $W(\rho)$ for different values of $\gamma \in [\beta, 0]$.

7. Conclusion

Apparently, the first closed-form solution has been derived for the free vibrations of inhomogeneous circular plates supported by a translational spring along plate's boundary.

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