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Short Communication

Fundamental frequency of a circular plate supported by a partial elastic foundation

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Abstract

The fundamental frequency of a thin circular plate supported by a concentric partial foundation is studied. For free or sliding edge conditions, the fundamental mode may not be axisymmetric.

1. Introduction

The vibration of a plate supported laterally by an elastic foundation was discussed on the first page of Leissa's celebrated book [1]. Leissa deduced that the effect of a full (Winkler) foundation merely increases the square of the natural frequency of the plate by a constant. The same conclusion was conjectured by Salari et al. [2]. The vibration of a plate supported by a *partial* elastic foundation was considered by Laura et al. [3], in which case a simple frequency relation no longer holds.

The aims of this communication are several. Firstly, we shall derive exact frequency determinants to confirm and extend Laura's approximate results for the clamped and simply supported plates. Secondly we study, for the first time, the plate with free and sliding edge conditions. It will be shown that in these cases the axisymmetric fundamental mode assumed by previous authors may not be appropriate.

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2. Formulation

Consider a circular plate of radius R, density ρ and flexural rigidity D supported in the interior by a foundation of radius bR. We designate the outer unsupported region by subscript I and the inner foundation region by subscript II. For region I the (Kirchhoff) thin plate equation is

$$\nabla^4 w_{\mathrm{I}} - k^4 w_{\mathrm{I}} = 0, \tag{1}$$

where all lengths are normalized by R and $k^4 \equiv R^4\omega^2\rho/D$ is a parameter representing the frequency ω. For region II the equation is [1]

$$\nabla^4 w_{\mathrm{II}} - k^4 w_{\mathrm{II}} + \alpha^4 w_{\mathrm{II}} = 0, \tag{2}$$

where $\alpha^4 \equiv R^4 K/D$ represents the stiffness K of the foundation. It is seen from Eqs. (1) and (2) that, if the plate is fully supported, the frequency satisfies the relation

$$k^4 = k_0^4 + \alpha^4, \tag{3}$$

where k_0 is the frequency parameter when the foundation is absent. For partial foundations, Eq. (3) fails since the frequencies, stresses, displacements, etc. need to be reconciled between regions I and II. Let

$$w = u(r)\cos(n\theta), \tag{4}$$

where n is the number of nodal diameters. The general solution to Eq. (1) is a linear combination of the Bessel functions

$$u_{\mathrm{I}}(r) = C_1 J_n(kr) + C_2 Y_n(kr) + C_3 I_n(kr) + C_4 K_n(kr). \tag{5}$$

The general solution to Eq. (2), bounded at the center, is more complicated. If $k > \alpha$ the solution is

$$u_{\mathrm{II}}(r) = C_5 J_n(\hat{k}r) + C_6 I_n(\hat{k}r), \tag{6}$$

where $\hat{k} = (k^4 - \alpha^4)^{1/4}$. If $k = \alpha$ the solution is

$$u_{\mathrm{II}}(r) = C_5 r^n + C_6 r^{n+2}. \tag{7}$$

If $k < \alpha$ the solution is

$$u_{\mathrm{II}}(r) = C_5 \,\mathrm{Re}[J_n(\sqrt{\mathrm{i}}\tilde{k}r)] + C_6 \,\mathrm{Im}[J_n(\sqrt{\mathrm{i}}\tilde{k}r)], \tag{8}$$

where $\mathrm{i} \equiv \sqrt{-1}$ and $\tilde{k} = (\alpha^4 - k^4)^{1/4}$.

Since the plate is continuous, the matching conditions are

$$u_{\mathrm{I}}(b) = u_{\mathrm{II}}(b), \quad u'_{\mathrm{I}}(b) = u'_{\mathrm{II}}(b), \tag{9,10}$$

$$u''_{\mathrm{I}}(b) = u''_{\mathrm{II}}(b), \quad u'''_{\mathrm{I}}(b) = u'''_{\mathrm{II}}(b). \tag{11,12}$$

At the outer edge, the plate may be clamped,

$$u_{\mathrm{I}}(1) = 0, \quad u'_{\mathrm{I}}(1) = 0, \tag{13}$$

or simply supported,

$$u_{\mathrm{I}}(1) = 0, \quad u''_{\mathrm{I}}(1) + \nu u'_{\mathrm{I}}(1) = 0, \tag{14}$$

where v is the Poisson ratio (taken as 0.3 in this paper), or free,

$$u_I''(1) + v[u_I'(1) - n^2 u_I(1)] = 0,$$
$$u_I'''(1) - u_I'(1)[1 + v + n^2(2 - v)] + 3n^2 u_I(1) = 0, \tag{15}$$

or sliding (movable),

$$u_I'(1) = 0, \quad u_I'''(1) + u_I''(1) + n^2(3 - v)u_I(1) = 0. \tag{16}$$

For given b, n and α, Eqs. (9)–(12) and one set of Eqs. (13)–(16) give an exact characteristic equation for non-trivial solutions of the coefficients $C_1, \ldots, C_6$. The equation is then solved by simple bisection for the parameter k. We are interested in the fundamental or lowest frequency.

3. Results for the clamped and simply supported plate

For the plate with a clamped edge, Eq. (13) is used. The results are shown in Fig. 1 for α up to 20. It was found that the $n = 0$ axisymmetric mode gives the fundamental frequency. All three forms of Eqs. (6)–(8) are used. When $b = 0$, the foundation is absent and the frequency is governed by the clamped circular plate, i.e. $k = 3.19623$ or the first root of

$$J_0(k)I_1(k) + I_0(k)J_1(k) = 0. \tag{17}$$

When $b = 1$, the plate has full foundation support, and the frequency can be obtained from Eq. (3) with $k_0 = 3.19623$. For large α (very stiff foundation) the curves will be asymptotic to

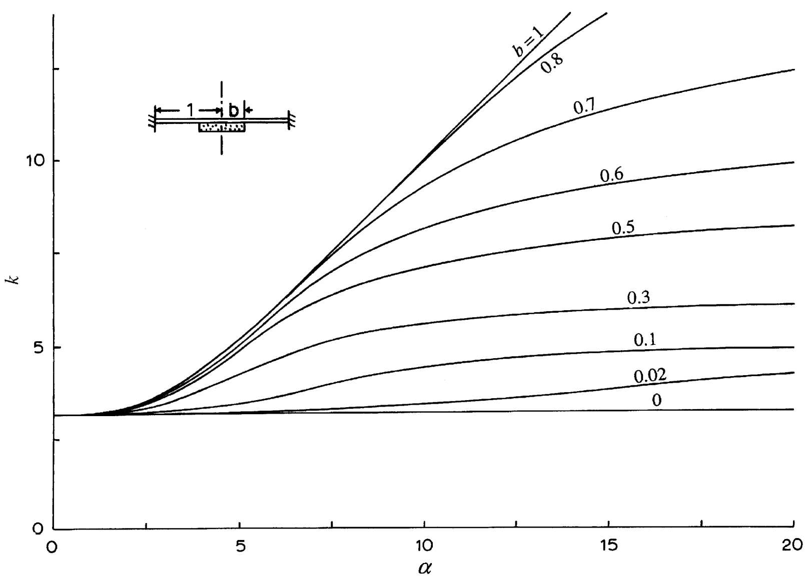

Fig. 1. Fundamental frequency of plate with clamped edge.

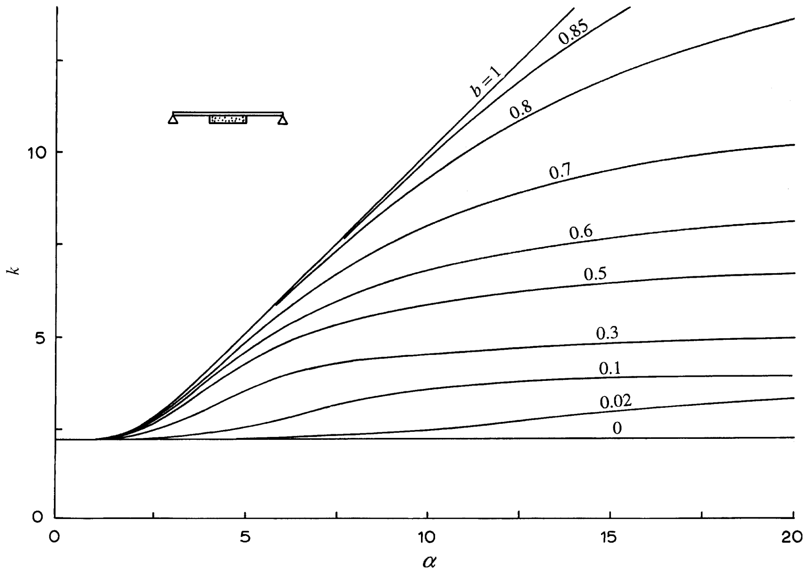

Fig. 2. Fundamental frequency of plate with simply supported edge.

Table 1
Comparison of exact values with approximate values from Ref. [3] (clamped edge)

	$b = 0.3$	$b = 0.3$	$b = 0.6$	$b = 0.6$
α	2.1147	3.1623	2.1147	3.1623
k (exact)	3.25573	3.46124	3.32706	3.73672
Ritz [3]	3.2558	3.4615	3.3275	3.7367
F.E. [3]	3.2558	3.4771	3.1257	3.6576

the clamped–clamped annulus. The exact characteristic equation was given by McLeod and Bishop [4].

Table 1 shows a comparison of our exact values with the values obtained by Laura et al. [3] by Ritz and finite element methods.

The simply supported edge conditions are given in Eq. (14). The results are shown in Fig. 2. Again the $n = 0$ axisymmetric mode prevails. When $b = 0$ the fundamental frequency is 2.22152 which is the first root of

$$\frac{J_1(k)}{J_0(k)} + \frac{I_1(k)}{I_0(k)} - \frac{2k}{1 - \nu} = 0. \tag{18}$$

When $b = 1$, Eq. (3) is used. The asymptotes can be found from an exact determinant given in Ref. [4]. Table 2 shows a comparison with the data from Laura et al. [3].

In both cases, the Ritz method seems to be accurate for the low values of the stiffness parameters considered.

Table 2
Comparison of exact values with approximate values from Ref. [3] (simply supported edge)

	$b = 0.3$	$b = 0.3$	$b = 0.6$	$b = 0.6$
α	2.1147	3.1623	2.1147	3.1623
k (exact)	2.33844	2.67274	2.51384	3.17204
Ritz [3]	2.339	2.677	2.514	3.1724
F.E. [3]	2.349	2.702	2.536	3.2249

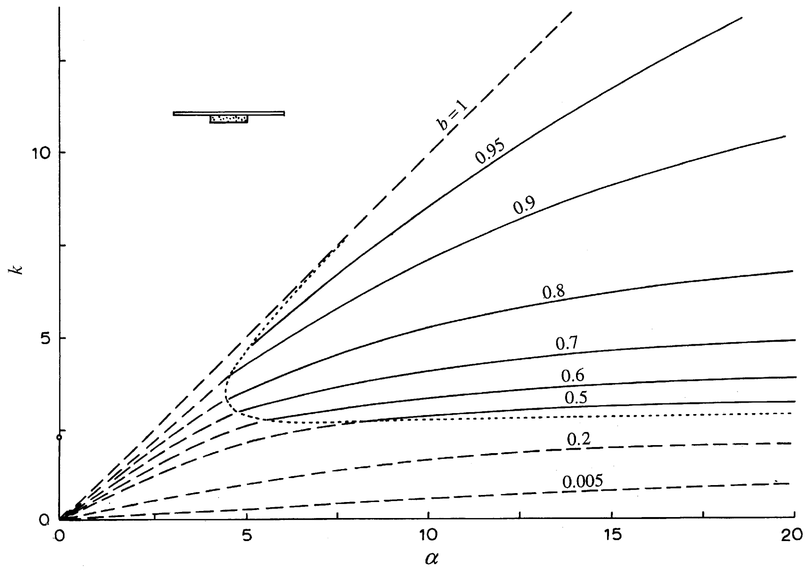

Fig. 3. Fundamental frequency of plate with free edge. Solid lines are from the $n = 0$ mode, dashed lines are from the $n = 1$ mode, and dotted line denote the switching of fundamental modes. Small circle represent the traditional $n = 2$ mode when $\alpha = 0$.

4. Results for plates with a free edge and with a sliding edge

The plate with a free edge is solely supported by the partial elastic foundation. If the foundation were absent, the fundamental frequency of a totally free plate is 2.3148 from the $n = 2$ mode [1], since the $n = 0$ and 1 modes have zero frequency and represent rigid motions. However, the effect of a foundation is to promote the $n = 0$ and 1 modes. Using Eq. (15) the exact characteristic equation is obtained and solved for the lowest frequency. Fig. 3 shows the results.

We found that all curves start from zero. From Eq. (3) the $b = 1$ curve is the straight line $k = \alpha$. The $n = 2$ mode is no longer the fundamental vibration mode. For small α ($\alpha < 4.45$) or small b ($b < 0.349$) the fundamental frequency is governed by the $n = 1$ mode. For larger α or b the fundamental frequency is given by the symmetric $n = 0$ mode. The locus where the modes switch depends on b and is shown by the dotted curve. If stiffness is infinite, the plate is equivalent to a

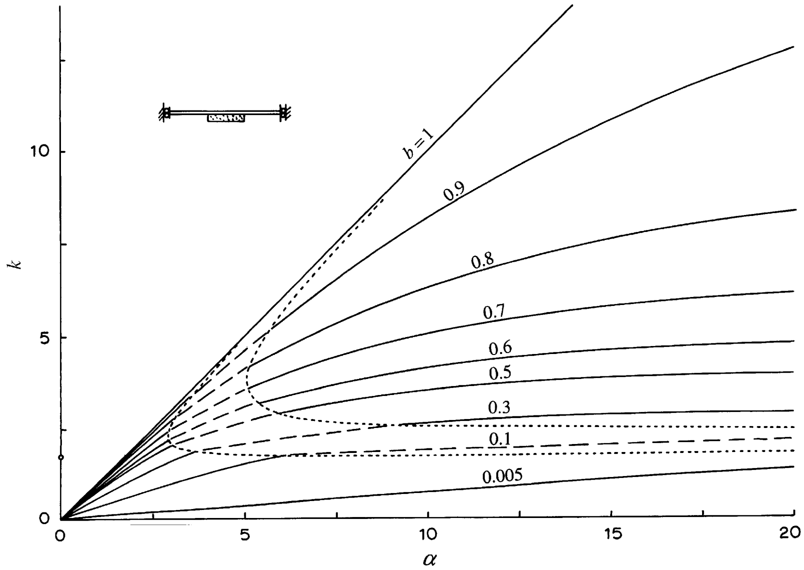

Fig. 4. Fundamental frequency of plate with sliding edge. Solid lines are from the $n = 0$ mode, dashed lines are from the $n = 1$ mode, and dotted lines denote the switching of fundamental modes. Small circle represent the traditional $n = 1$ mode when $\alpha = 0$.

clamped–free annulus, whose frequency values are tabulated by, e.g. Ref. [5]. Our computations show the switch from $n = 1$ to 0 mode when $\alpha \to \infty$ is at $b = 0.349$, $k = 2.814$.

The last case is the plate with a sliding or movable edge, where the slope and shear are zero on the boundary. If the foundation is absent, we find that the fundamental frequency is 1.75570 given by the $n = 1$ mode. However, the slightest foundation stiffness causes the fundamental frequency to be determined by the $n = 0$ mode. Fig. 4 shows for $0 < \alpha < 2.90$ that the plate vibrates with the $n = 0$ mode. Then, for given b and increasing α, the fundamental mode switches to the $n = 1$ mode and back to the $n = 0$ again at higher stiffness. Thus there exists a V-shaped corridor of $n = 1$ fundamental modes. When stiffness is infinite, we find the $n = 1$ mode occurs for $0 < b < 0.1177$ ($k < 2.4700$) and the $n = 0$ mode dominates for $b > 0.1177$.

5. Discussions

Using exact characteristic equations, the fundamental frequency is found for a partially supported plate for all four types of edge conditions. The approximate energy or finite element methods would have scaling difficulties when the radius of the supported region is very small or very close to unity.

For the plates with free and sliding edges, the fundamental frequencies are no longer related to the traditional $n = 2$ and 1 modes when the support is absent. The fundamental frequencies are complicated functions of the stiffness and the radius of the support corresponding to both $n = 0$ and 1 modes, as depicted in Figs. 3 and 4.

Is a partially supported plate equivalent to a stepped plate where two regions of different properties are similarly joined? The answer is no. For a stepped plate complete continuity of Eqs. (9)–(12) does not hold. A free stepped plate, like a free homogeneous plate, would have a fundamental mode with $n = 2$, unlike the $n = 0$ or 1 modes for partially supported plates.

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