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Short Communication

Comparison of two analytical methods used to calculate sound radiation from radial vibration modes of a thick annular disk

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Abstract

Two analytical solutions for acoustic radiation from the radial structural modes of a thick annular disk with free boundaries are described. The far-field modal sound pressure is calculated first by using the Rayleigh integral formula and then obtained by treating the radiating surfaces as two cylindrical radiators. Modal sound power, radiation efficiency and directivity predictions are confirmed by using a boundary element code. Measured frequency responses also support the propsed theory.

Introduction

Sound radiation from thin circular and annular disks has been examined by several investigators [1–5], with focus on either flexural vibration modes or rigid body piston motions. In such studies, sound radiation from the in-plane modes of a disk has been assumed to be negligible compared to that from the out-of-plane modes. But, if the thickness (h) of a disk is beyond the range of thin plate (shell) theory, radial vibration could generate sufficient sound, given proper structural excitation. In this communication, we develop two analytical solutions for acoustic radiation from the radial structural modes (with index q) of a thick annular disk that is described in the cylindrical coordinates (r, φ, z). Primary assumptions are as follows: (1) Disk is

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free at the inner ($r = b$) and outer ($r = a$) edges and the associated scattering effect is negligible. (2) Harmonic sound pressure amplitude (P) at the observation point ($\vec{r}_p$) in the free field is generated only by the modal radial velocities on $r = b$ and $r = a$ edges, and the z-direction surfaces do not contribute to P. (3) Vibration amplitudes of the radial surfaces due to radial modes (at natural frequency ω_q) are uniform in the z-direction. The far-field modal sound pressure is calculated first by using the Rayleigh integral formula and then obtained by treating the radiating surfaces as two cylindrical radiators of length h. Predictions are confirmed by using a commercial boundary element (BEM) code as well as by the frequency response measurements.

2. Sound radiation calculation methods

With reference to Fig. 1(a), $P(\vec{r}_p)$ in the far and free fields due to a vibrating structure with harmonic acceleration can be expressed by the Helmholtz integral equation [6].

$$P(\vec{r}_p) = -\int_{S_S} \left(P\frac{\partial g}{\partial \eta} + \rho_0 \ddot{U}(\vec{r}_s) g \right) \mathrm{d}S(\vec{r}_s). \tag{1}$$

Here, g is the free space Green's function, ρ_0 is the medium density, $\ddot{U}(\vec{r}_s)$is the surface acceleration at $\vec{r}_s$ and S_s is the source surface. The first and second terms in Eq. (1) represent the partial sound pressures generated at $\vec{r}_p$ by the surface pressure at $\vec{r}_s$ and surface acceleration at $\vec{r}_s$, respectively. If the field point is sufficiently far from the source ($k|\vec{r}_p| \gg 1$), one can express the amplitude of acoustic particle velocity as $P/\rho_0 c_0$, where $k = \omega/c_0$ is the acoustic wavenumber and c_0 is the medium sonic speed. Furthermore, since particle velocity is in-phase with the sound pressure in the far field, the sound intensity (I) at the same location can be uniquely defined as $I = P^2/2\rho_0 c_0$. The sound power W from a vibrating structure can be found by integrating the far-field sound intensity over the control surface S_v that surrounds the source. The acoustic radiation resistance $\Re$ is then obtained from W and spatially averaged mean square radial velocity $\langle \dot{u}^2 \rangle_{t,s}$ as follows [6], where σ is the acoustic radiation efficiency, A_s is the area of the radiator, and $\langle\rangle_{t,s}$ is a

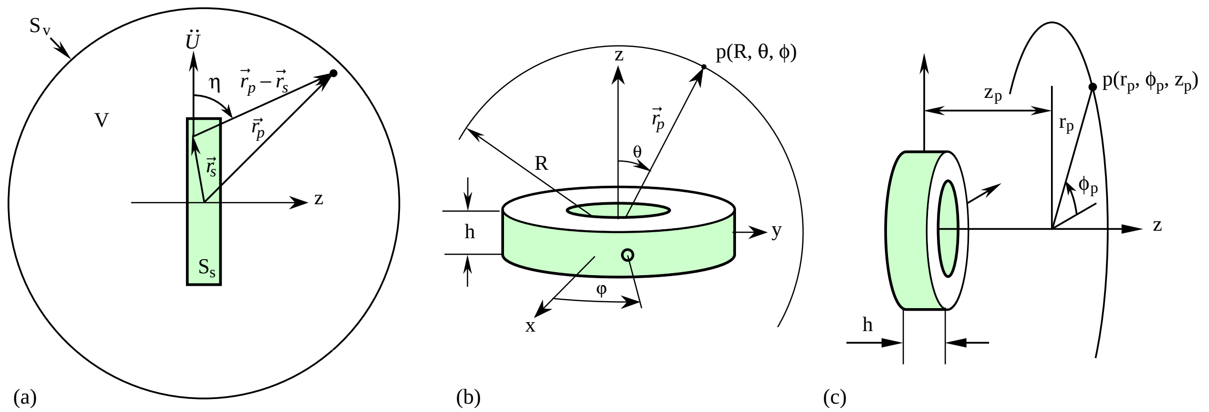

Fig. 1. Sound radiation from the radial vibration modes of a thick annular disk. (a) spherical radiation from unbaffled disk; (b) radiation expressed in the spherical coordinate system; (c) radiation expressed in the cylindrical coordinate system.

temporal and spatial average operator:

$$\Re = \frac{W}{\langle \dot{u}^2 \rangle_{t,s}} = \sigma \rho_0 c_0 A_s. \quad (2)$$

2.1. Method I: Rayleigh integral approach

Without restrictions on the source configuration and frequency range, the surface pressure distribution must be obtained through a numerical calculation. If the frequency range is, however, restricted to the short-wavelength limit, the solution to the original Helmholtz integral equation can be circumvented. With this assumption, Eq. (1) is simplified to the following expression that can be solved without using a numerical method [6]; refer to Fig. 1(a) for the configuration.

$$P(\vec{r}_p) = \frac{\rho_0 c k}{4\pi} - \int_{S_S} \frac{e^{ik|\vec{r}_p - \vec{r}_s|} \ddot{U}(\vec{r}_s)}{|\vec{r}_p - \vec{r}_s|} (1 + \cos\eta) \, dS(\vec{r}_s). \quad (3)$$

In our study, sound radiation from the qth radial mode of a thick annular disk is calculated by assuming the unbaffled condition. Since we assume that the acceleration amplitude is constant in the z-direction, normal accelerations on the outer (O) and inner (I) radial surfaces are expressed as

$$\begin{aligned} \ddot{U}_{qO}(\varphi) &= |\ddot{u}_{qO}| \cos(q\varphi) = -\omega_q^2 |u_{qO}| \cos(q\varphi), \\ \ddot{U}_{qI}(\varphi) &= |\ddot{u}_{qI}| \cos(q\varphi) = -\omega_q^2 |u_{qI}| \cos(q\varphi). \end{aligned} \quad (4)$$

If the sound-generating surfaces are discretized into small elements dS of constant $\ddot{u}_q, P(\vec{r}_p)$ from the structure can be easily calculated using Eq. (3). For the annular disk case of Fig. 1(b), $dS(\vec{r}_s)$ in Eq. (3) is expressed by $dS(\vec{r}_s) = a\,d\varphi\,dz$ and $dS(\vec{r}_s) = b\,d\varphi\,dz$ for the outer and inner radial surfaces, respectively. The total sound pressure P at $\vec{r}_p$ can be calculated by integrating the sound pressure generated by each element over the entire source surface. In our study, numerical integration is used to solve for the sound pressure distribution. The size of a dS element should be selected according to the frequency of vibration. If the characteristic dimension of the element is larger than π/k, P will have some errors and consequently acoustic radiation properties including the directivity patterns will be distorted. In our study, observation positions are defined in the spherical coordinates (R, ϕ, θ) by a group of points having equal angular increments ($\Delta\phi$, $\Delta\theta$) on a sphere that is centered at the disk center. With computed modal $P_q(\vec{r}_p)$ data, the modal directivity function $D_q(\theta, \phi)$ at ω_q is defined as follows, where $R = |\vec{r}_p|$ is the radius of sphere on which observation positions are defined:

$$P_q(R, \theta, \phi) = \frac{e^{ik_q R}}{R} D_q(\theta, \phi). \quad (5)$$

From the far-field approximation, the modal power W_q for the qth radial mode is then calculated from modal pressures on a sphere surrounding the disk by using the following equation, where θ and ϕ are the cone and azimuthal angles of the observation positions:

$$W_q = \langle I_{sq} S \rangle_s = \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} \frac{P_q^2}{\rho_0 c_0} R^2 \sin\theta \, d\theta \, d\phi. \quad (6)$$

Modal acoustic radiation resistance $\Re_q$ is calculated using Eqs. (2) and (6), where

$$\left\langle |\dot{u}_q|^2 \right\rangle_{t,s} = \frac{1}{4\pi h(a+b)} \int_{-h/2}^{h/2} \int_0^{2\pi(a+b)} \dot{U}_q^2 \, \mathrm{d}l \, \mathrm{d}z. \tag{7}$$

Based on Eq. (2), the modal radiation efficiency σ_q of an annular disk is determined as follows, where $2\pi h(a+b)$ is the total area of radiating surfaces.

$$\sigma_q = \frac{\Re_q}{2\rho_0 c_0 \pi h(a+b)}. \tag{8}$$

2.2. Method II: cylindrical radiator

Outer and inner radial surfaces of the annular disk are treated as two separate cylindrical radiators of identical length h that have uniform surface acceleration amplitudes in the thickness direction (z in Fig. 1(c)). In our approach, the far-field sound pressure is calculated based on the procedure proposed by Junger and Feit [6], along with the approximation for an unbaffled cylindrical radiator in terms of a baffled one as suggested by Sandman [7]. Junger and Feit [6] analyzed a cylindrical radiator of length h that has arbitrary acceleration amplitude distribution $Z(z)$ in the z-direction and a sinusoidal distribution ($\cos n\varphi$) in the circumferential direction φ. Surface acceleration on the radial surface is expressed in the cylindrical coordinate system (excluding time dependency) as

$$\ddot{U}(z,\varphi) = |\ddot{u}| Z(z) \cos n\varphi. \tag{9}$$

Application of the Fourier transform [$\Im$] in the z-direction to the Helmholtz equation expressed in cylindrical coordinates, followed by the inverse Fourier transform and the stationary phase approximation, yields P due to the vibration of Eq. (9) as

$$P(R,\theta,\phi) = \frac{\rho_0 \mathrm{e}^{\mathrm{i}kR}}{\pi k R \sin\theta} |\ddot{u}| \frac{\tilde{Z}(k_z)(-\mathrm{i})^{n+1}}{H_n'(kr\sin\theta)} \cos n\phi. \tag{10}$$

Next, consider a thick annular disk case in which the modal surface accelerations of two radial surfaces are given by Eq. (4). In this case, variation in the z-direction can be expressed via a square pulse (rectangular) function. Accordingly, $Z(z)$ can be expressed as Eq. (11) and the Fourier transform of this equation is obtained by Eq. (12):

$$Z(z) = \begin{cases} 1, & |z| < h/2, \\ 0, & |z| > h/2, \end{cases} \tag{11}$$

$$\tilde{Z}(k_z) = \mathrm{Im}[Z(z)] = 2\frac{\sin(k_z h/2)}{k_z} = h\frac{\sin(k_z h/2)}{k_z h/2} = h\,\mathrm{Sinc}(k_z h/2), \tag{12}$$

where the Sinc function is defined as $\mathrm{Sinc}(x) = \sin(x)/x$. Sound pressure P_{qO} from the outer radial surface and P_{qI} from the inner radial surfaces are, respectively, generated by diverging and converging waves, respectively. These are expressed by the Hankel functions (H) of the first and the second kinds respectively [8]. Therefore, using Eqs. (10) and (12), we get

$$P_{qO}(R,\theta,\phi) = \frac{\rho_0 e^{ik_q R}}{\pi k_q R \sin\theta} |\ddot{u}_{qO}| h \frac{\mathrm{Sinc}(k_q \sin\theta h/2)(-i)^{q+1}}{H_q^{1'}(k_q a \sin\theta)} \cos q\phi, \quad (13a)$$

$$P_{qI}(R,\theta,\phi) = \frac{\rho_0 e^{ik_q R}}{\pi k_q R \sin\theta} |\ddot{u}_{qI}| h \frac{\mathrm{Sinc}(k_q \sin\theta h/2)(-i)^{q+1}}{H_q^{2'}(k_q b \sin\theta)} \cos q\phi. \quad (13b)$$

Also, the total modal sound pressure is given by the sum

$$P_q(R,\theta,\phi) = P_{qI}(R,\theta,\phi) + P_{qO}(R,\theta,\phi). \quad (14)$$

Other modal radiation properties such as W_q, $\Re_q$ and σ_q are calculated based on $P_q(R, \theta, \phi)$ using Eqs. (2), (6) and (8).

3. Conclusion

Modal radiation properties such as $D_q(\theta,\phi)$, W_q, and σ_q of the sample annular disk (with $a = 151.5$ mm, $b = 87.5$ mm, $h = 31.5$ mm) are obtained by using the two analytical methods of Section 2. Further, the same radiation properties are calculated with an uncoupled, direct, exterior, and unbaffled BEM analysis [9]. In the computational model, 4400 nodes and 6600 structural elements are used to describe the disk. Also, 6146 acoustic field points and 6144 elements are defined on the sphere surrounding the disk. The center of this sphere coincides with the disk center. This BEM model is excited by the structural accelerations at $r = a$ and $r = b$ surfaces that are numerically calculated from a forced vibration analysis. Results are illustrated in

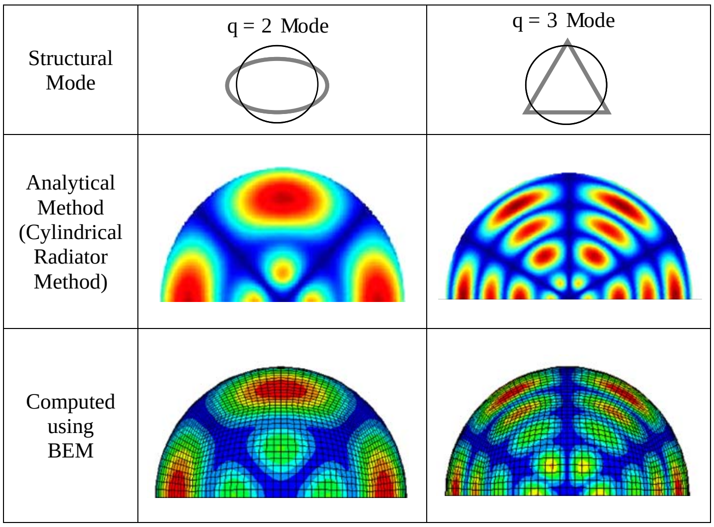

Fig. 2. Comparison of the directivity patterns for $q = 2$ and 3 modes.

Table 1
Comparison of modal acoustic power and radiation efficiency levels

Radiation property	Mode q	Computed using BEM	Analytical methods	
			Rayleigh integral	Cylindrical radiator model
W_q (dB re 1 pW)	2	66.5	66.8	66.0
	3	67.5	67.2	67.5
σ_q (dB re 1)	2	−4.0	−2.3	−4.0
	3	−1.0	−2.2	−2.0

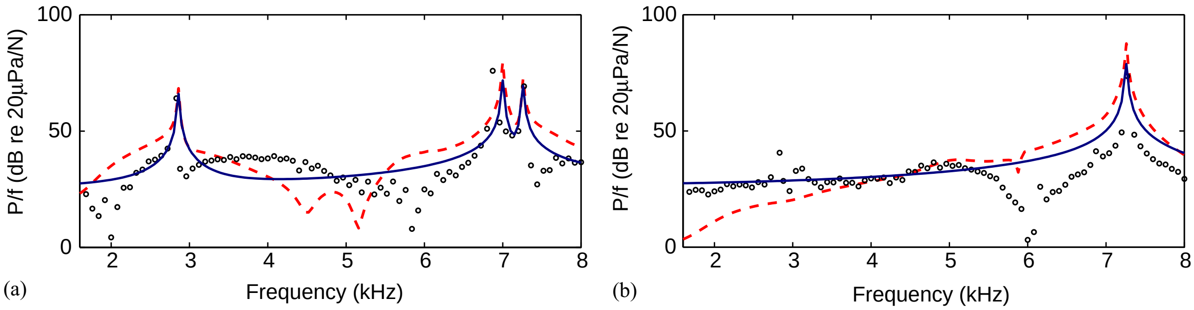

Fig. 3. Acoustic frequency response functions $P/f(\omega)$, given force excitation in the radial direction. (a) $\theta = \pi/2$ and $\phi = 0$; (b) $\theta = 0$ and $\phi = 0$. Key: ◦ ◦ ◦, measured; - - -, computed using BEM; —, analytical calculation.

Fig. 2 where $P_q(\theta, \phi)$ results are compared with computed values for the $q = 2$ and $q = 3$ modes. The W_q and σ_q predictions for the first two radial modes (Table 1) compare well with the BEM code and measured results. Since predictions are within 1 dB for W_q and 2 dB for σ_q, one may conclude that analytical solutions are sufficiently accurate in predicting W_q and σ_q, though the cylindrical radiator method appears to be better in predicting the modal radiation. When the proposed theory for radial modes is combined with the analytical radiation solutions for out-of-plane flexural modes [10] as well as their interactions, the total sound radiation from a thick annular disk given a multidimensional force excitation can be formulated in an efficient manner [11]. For instance, examine the results of Fig. 3 where the theoretical solutions of this paper are used to predict the acoustic frequency response functions, $P/f(\omega)$, given harmonic force excitation in the radial direction; calculations are based on the multi-modal sound radiation concepts [11]. Analytical formulations agree well with computed and measured data.

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