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Short Communication

Analytical study on a Duffing-harmonic oscillator

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Conservative nonlinear oscillatory systems are often modeled by potentials having a rational form for the potential energy [1–3], which lead to the equations of motion:

$$\frac{d^2y}{d\tau^2}+\frac{\alpha y^3}{\beta+\gamma y^2}=0. \tag{1}$$

Here, y is the displacement. τ is the time. α, β and γ are non-negative parameters. It should be noted that Eq. (1) was named the Duffing-harmonic oscillator by Mickens [3]. Defining $y=\sqrt{\beta/\gamma}x$ and $\tau=\sqrt{\gamma/\alpha}t$, Eq. (1) is reduced to the following non-dimensional equation (which is free of non-essential parameters given by Mickens [3]):

$$\ddot{x}+x^3(1+x^2)^{-1}=0. \tag{2}$$

Here, overdots denote differentiation with respect to time, t. For small x, the equation of motion (2) is that of a Duffing-type nonlinear oscillator (i.e., $\ddot{x}+x^3\cong 0$), while for large x, the equation of motion (2) approximates that of a linear harmonic oscillator (i.e., $\ddot{x}+x\cong 0$). Hence, Eq. (2) is referred as the Duffing-harmonic oscillator [3]. The restoring force in the equation is the same for both negative and positive amplitudes.

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By applying the method of harmonic balance [4] with just the first harmonic present:

$$x(t) \cong x_0 \cos(\omega t), \tag{3}$$

which satisfies the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = 0, \tag{4}$$

the angular frequency, ω for Eq. (2) is obtained as [3]

$$\omega^2 = \frac{3}{4} x_0^2 \left(1 + \frac{3}{4} x_0^2\right)^{-1}. \tag{5}$$

It is interesting to note that $\frac{3}{4}x_0^2$ is the squared angular frequency for the equation of motion: $\ddot{x} + x^3 = 0$, which is obtained by applying the method of harmonic balance in lowest order harmonics (3). The exact squared angular frequency for this equation is $\phi^2 x_0^2$. Here, $\phi = \pi\{2F(1/\sqrt{2}, \pi/2)\}^{-1} \cong 0.8472$, and $F(1/2, \pi/2) \cong 1.8541$, is the complete elliptic integral of the first kind. Replacing $\frac{3}{4}x_0^2$ in Eq. (5) by $\phi^2 x_0^2$, the conjectured exact angular frequency (ω) of Mickens [3] for Eq. (2) is:

$$\omega^2 = \phi^2 x_0^2 (1 + \phi^2 x_0^2)^{-1}. \tag{6}$$

In Ref. [3], two non-standard finite difference schemes were constructed and the explicit scheme was used to numerically integrate the equation of motion (2). It is noted from Ref. [3] that further work on the Duffing-harmonic equation will center on trying to prove the relation conjectured in Eq. (6). Infact, the frequency–amplitude relations (5) and (6) provided by Mickens [3] are approximate. The exact angular frequency estimates to Eq. (2) is possible only through numerical integration.

The objective of this letter is to present an approximate frequency–amplitude relation close to the exact, assuming a single-term solution (3) and following the Ritz procedure [5]. This letter also presents the periodic solution for Eq. (2) without numerical integration, which is achieved by applying a rational harmonic balance approximation to the equation of motion (2) as in Refs. [6,7].

Table 1
Comparison of the angular frequency (ω) estimates of Eq. (2) for the specific amplitude, x_0

Amplitude x_0	Mickens [3]		Present study	
	Eq. (5)	Eq. (6)	Eq. (7)	Exact solution[a]
0.1	0.0863	0.0844	0.0862	0.0844
1	0.6547	0.6464	0.6436	0.6368
10	0.9934	0.9931	0.9910	0.9909

[a]Results from the exact angular frequency–amplitude relation of Ref. [8].

An approximate frequency–amplitude relation obtained to Eq. (2) assuming a single-term solution (3) and following the Ritz procedure [5] is:

$$\omega^2 = 1 + \left(\frac{2}{x_0^2}\right)\left\{\frac{1}{\sqrt{1+x_0^2}} - 1\right\}, \tag{7}$$

which satisfies the limits

$$x_0 \text{ is small}: \quad [\omega(x_0)]^2 = \frac{3}{4}x_0^2 + O(x_0^4), \quad x_0 \text{ large}: \quad [\omega(x_0)]^2 = 1 + O\left(\frac{1}{x_0^2}\right).$$

The exact angular frequency relation to Eq. (2) was derived in Ref. [8], and the integral in the derived relation was numerically evaluated following the general procedure of Ref. [9]. Table 1 gives comparison of the angular frequency estimates for the specified amplitudes, x_0, from the frequency–amplitude relations (5), (6) and (7) as well as with those obtained from the exact angular frequency relation of Ref. [8] through numerical integration using a ten-point Gauss rule.

It is evident from the results presented in the Table 1 that Eq. (6) gives the exact solution as expected for small x_0. Because, for small x, Eq. (2) is that of the Duffing-type nonlinear oscillator ($\ddot{x} + x^3 \cong 0$) for which the exact angular frequency, $\omega = \phi x_0$. This result was utilized while

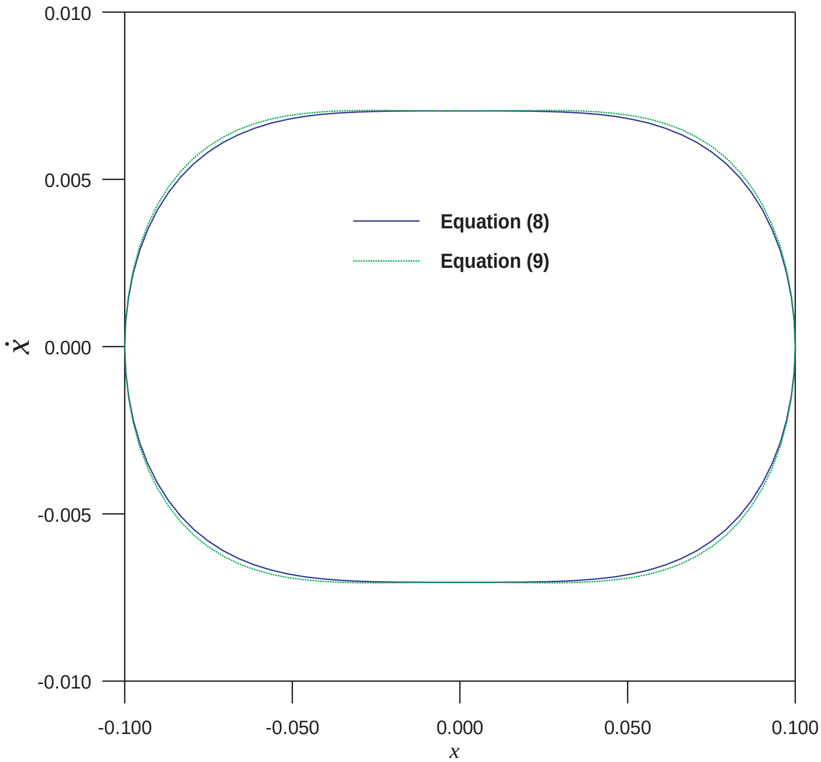

Fig. 1. Phase-space curve ($\dot{x}$ versus x curve) of Eq. (2) for the amplitude $x_0 = 0.1$.

constructing the frequency–amplitude relation (6). However, Eq. (7) gives a good estimate for the angular frequency.

Multiplying Eq. (2) by $2\dot{x}$ and using the initial conditions (4), after integration one obtains the energy relation:

$$(\dot{x})^2 = I(x_0) - I(x), \tag{8}$$

where $I(x) = x^2 - \ln(1 + x^2)$. Radhakrishan et al. [8] have examined the uniqueness of angular frequency using harmonic balance from the equation of motion (2) and the energy relation (8). Mickens and Semwogerere [6] have recommended a rational function,

$$x(t) = A\cos(\omega t)(1 + B\cos(2\omega t))^{-1}, \tag{9}$$

for the nonlinear one-dimensional oscillator differential equation. The constants A and B in Eq. (9) are derived here using the initial conditions (4) and the values $x = 0$ and $\dot{x} = -\sqrt{I(x_0)}$ at the quarter-period (i.e., $t = T/4 = \pi/(2\omega)$) from Eq. (8). These are:

$$A = 2\sqrt{I(x_0)}x_0(\sqrt{I(x_0)} + \omega x_0)^{-1} \quad \text{and} \quad B = (\sqrt{I(x_0)} - \omega x_0)(\sqrt{I(x_0)} + \omega x_0)^{-1}.$$

Figs. 1–3 show the phase-space curves ($\dot{x}$ versus x curve) of the equation of motion (2) generated from Eqs. (8) and (9) for amplitudes, $x_0 = 0.1, 1$ and 10. The phase-space curve generated from Eq. (9) is close to that of the exact curve from Eq. (8).

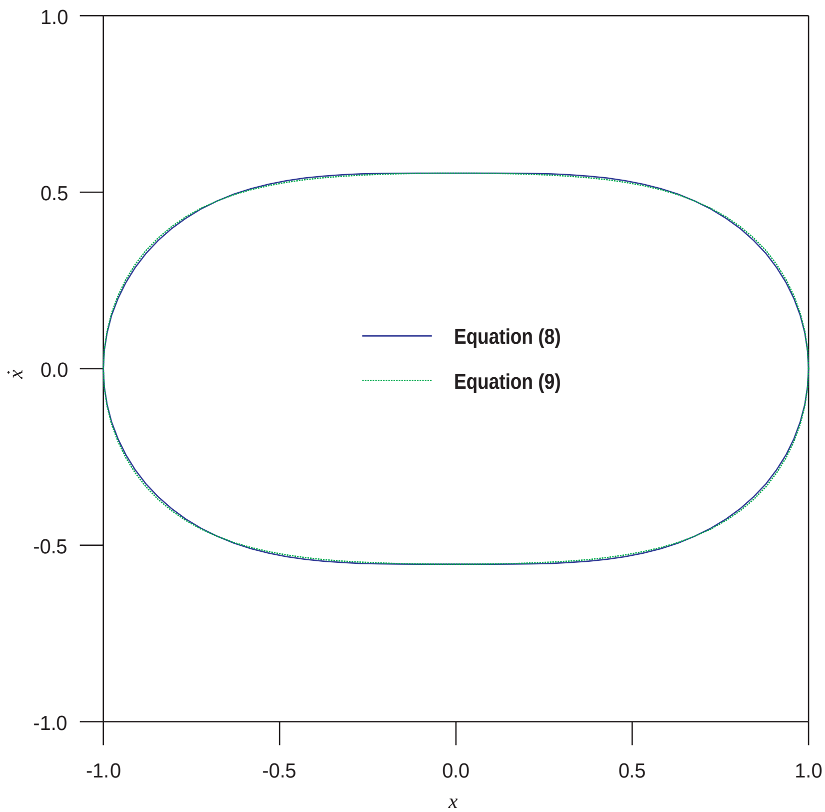

Fig. 2. Phase-space curve ($\dot{x}$ versus x curve) of Eq. (2) for the amplitude $x_0 = 1$.

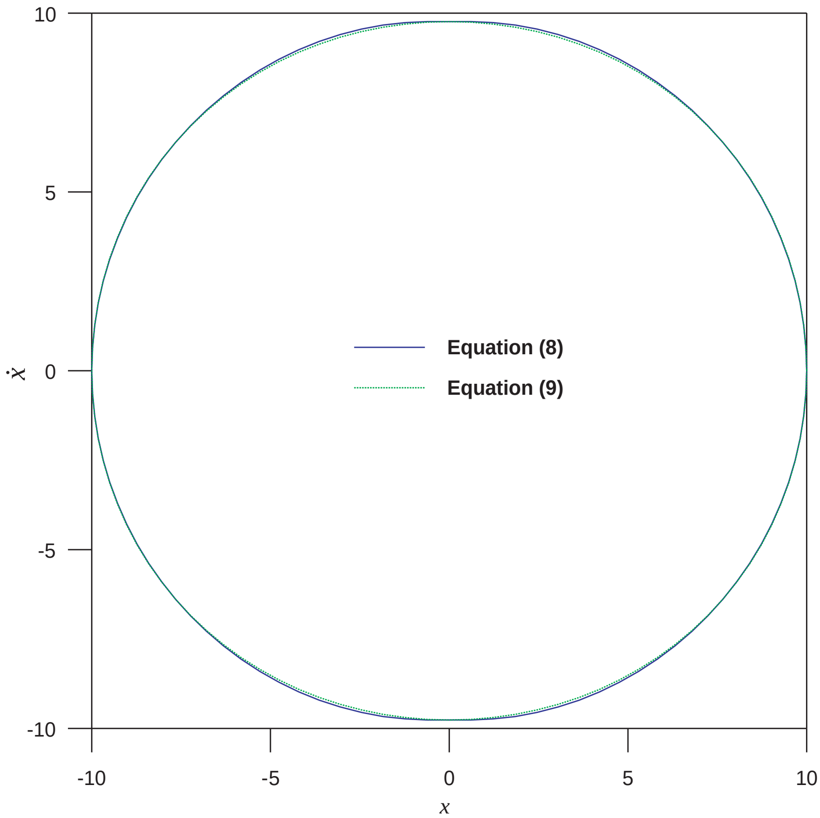

Fig. 3. Phase-space curve ($\dot{x}$ versus x curve) of Eq. (2) for the amplitude $x_0 = 10$.

In summary, this letter provides an approximate frequency–amplitude relation (7) for the equation of motion (2), which gives the results close to the exact solution. It also provides accurate periodic solution (9) to Eq. (2) without numerical integration by applying a rational harmonic balance approximation.

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