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Dynamics of coupled linear and essentially nonlinear oscillators with substantially different masses

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Abstract

The dynamics of a linear oscillator, coupled to an essentially nonlinear attachment of substantially lower mass, is investigated. The essential (nonlinearizable) nonlinearity of the attachment enables it to resonate with the oscillator, leading to energy pumping phenomena, e.g., passive, almost irreversible transfer of energy from the substructure to the attachment. Feasibility of this process for possible applications depends on relative mass of the attachment, the obvious goal being to minimize it while preserving the efficiency of the pumping. Two different models of the attachment coupled to the main single-degree-of-freedom body are proposed and analyzed both analytically and numerically. It is demonstrated that efficient energy pumping may be obtained for a rather small value of the attachment mass. Two mechanisms of energy pumping in the system under consideration are revealed. The first one is similar to previously studied resonance capture; a novel analytic framework allowing explicit account of the damping is proposed. The second mechanism is related to nonresonant excitation of high-frequency vibrations of the attachment. Both mechanisms are demonstrated numerically for a model consisting of a linear chain with a nonlinear attachment.

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1. Introduction

In this work we study the dynamics of a linear oscillator weakly coupled to a local nonlinear attachment with small mass, possessing essential stiffness nonlinearity. It was shown recently [1–4] that under certain conditions this type of essentially nonlinear attachment can passively absorb energy from a linear nonconservative (damped) structure, in essence, acting as nonlinear energy sink. Then, *energy pumping* from the linear structure to the attachment occurs, namely, almost one-way, irreversible transfer of energy. More exactly, the major part of the energy of the system is transferred to the attachment within a time scale less than the characteristic time of energy damping [1]. In previous studies of two degrees of freedom (2dof) systems the masses of the oscillators have been assumed equal. The corresponding phenomenon in systems where the attachment has relatively small mass has not been considered before and is the primary objective of the present paper.

As discussed in Ref. [3] and experimentally observed in Ref. [5], energy pumping from the linear nonconservative structure to the attachment may occur due to *resonance capture*. This is a transient dynamical phenomenon that has been theoretically studied in previous works [6–11]. It occurs (among other types of dynamical systems) in coupled nonconservative oscillators and leads to transient capture of the dynamical flow on a resonance manifold of the system. In the above-mentioned papers it was demonstrated that the physics of the energy pumping/resonance capture in the nonconservative systems under consideration could be understood and interpreted by studying the topological structure of the branches of nonlinear free periodic solutions (nonlinear normal modes (NNMs) [12]) of the corresponding conservative system obtained when all damping forces are removed.

Generally, it is correct also for the system with light attachment, although the picture of the NNMs is somewhat different; it will be demonstrated that explicit account of damping terms reveals new important qualitative features of dynamical behavior of the system. Besides, some new effects related to the energy pumping are described that cannot be treated in the framework of the *resonance capture* approach. It will also be demonstrated that the regimes revealed for 2dof system with light nonlinear attachment are relevant also for the case of linear chain with nonlinear attachment, which is a natural generalization of the system considered.

2. The system with linear coupling

The most straightforward way for design of the system with light strongly nonlinear attachment is to use relatively weak linear coupling between the oscillators along with strongly nonlinear ground spring of the attachment. The equations describing such a system are written as

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}t^2} + \varepsilon^2\lambda_1\frac{\mathrm{d}u_1}{\mathrm{d}t} + u_1 + c\varepsilon^2(u_1 - u_2) = 0$$
$$\varepsilon\frac{\mathrm{d}^2u_2}{\mathrm{d}t^2} + \varepsilon^2\lambda_2\frac{\mathrm{d}u_2}{\mathrm{d}t} + 8\varepsilon u_2^3 + c\varepsilon^2(u_2 - u_1) = 0 \qquad (1)$$

where $u_{1,2}$ are the displacements of the oscillators, the mass of the first oscillator is supposed to be unity, the mass of the second one is $\varepsilon \ll 1$, $\varepsilon\lambda_{1,2}$ are the coefficients of the linear damping for both oscillators, $c\varepsilon^2$ is the coupling coefficient. The stiffness of the first oscillator was chosen to be unity and the stiffness of the nonlinear oscillator was chosen to be equal to 8ε. The latter propositions are of no physical significance, since proper rescaling of time and amplitudes may change them. The relative orders of the other terms were chosen in order to ensure the possibility of nontrivial behavior.

The standard framework for analysis of the resonance capture [13] involves transition to complex variables [14]:

$$\psi_1 = \dot{u}_1 + \mathrm{i}u_1, \quad \psi_2 = \dot{u}_2 + \mathrm{i}u_2 \tag{2}$$

Initial system (1) thus is rewritten as

$$\begin{aligned} &\dot{\psi}_1 - \mathrm{i}\psi_1 + \frac{\varepsilon^2\lambda_1}{2}(\psi_1 + \psi_1^*) + \frac{\mathrm{i}c\varepsilon^2}{2}(\psi_2 - \psi_2^* - \psi_1 + \psi_1^*) = 0 \\ &\varepsilon\left(\dot{\psi}_2 - \frac{\mathrm{i}}{2}(\psi_2 + \psi_2^*)\right) + \frac{\varepsilon^2\lambda_2}{2}(\psi_2 + \psi_2^*) + \mathrm{i}\varepsilon(\psi_2 - \psi_2^*)^3 - \frac{\mathrm{i}c\varepsilon^2}{2}(\psi_2 - \psi_2^* - \psi_1 + \psi_1^*) = 0 \end{aligned} \tag{3}$$

We introduce the change of variables related with "fast" time scale

$$\psi_k = \varphi_k \exp(\mathrm{i}t), \quad k = 1, 2$$

System (3) is reduced to the form,

$$\begin{aligned} &\dot{\varphi}_1 + \frac{\varepsilon^2\lambda_1}{2}(\varphi_1 + \varphi_1^* \exp(-2\mathrm{i}t)) + \frac{\mathrm{i}c\varepsilon^2}{2}(\varphi_2 - \varphi_2^* \exp(-2\mathrm{i}t) - \varphi_1 + \varphi_1^* \exp(-2\mathrm{i}t)) = 0 \\ &\dot{\varphi}_2 + \varepsilon\delta\left[\frac{\mathrm{i}}{2}(\varphi_2 - \varphi_2^* \exp(-2\mathrm{i}t)) + \mathrm{i}(\varphi_2 \exp(\mathrm{i}t) - \varphi_2^* \exp(-\mathrm{i}t))^3 \exp(-\mathrm{i}t)\right] \\ &\quad + \frac{\varepsilon\lambda_2}{2}(\varphi_2 + \varphi_2^* \exp(-2\mathrm{i}t)) - \frac{\mathrm{i}c\varepsilon}{2}(\varphi_2 - \phi_2^* \exp(-2\mathrm{i}t) - \varphi_1 + \varphi_1^* \exp(-2\mathrm{i}t)) = 0 \end{aligned} \tag{4}$$

The terms $\mathrm{i}/2(\varphi_2 - \varphi_2^* \exp(-2\mathrm{i}t))$ and $\mathrm{i}(\varphi_2 \exp(\mathrm{i}t) - \varphi_2^* \exp(-\mathrm{i}t))^3 \exp(-\mathrm{i}t)$ are not small individually, but their sum should be of order ε to provide the possibility of the resonance motion. This requirement means that the phase trajectory of the system is adjacent to the resonance surface and the distance between them is of order ε. This fact is formalized by introducing the bookkeeping coefficient $\delta = 1/\varepsilon$ which is considered to be of order unity in the forthcoming asymptotic analysis.

Multiple scales are introduced with the help of the following change of variables:

$$\begin{aligned} &\varphi_k = \varphi_{k0} + \varepsilon\varphi_{k1} + \varepsilon^2\varphi_{k2} + \cdots \quad k = 1, 2 \\ &\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1} + \varepsilon^2\frac{\partial}{\partial\tau_2} + \cdots \\ &\tau_0 = t, \tau_1 = \varepsilon t, \tau_2 = \varepsilon^2 t \end{aligned} \tag{5}$$

The combination of Eqs. (4) and (5) allows the asymptotic analysis of the system. The terms proportional to ε^0 constitute the zero approximation of system (4):

$$\frac{\partial\varphi_{10}}{\partial\tau_0} = 0 \Rightarrow \varphi_{10} = \varphi_{10}(\tau_1, \tau_2, \ldots)$$
$$\frac{\partial\varphi_{20}}{\partial\tau_0} = 0 \Rightarrow \varphi_{20} = \varphi_{20}(\tau_1, \tau_2, \ldots) \quad (6)$$

The equations of order ε^1 are written as

$$\frac{\partial\varphi_{10}}{\partial\tau_1} + \frac{\partial\varphi_{11}}{\partial\tau_0} = 0$$
$$\frac{\partial\varphi_{20}}{\partial\tau_1} + \frac{\partial\varphi_{21}}{\partial\tau_0} + \delta\left[\frac{\mathrm{i}}{2}(\varphi_{20} - \varphi_{20}^{*}\exp(-2\mathrm{i}\tau_0)) + \mathrm{i}(\varphi_{20}^{3}\exp(2\mathrm{i}\tau_0) - 3|\varphi_{20}|^2\varphi_{20}\right.$$
$$\left. + 3|\varphi_{20}|^2\varphi_{20}^{*}\exp(-2\mathrm{i}\tau_0) - \varphi_{20}^{*^3}\exp(-4\mathrm{i}\tau_0))\right] + \frac{\lambda_2}{2}(\varphi_{20} + \varphi_{20}^{*}\exp(-2\mathrm{i}\tau_0))$$
$$- \frac{\mathrm{i}c}{2}(\varphi_{20} - \phi_{20}^{*}\exp(-2\mathrm{i}\tau_0) - \varphi_{10} + \varphi_{10}^{*}\exp(-2\mathrm{i}\tau_0)) = 0 \quad (7)$$

The condition of absence of secular terms and account of the first equation of Eq. (6) allow immediate solution of the first equation of Eq. (7). The solution is written as

$$\varphi_{10} = \varphi_{10}(\tau_2, \tau_3, \ldots), \quad \varphi_{11} = \varphi_{11}(\tau_1, \tau_2, \tau_3, \ldots) \quad (8)$$

The second equation of Eq. (7) implies that secular terms with respect to τ_0 will be absent in the expression for φ_{21} if the following condition holds:

$$\frac{\partial\varphi_{20}}{\partial\tau_1} + \delta\left[\frac{\mathrm{i}}{2}\varphi_{20} - 3\mathrm{i}|\varphi_{20}|^2\varphi_{20}\right] + \frac{\lambda_2}{2}\varphi_{20} - \frac{\mathrm{i}c}{2}(\varphi_{20} - \varphi_{10}) = 0 \quad (9)$$

The equation for the first oscillator for order ε^2 is written as

$$\frac{\partial\varphi_{10}}{\partial\tau_2} + \frac{\partial\varphi_{11}}{\partial\tau_1} + \frac{\partial\varphi_{12}}{\partial\tau_0} + \frac{\lambda_1}{2}(\varphi_{10} + \varphi_{10}^{*}\exp(-2\mathrm{i}\tau_0))$$
$$+ \frac{\mathrm{i}c}{2}(\varphi_{20} - \varphi_{20}^{*}\exp(-2\mathrm{i}\tau_0) - \varphi_{10} + \varphi_{10}^{*}\exp(-2\mathrm{i}\tau_0)) = 0 \quad (10)$$

With account of Eq. (8) it is easy to conclude that the necessary condition for absence of the secular terms with respect to τ_0 may be expressed as

$$\frac{\partial\varphi_{10}}{\partial\tau_2} + \frac{\lambda_1}{2}\varphi_{10} + \frac{\mathrm{i}c}{2}(\varphi_{20} - \varphi_{10}) = 0 \quad (11)$$

Eqs. (9) and (11) constitute the main approximation. In contrast to previously explored cases [13], this main approximation describes the evolution of two variables with respect to different time scales. According to Eq. (8) function φ_{10} is constant with respect to time scale τ_1. In Eq. (9) φ_{10} is to be considered as constant. The dissipative term in Eq. (9) ensures that variable φ_{20} exponentially (with respect to τ_1) tends to "stationary point" regime described by the following equation:

$$\delta\left[\frac{\mathrm{i}}{2}\varphi_{20} - 3\mathrm{i}|\varphi_{20}|^2\varphi_{20}\right] + \frac{\lambda_2}{2}\varphi_{20} - \frac{\mathrm{i}c}{2}(\varphi_{20} - \varphi_{10}) = 0 \quad (12)$$

with all variables depending only on τ_2. In other words, for the sake of solution of Eq. (11) we are to consider the "fast process" described by Eq. (9) as already elapsed and the relationship between the variables—as corresponding to "stationary point condition" for Eq. (9), i.e. to Eq. (12). Therefore with respect to time scale τ_2, the main approximation is reduced to combination of the algebraic relationship between two variables (12) and differential equation (11). Such situation is very typical when speaking about NNMs [12], but the novel feature of the considered case is the account of the dissipation terms. System (11)–(12) is solved by splitting the complex variables to modulus and argument parts:

$$\varphi_{10} = R_1 \exp(\mathrm{i}\vartheta_1), \quad \varphi_{20} = R_2 \exp(\mathrm{i}\vartheta_2)$$

where R_i, ϑ_i, $i = 1, 2$ are functions depending on τ_2.

System (11)–(12) is transformed to the following form:

$$\begin{aligned}
&\frac{\partial R_1}{\partial \tau_2} + \frac{\lambda_1}{2} R_1 - \frac{c}{2} R_2 \sin\vartheta = 0 \\
&\frac{\partial \vartheta_1}{\partial \tau_2} + \frac{c}{2} - \frac{c}{2}\frac{R_2}{R_1} \cos\vartheta = 0 \\
&\lambda_2 R_2 + c R_1 \sin\vartheta = 0 \\
&2\delta\left(\frac{R_2}{2} - 3R_2^3\right) - cR_2 + cR_1 \cos\vartheta = 0 \\
&\vartheta = \vartheta_1 - \vartheta_2
\end{aligned} \tag{13}$$

If $\sin\vartheta$ is eliminated from the first and the third equation of system (13) we obtain

$$R_1 \frac{\partial R_1}{\partial \tau_2} + \frac{\lambda_1}{2} R_1^2 + \frac{\lambda_2}{2} R_2^2 = 0 \tag{14}$$

From the third and the fourth equations of Eq. (13) we obtain:

$$c^2 R_1^2 = R_2^2 \left(\lambda_2^2 + \left(c - 2\delta\left(\tfrac{1}{2} - 3R_2^2\right)\right)^2\right) \tag{15}$$

By substituting Eq. (15) to Eq. (14) we finally obtain

$$\frac{\partial}{\partial \tau_2} Z\left(\lambda_2^2 + \left(c - 2\delta\left(\frac{1}{2} - 3Z\right)\right)^2\right) + \lambda_1 Z\left(\lambda_2^2 + \left(c - 2\delta\left(\frac{1}{2} - 3Z\right)\right)^2\right) + c^2 \lambda_2 Z = 0 \tag{16}$$

where $Z = R_2^2$ is a measure of energy in the second oscillator.

Ordinary differential equation (16) is trivially reduced to standard integral of rational function:

$$\mathrm{d}\tau_2 = -\mathrm{d}Z \frac{\lambda_2^2 + (c-\delta)^2 + 24(c-\delta)\delta Z + 108\delta^2 Z^2}{\lambda_1 Z\left(\lambda_2^2 + (c-\delta)^2 + 12(c-\delta)\delta Z + 36\delta^2 Z^2\right) + c^2 \lambda_2 Z} \tag{17}$$

It is an awkward task to present the integral of Eq. (17) explicitly, but it is easy to perform such computation for any specified set of parameters. The example is presented in Fig. 1.

It should be mentioned that the solution has two points with divergent derivative, corresponding to roots of numerator in Eq. (17). The curve in Fig. 1 with two saddle-node bifurcations is very typical for many problems in the theory of nonlinear vibrations. In our

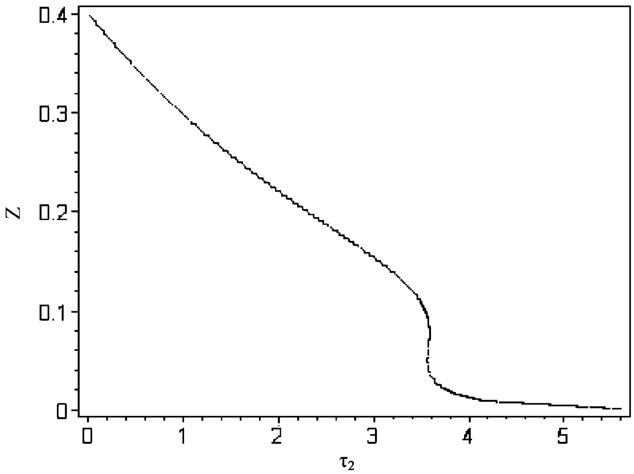

Fig. 1. Plot of analytic solution of Eq. (17), Z versus τ_2. The parameters are $c = 1.4$, $\lambda_1 = \lambda_2 = 1$, $\varepsilon = 0.3$, $Z(0) = 0.4$. Computation of the breakdown point according to Eq. (18) gives $Z = 0.078$.

problem the bifurcation may be associated with the breakdown of the resonant regime of vibrations. The amplitude condition for this point is determined by equation

$$\lambda_2^2 + (c-\delta)^2 + 24(c-\delta)\delta Z + 108\delta^2 Z^2 = 0 \tag{18}$$

The solution for the phase variable v is obtained in similar manner. By excluding $\cos\vartheta$ from the second and the fourth equation of Eq. (13) and by using Eq. (15) we obtain the following equation for phase variable v_1:

$$\frac{\partial\vartheta_1}{\partial\tau_2} + \frac{c}{2} - \frac{c^2(6\delta Z + c - \delta)}{2\left(\lambda_2^2 + (c-\delta+6\delta Z)^2\right)} = 0 \tag{19}$$

In order to verify the relevance of the computation procedure described above it is necessary to compare the results with independent numerical simulations of system (1). The result for maximum amplitude of the first oscillator y_1 is presented in Fig. 2.

The agreement shown in Fig. 2 is rather satisfactory. Similar comparison for the u_2 variable is presented in Fig. 3.

The coincidence here is rather satisfactory for late stages of the process ($t > 10$). This result is quite natural as for early stage of the process the derivative with respect to τ_1 in Eq. (9) may not be neglected. Similar correction is required if one wants to describe phase variables in a reliable manner in the whole range of t. The easiest way of such correction is straightforward and uses Taylor expansions of the unknown functions at early stages of the process; afterwards the solution obtained from Eqs. (13)–(19) is matched with the early-time expansion. Technical details for obtaining the early-time expansion are obvious but awkward and therefore we omit them. It should be mentioned that usually (and in the example below) this early stage of the

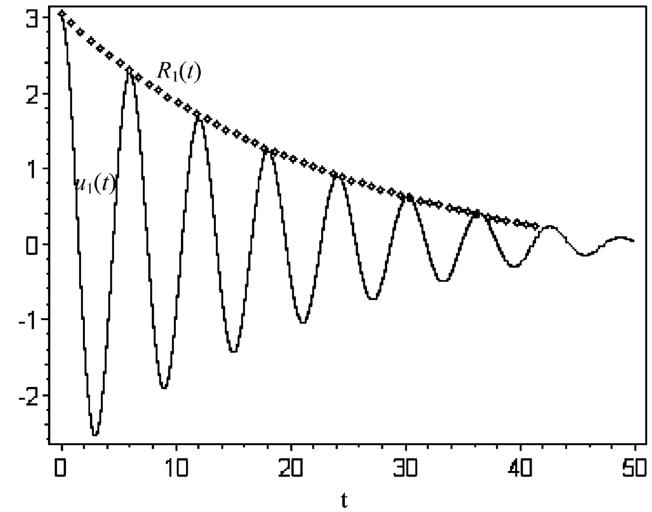

Fig. 2. Plot of analytic solution of Eqs. (17) and (15) for $R_1(t)$ (open symbols, converted to time scale t according to Eq. (5)), numeric solution for $u_1(t)$ (solid line). The parameters are $c = 1.4$, $\lambda_1 = \lambda_2 = 1$, $\varepsilon = 0.3$. Initial conditions are $u_1(0) = 3, \mathrm{d}u_1/\mathrm{d}t(0) = u_2(0) = \mathrm{d}u_2/\mathrm{d}t(0) = 0$. The analytic solution is presented until the breakdown bifurcation point determined by Eq. (18).

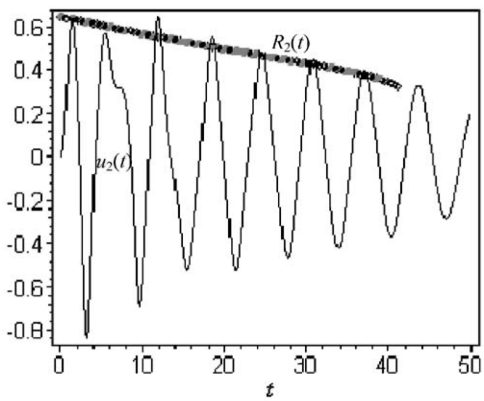

Fig. 3. Plot of analytic solution of Eqs. (17) and (15) for $R_2(t)$ (open symbols, converted to time scale t according to Eq. (5)), numeric solution for $u_2(t)$ (solid line). The parameters are $c = 1.4$, $\lambda_1 = \lambda_2 = 1$, $\varepsilon = 0.3$. Initial conditions are $u_1(0) = 3$, $\mathrm{d}u_1/\mathrm{d}t(0) = u_2(0) = \mathrm{d}u_2/\mathrm{d}t(0) = 0$. The analytic solution is presented until the breakdown bifurcation point determined by Eq. (18).

process occurs at time scale τ_0, i.e. takes about half-period of the fast vibrations. The result for u_1 is presented in Fig. 4.

The process of energy pumping involves fast and irreversible to the possible extent transition of energy from the heavy to the light oscillator. As shown earlier, the energy distribution in the stationary with respect to τ_1 regime is governed by Eqs. (13)–(19). Rough estimation of the energy

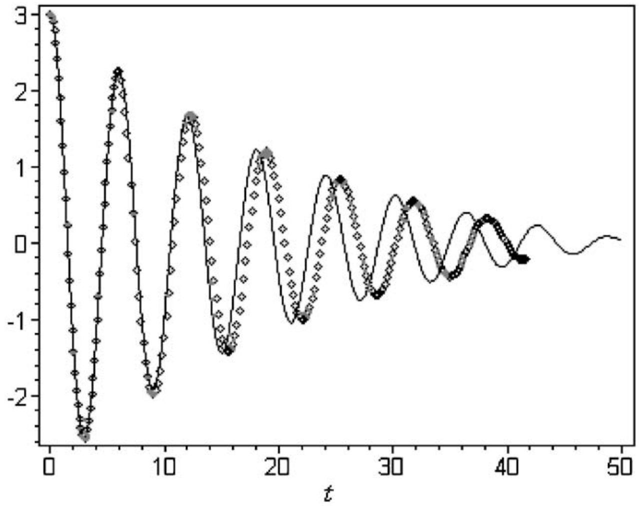

Fig. 4. Plot of matched analytic approximations for $y_1(t)$ (open symbols), numeric solution for $u_1(t)$ (solid line). The parameters are $c = 1.4$, $\lambda_1 = \lambda_2 = 1$, $\varepsilon = 0.3$. Initial conditions are $u_1(0) = 3$, $\mathrm{d}u_1/\mathrm{d}t(0) = u_2(0) = \mathrm{d}u_2/\mathrm{d}t(0) = 0$.

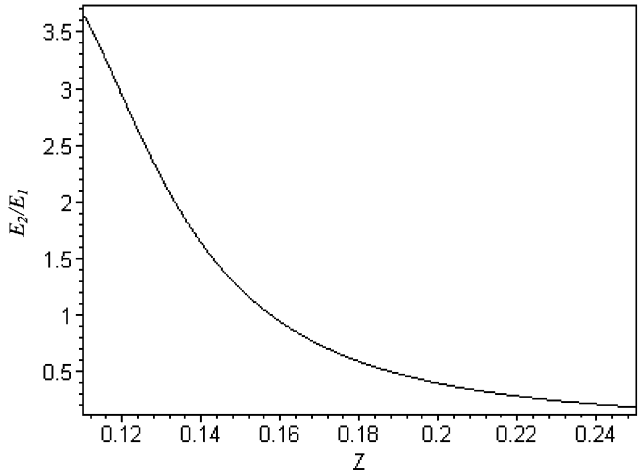

Fig. 5. Average energy ratio E_2/E_1 depending on $R_2^2 = Z$. The function descends monotonously in the range above the breakdown bifurcation. Parameters of the system are $c = 2$, $\lambda_1 = \lambda_2 = 1$, $\varepsilon = 0.2$. Initial conditions are $u_1(0) = 0.4$, $\mathrm{d}u_1/\mathrm{d}t(0) = u_2(0) = \mathrm{d}u_2/\mathrm{d}t(0) = 0$.

ratio for such stationary regime may be obtained from condition (15) (for this sake we have to average the process over the fast variable τ_0):

$$\left\langle \frac{E_2}{E_1} \right\rangle_{\text{fast time}} \propto \frac{R_2^2}{R_1^2} = \frac{c^2}{\lambda_2^2 + \left(c - 2\delta\left(\frac{1}{2} - 3R_2^2\right)\right)^2} \tag{20}$$

with E_i, $i = 1, 2$ are the energies of the respective oscillators. Typical shape of such ratio dependence on the amplitude is presented in Fig. 5.

It is clear that the desired value of the average energy ratio grows as the system approaches the breakdown bifurcation point. It means that in order to optimize the rate of the energy pumping the initial conditions should bring the system close to the bifurcation point as fast as possible.

Still, there exists one restriction. The "stationary" regime (13)–(19) is formed as a result of evolution of the system according to Eq. (9) with respect to time scale τ_1 and within this scale the system should have enough time for relaxation and transition to this regime. The evolution of φ_{10} occurs at scale τ_2; therefore for the analysis of formation of initial slope this variable in Eq. (9) can be taken as a constant determined by the initial conditions. Thus, we have to analyze the following equation:

$$\frac{\partial\varphi_{20}}{\partial\tau_1}+\delta\left[\frac{i}{2}\varphi_{20}-3i|\varphi_{20}|^2\varphi_{20}\right]+\frac{\lambda_2}{2}\varphi_{20}-\frac{ic}{2}(\varphi_{20}-A)=0 \tag{21}$$

where $A=\mathrm{d}u_1/\mathrm{d}t(0)+\mathrm{i}u_1(0)$ and the initial condition is $\varphi_{20}(0)=0$. Eq. (21) describes the evolution of the system from the initial state to the "stationary" regime mentioned above. A typical solution for $\mathrm{Im}(\varphi_{20})$ (i.e. for modulation amplitude for the light oscillator) is presented in Fig. 6.

The time for the slope formation may be estimated as the characteristic frequency of the vibrations around the stationary point. As is clear in Fig. 6, this frequency is not constant due to nonlinearity of Eq. (21). Still, within the accuracy required for estimating the characteristic time of modulation it is enough to consider Eq. (21) linearized around the stationary point. Eq. (21) may be rewritten in the following form:

$$\begin{aligned}&\frac{\partial R_2}{\partial\tau_1}+\frac{\lambda_2}{2}R_2-\frac{cP_0}{2}\cos\vartheta_2=0\\&\frac{\partial\vartheta_2}{\partial\tau_1}+\delta\left(\frac{1}{2}-3R_2^2\right)-\frac{c}{2}+\frac{cP_0}{2R_2}\sin\vartheta_2=0\end{aligned} \tag{22}$$

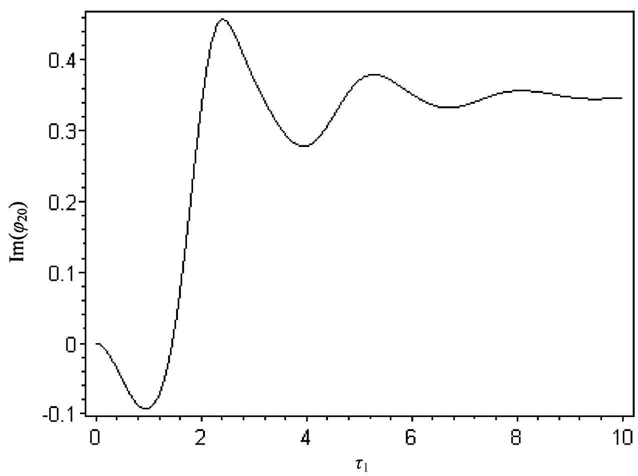

Fig. 6. $\mathrm{Im}(\varphi_{20})$ versus τ_1. Parameters of the system are $c=2$, $\lambda_1=\lambda_2=1$, $\varepsilon=0.2$. Initial conditions are $u_1(0)=0.4$, $\mathrm{d}u_1/\mathrm{d}t(0)=u_2(0)=\mathrm{d}u_2/\mathrm{d}t(0)=0$.

where for the sake of simplicity we take $u_1(0) = P_0, \mathrm{d}u_1/\mathrm{d}t(0) = 0$. Initial conditions for Eq. (22) are $R_2(0) = 0,\ \vartheta_2(0) = \pi/2$. The linearization of Eq. (22) around the stationary point leads to the following equations:

$$\frac{\partial r_2}{\partial \tau_1} + \frac{\lambda_2}{2} r_2 - \frac{cP_0}{2}\gamma_2 \sin\vartheta_2^* = 0$$
$$\frac{\partial \gamma_2}{\partial \tau_1} - 6\delta R_2^* r_2 - \frac{cP_0}{2}\left(\sin\vartheta_2^* \frac{r_2}{R_2^{*2}} - \frac{\cos\vartheta_2^*}{R_2^*}\gamma_2\right) = 0 \tag{23}$$

with $r_2 = R_2 - R_2^*,\ \gamma_2 = \vartheta_2 - \vartheta_2^*$. Stationary point of Eq. (22) is determined by the following equations:

$$c^2P_0^2 = R_2^{*2}\left(\lambda_2^2 + \left(c - 2\delta\left(\frac{1}{2} - 3R_2^{*2}\right)\right)^2\right)$$
$$\cos\vartheta_2^* = \frac{\lambda_2}{cP_0}R_2^* \tag{24}$$

Eigenfrequencies of Eq. (23) are determined by the secular equation

$$\left(\frac{\lambda_2}{2} + \mathrm{i}\omega\right)\left(\frac{cP_0\cos\vartheta_2^*}{2R_2^*} + \mathrm{i}\omega\right) + \frac{cP_0}{2}\sin\vartheta_2^*\left(6\delta R_2^* + \frac{cP_0\sin\vartheta_2^*}{2R_2^{*2}}\right) = 0 \tag{25}$$

The period of modulation described by Eq. (21) thus can be evaluated as

$$T \propto \frac{2\pi}{\mathrm{Re}(\omega)} \tag{26}$$

The bifurcation points of the breakdown of the stationary regime are determined by Eq. (18). In order to ensure the efficient formation of the resonance slope the initial conditions for u_1 should lead to time interval after the beginning of the process

$$\Delta\tau_2 \sim \varepsilon T \tag{27}$$

before the first bifurcation point will be achieved. For every concrete set of parameters equations (24)–(26) can be easily integrated analytically and estimation (27) is to be substituted in the solution of Eq. (18). Thus, the time elapsed between the beginning of the process and the first bifurcation point should be set equal to εT. For one of specific sets of parameters used above ($c = 2,\ \lambda_1 = \lambda_2 = 1,\ \varepsilon = 0.2$) the computation gives critical value $P_0 = 0.405$.

The numerical simulation supporting these results are presented in Figs. 7a and b. At both pictures the value describing the relative amount of energy localized at the second oscillator

$$\frac{E_2}{E_1 + E_2} = \frac{\varepsilon\dot{u}_2(t)^2 + 4\varepsilon u_2(t)^4 + c\varepsilon^2(u_1(t) - u_2(t))^2}{\dot{u}_1(t)^2 + u_1(t)^2 + \varepsilon\dot{u}_2(t)^2 + 4\varepsilon u_2(t)^4 + c\varepsilon^2(u_1(t) - u_2(t))^2} \tag{28}$$

is plotted versus time t for initial conditions below and above the threshold value P_0:

It is clear that above the threshold initial amplitude the dynamics of the system undergoes essential change. For the initial conditions corresponding to Fig. 7a the energy primarily is kept

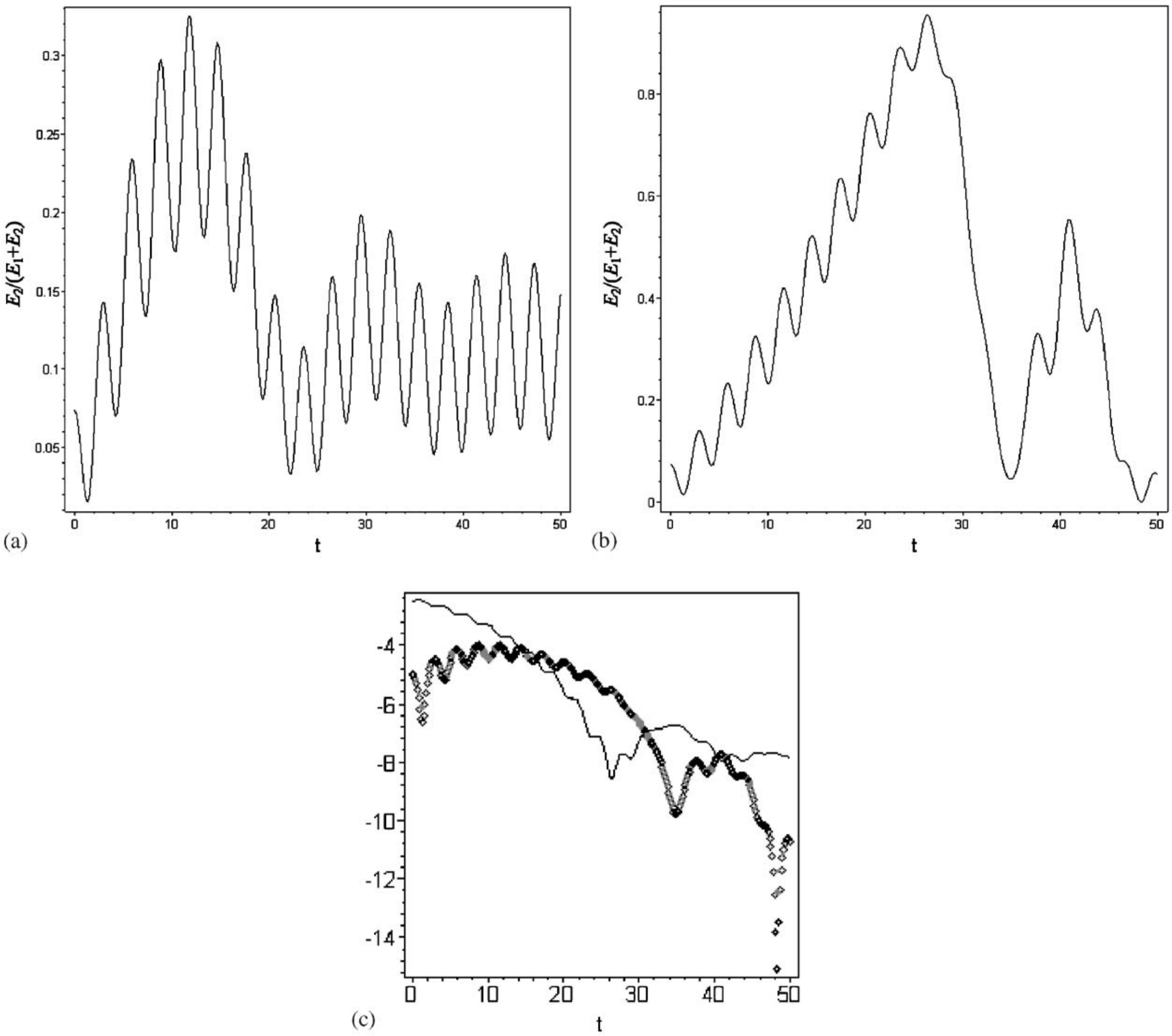

Fig. 7. Relative amount of energy in attachment (28) versus time t for initial conditions: (a) $u_1(0) = 0.31$, $du_1/dt(0) = u_2(0) = du_2/dt(0) = 0$; (b) $u_1(0) = 0.41$, $du_1/dt(0) = u_2(0) = du_2/dt(0) = 0$; (c) partial energies of the oscillators for initial conditions $u_1(0) = 0.41$, $du_1/dt(0) = u_2(0) = du_2/dt(0) = 0$.——, $\log(E_2)$; $\diamond\diamond\diamond\diamond\diamond\diamond\diamond$, $\log(E_1)$.

on the heavy oscillator, for the initial conditions corresponding to Fig. 7b very essential amount of energy (up to 95%) is pumped to the light oscillator. To obtain further verification of this fact, the logarithms of energies of individual oscillators are plotted separately in Fig. 7c. It is clear that the energy of the second oscillator grows in the domain of interest despite overall damping present in the system.

Thus, the process of the energy pumping to the light oscillator can be realized for design (1) of the system with 2dof and the analytic approach presented above allows rather efficient evaluation of the desired diapason of the system parameters.

3. The system with strongly nonlinear coupling

Alternative way of introducing the strong nonlinearity to the system is the consideration of nonlinear coupling between the oscillators. Such coupling was studied in previous papers devoted to 2dof systems with equal masses [1,2]. Generally the nonlinear coupling leads to higher performance of the "energy sink" than the linear one. It is therefore reasonable to consider the system of light and heavy mass coupled by the nonlinear spring. Such system is described by the following system of equations:

$$
\begin{aligned}
&\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + \lambda_1 \frac{\mathrm{d}y_1}{\mathrm{d}t} + y_1 + 8\varepsilon(y_1 - y_2)^3 = 0 \\
&\varepsilon \frac{\mathrm{d}^2 y_2}{\mathrm{d}t^2} + \varepsilon\lambda_2 \frac{\mathrm{d}y_2}{\mathrm{d}t} + 8\varepsilon(y_2 - y_1)^3 = 0
\end{aligned} \tag{29}
$$

In this system $\varepsilon \ll 1$ again characterizes the mass ratio, $y_{1,2}$ denote the displacements of the heavy and the light oscillator, respectively, from their equilibrium positions. The stiffness of the linear oscillator and the coefficient of the nonlinear term are the same as in Eq. (1), again determining the scales of the dependent and the independent variables. The difference is that the orders of damping coefficients $\lambda_{1,2}$ is not yet prescribed (it will be accomplished later) and the term ε in the damping coefficient of the light oscillator is introduced for the sake of convenience only.

It turns out to be useful to introduce new variables in Eq. (29) in the following way:

$$
v = \frac{y_1 + \varepsilon y_2}{1+\varepsilon}, \quad w = \frac{y_1 - y_2}{1+\varepsilon} \tag{30}
$$

Change of variables (30) physically corresponds to consideration of the center of masses and internal displacement of the system of oscillators. Eq. (29) are transformed to the form,

$$
\begin{aligned}
&\ddot{v} + \frac{\lambda_1 + \varepsilon\lambda_2}{1+\varepsilon}\dot{v} + \frac{v}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon}w + \frac{\varepsilon(\lambda_1 - \lambda_2)}{1+\varepsilon}\dot{w} = 0 \\
&\ddot{w} + \frac{\lambda_2 + \varepsilon\lambda_1}{1+\varepsilon}\dot{w} + \frac{\varepsilon}{1+\varepsilon}w + 8(1+\varepsilon)^3 w^3 + \frac{v}{1+\varepsilon} + \frac{(\lambda_1 - \lambda_2)}{1+\varepsilon}\dot{v} = 0
\end{aligned} \tag{31}
$$

Additional change of variables in Eq. (31)

$$
v = \chi V, \quad \varepsilon = \chi^3, \quad \lambda_1 = \chi^2 \mu_1, \quad \lambda_2 = \chi \mu_2 \tag{32}
$$

leads to the following system of equations:

$$
\begin{aligned}
&\ddot{V} + \chi^2 \frac{\mu_1 + \chi^2 \mu_2}{1+\chi^3}\dot{V} + \frac{V}{1+\chi^3} + \frac{\chi^2}{1+\chi^3}w + \frac{\chi^3(\chi\mu_1 - \mu_2)}{1+\chi^3}\dot{w} = 0 \\
&\ddot{w} + \chi \frac{\mu_2 + \chi^4 \mu_1}{1+\chi^3}\dot{w} + \frac{\chi^3}{1+\chi^3}w + 8(1+\chi^3)^3 w^3 + \frac{\chi V}{1+\chi^3} + \frac{\chi^2(\chi\mu_1 - \mu_2)}{1+\chi^3}\dot{V} = 0
\end{aligned} \tag{33}
$$

It is easy to see that system (33) qualitatively coincides with system (1), i.e., main approximations for damping coefficients, coupling and nonlinearity have the same orders of magnitude with respect to parameter χ, as corresponding terms in Eq. (1)—with respect to ε. Additional terms have higher orders with respect to χ; besides it is clear that they can be easily treated within the computational framework for investigation of the resonant regime developed in the previous section. It means that the dynamics of such resonant regime of the system with the nonlinear coupling is equivalent to those described for system (1) with account of change of variables (30), (32) and no additional analysis for this system is required.

It should be mentioned that the small parameter of system (1) $\varepsilon = m_2/m_1$ with $m_{1,2}$ denoting the masses of the heavy and the light oscillator, respectively, is substituted in Eq. (33) by $\chi = \varepsilon^{1/3}$. It means that we can expect in Eq. (33) that the effects, which depend on the value of ε (for instance, the pumping of energy described in the end of the previous section) will occur for very small values of the mass ratio as compared to system (1)—effective coupling and damping will be the same. Physically it means that the "energy sink" with nonlinear coupling may be designed for much smaller mass of the attachment. This conclusion, of course, may be of major practical importance.

In order to illustrate the above result we present the numerical simulations of the system with the nonlinear coupling in Figs. 8a–c for the following set of parameters:

$$\varepsilon = 0.05, \quad \lambda_1 = 0.027, \quad \lambda_2 = 0.2, \quad y_1(0) = 0, \quad \mathrm{d}y_1/\mathrm{d}t(0) = 0.207, \quad y_2(0) = \mathrm{d}y_2/\mathrm{d}t(0) = 0. \tag{34}$$

As in the above section, we explore the relative amount of energy stored in the second oscillator:

$$\frac{E_2}{E_1 + E_2} = \frac{\varepsilon \dot{y}_2(t)^2 + 4\varepsilon (y_1(t) - y_2(t))^4}{\dot{y}_1(t)^2 + y_1(t)^2 + \varepsilon \dot{y}_2(t)^2 + 4\varepsilon (y_1(t) - y_2(t))^4} \tag{35}$$

The next important issue is the rate of the energy damping in the system considered, as presented in Fig. 8c.

It is clear that the presence of light attachment greatly facilitates the damping of energy in the system by mechanism of the irreversible pumping. The resonant regime described above realizes itself almost regardless exact distribution of initial amplitude and initial velocity of the heavy oscillator, depending only on the overall level of input energy. This conclusion may be easily derived from the consideration presented in the previous section (additional argument will not change estimations (22)–(27)). Numerical simulations also support this conclusion. However it should be mentioned that for nonresonant regimes the behavior of system (29) would strongly depend on the type of the initial conditions. Particularly, additional possibilities for the energy pumping may arise.

Let us present the numerical simulation for the system with the nonlinear coupling described by Eq. (29) with parameters and ICs

$$\varepsilon = 0.05, \quad \lambda_1 = 0.027, \quad \lambda_2 = 0.2, \quad y_1(0) = 3, \quad \mathrm{d}y_1/\mathrm{d}t(0) = 0, \quad y_2(0) = \mathrm{d}y_2/\mathrm{d}t(0) = 0 \tag{36}$$

It should be mentioned that the input energy greatly (by two orders) exceeds the value necessary for efficient pumping via the resonant mechanism.

The system considered possesses additional efficient mechanism of the energy damping at the light oscillator. This mechanism exists due to high-frequency vibrations of the light oscillator at the initial stage of the process (Fig. 9a). These vibrations exist due to strong nonlinearity of the coupling function; their frequency decreases with decrease of the amplitude. This nonresonant

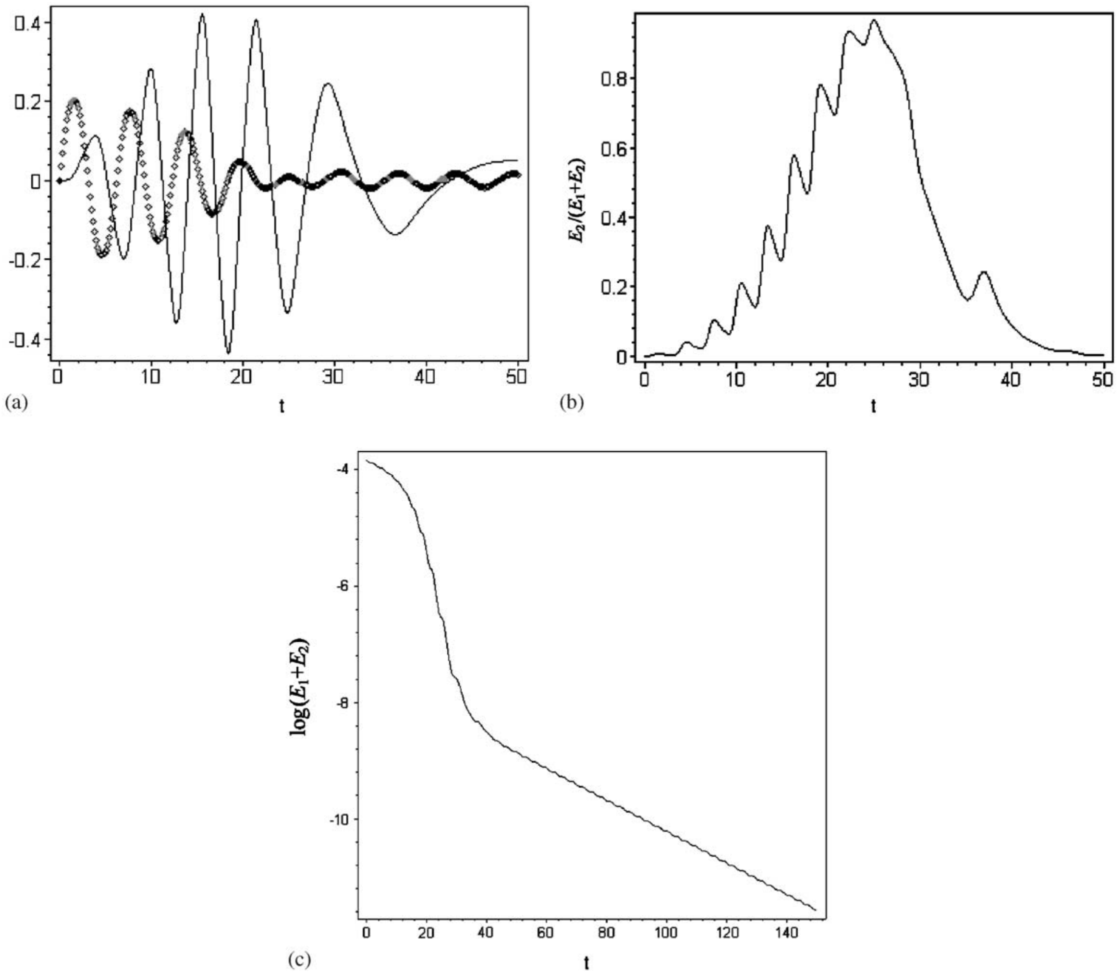

Fig. 8. (a) Trajectories of $y_1(t)$ (dotted line) and $y_2(t)$ (solid line) for system (29) with set of parameters and initial conditions (34). ——, $y_2(t)$; ⋄ ⋄ ⋄ ⋄ ⋄ ⋄ ⋄, $y_1(t)$. (b) Relative amount of energy in attachment (35) versus time for system (29) with set of parameters and initial conditions (34). (c) Logarithm of the sum of energies of the oscillators $E_1 + E_2$ versus time, for system (29) with set of parameters and initial conditions (34).

process elapses when the frequency of the light oscillator approaches the frequency of the heavy one, thus allowing the resonance capture and crossover to the resonant mechanism (such crossover is clearly demonstrated in Figs. 9b and c).

In order to estimate the characteristic time of the crossover we consider the main approximation for Eq. (33):

$$\begin{aligned} &\ddot{V} + \chi^2\mu_1\dot{V} + V + \chi^2 w + = 0 \\ &\ddot{w} + \chi\mu_2\dot{w} + 8w^3 + \chi V = 0 \end{aligned} \tag{37}$$

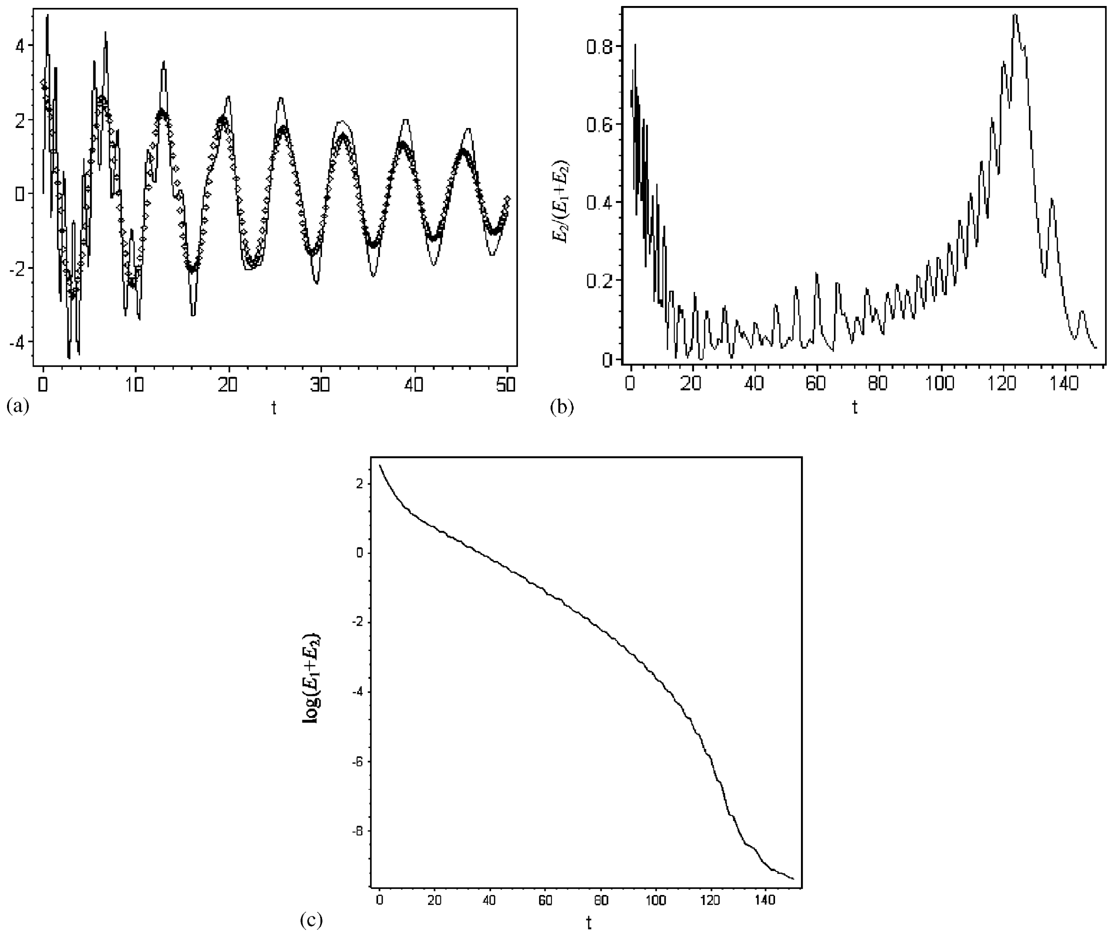

Fig. 9. (a) Trajectories of $y_1(t)$ (dotted line) and $y_2(t)$ (solid line) for system (29) with set of parameters and initial conditions (36). ——, $y_2(t)$; ◇ ◇ ◇ ◇ ◇ ◇ ◇, $y_1(t)$. (b) Relative amount of energy in attachment (35) versus time for system (29) with set of parameters and initial conditions (36). (c) Logarithm of the sum of energies of the oscillators $E_1 + E_2$ versus time, for system (29) with set of parameters and initial conditions (36).

with initial conditions derived from Eq. (36) in the main approximation:

$$\begin{aligned} V(0) &= y_1(0), \quad \dot{V}(0) = 0 \\ w(0) &= y_1(0), \quad \dot{w}(0) = 0 \end{aligned} \tag{38}$$

As it is demonstrated in Refs. [15–17], for sufficiently high level of the initial displacement the coupling may be neglected and the asymptotic behavior of $w(t)$ is described by the following formula:

$$w(t) = w(0)\exp\left(-\frac{1}{3}\chi\mu_2 t\right) cn\left(6\sqrt{2}\chi\mu_2 w(0)\exp\left(-\frac{1}{3}\chi\mu_2 t\right), \frac{\sqrt{2}}{2}\right) \tag{39}$$

where $cn(x,k)$ is Jacobi elliptic function and k is its modulus. From Eq. (39), it is easy to estimate the characteristic time of the crossover between two regimes of the anomalous dissipation—it is sufficient to suppose that the frequency approaches unity and the period of vibrations approaches 2π:

$$t^* \sim -\frac{3}{\chi\mu_2}\log\left(\frac{\pi}{6\sqrt{2}K(\sqrt{2}/2)\chi\mu_2 w(0)}\right) \sim 17 \tag{40}$$

in excellent agreement with the numerical simulation data (Fig. 9a).

The nonresonant mechanism presented above is rather efficient but for the system with 2dof, it is greatly restricted from the viewpoint of the initial conditions.

4. Chain of particles with nonlinear attachment

The results presented in the above sections allow to draw certain conclusions concerning the behavior of the systems with more than 2dof which are more complicated and close to practical implications. In this section, we present the results of numerical simulations, which demonstrate that the effects of efficient energy pumping and absorption in the light nonlinear attachment revealed for 2dof system may be detected in a linear chain of particles attached to the strongly nonlinear oscillator.

Let us consider the behavior of an N-particle chain with linear coupling and linear on-site potential, attached to the strongly nonlinear oscillator by weak linear coupling. The equations describing such a system may be written as follows:

$$\begin{aligned}
&\mu u_{0,tt} + \lambda_0 u_{0,t} + \alpha u_0^n + \varepsilon(u_0 - u_1) = 0\\
&m u_{1,tt} + \lambda_1 u_{1,t} + c(u_1 - u_2) + \omega^2 u_1 + \varepsilon(u_1 - u_0) = 0\\
&m u_{k,tt} + \lambda_k u_{k,t} + c(2u_k - u_{k-1} - u_{k+1}) + \omega^2 u_k = 0
\end{aligned} \tag{41}$$

where u_0 is the displacement of the attachment, u_i is the displacement of the ith particle of the linear chain, $i = 1,2,3,\ldots,N$. For the set of parameters: $N = 20$, $c = 0.2$, $\omega = 1.015$, $\alpha = 0.83$, $\mu = 0.3$, $\varepsilon = 0.04$, $n = 3$, $\lambda_i = 0.01$ for all i, $m = 1$ and initial conditions $u_{N,t}(0) = 10$, and the other initial velocities and displacements equal to zero, the energy is partially transferred to the attachment (see Fig. 10).

The frequencies of the attachment and the first particle of the chain coincide (see Fig. 11). Similar regime may be observed if the displacement of the end particle is nonzero and the velocity is zero.

More efficient energy pumping is observed for the set of parameters: $N = 20$, $c = 10$, $\omega = 0.224$, $\alpha = 1.6$, $\mu = 0.2$, $\varepsilon = 0.08$, $n = 3$, $\lambda = 0.04$, $m = 0.05$ and ICs $u_{N,t}(0) = 10$, with all others zero (Fig. 12).

Similar to 2dof system, the strongly nonlinear coupling provides more efficient mechanism for energy pumping to the attachment. Namely, the system is described by the equations

$$\begin{aligned}
&\mu u_{0,tt} + \lambda u_{0,t} + \alpha(u_0 - u_1)^3 = 0\\
&m u_{1,tt} + \lambda u_{1,t} + c(u_1 - u_2) + \omega^2 u_1 + \alpha(u_1 - u_0)^3 = 0\\
&m u_{k,tt} + \lambda u_{k,t} + c(2u_k - u_{k-1} - u_{k+1}) + \omega^2 u_k = 0
\end{aligned} \tag{42}$$

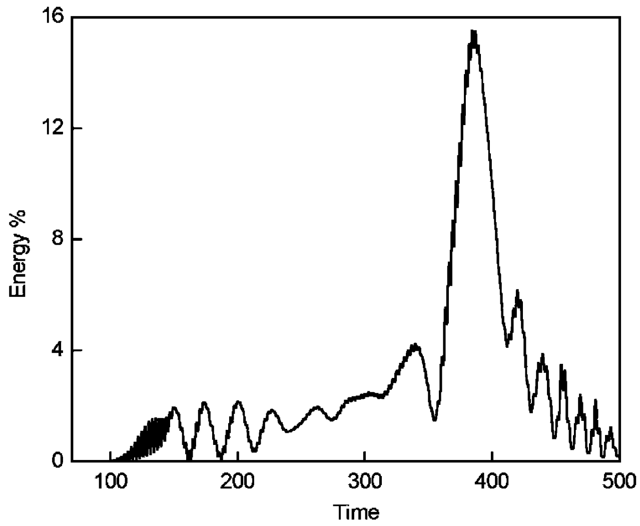

Fig. 10. The rate of overall energy of the chain stored in the attachment (percent).

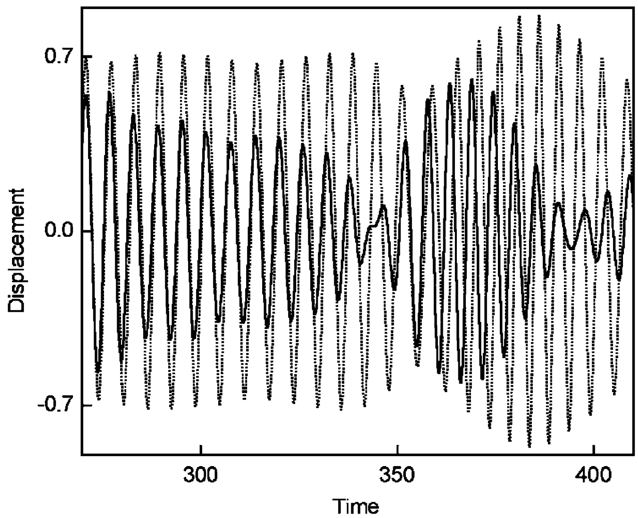

Fig. 11. The displacements of the attachment (dotted line) and the first particle (solid line).

The set of parameters $c = 200$, $\omega = 0.63$, $\mu = 0.01$, $\alpha = 210, \lambda = 0.01$, $m = 0.08, N = 20$ and ICs $u_{N,t}(0) = 20$, with all others zero, leads to the efficient energy pumping. The simulation demonstrates that about 42% of the initial energy is dissipated at the attachment. The dynamical regime is demonstrated in Fig. 13.

The dynamical regime is similar to the nonresonant process described in the end of Section 3. It should be mentioned, however, that unlike the 2dof system the efficiency of the nonresonant mechanism is not restricted by specific type of the initial conditions and depends only on the overall energy level.

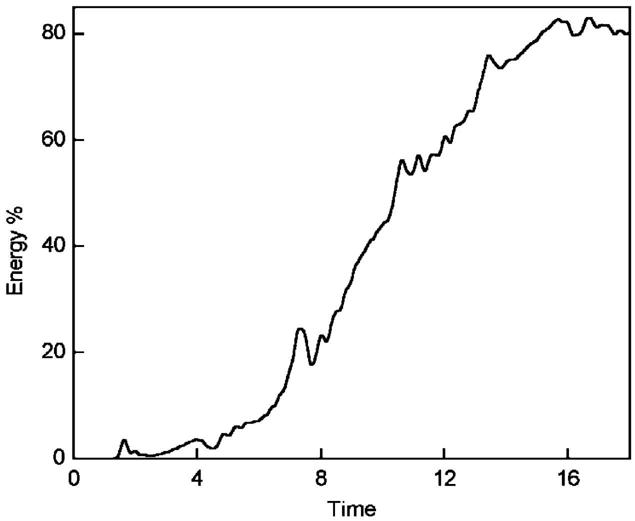

Fig. 12. The rate of overall energy of the chain stored in the attachment (percent).

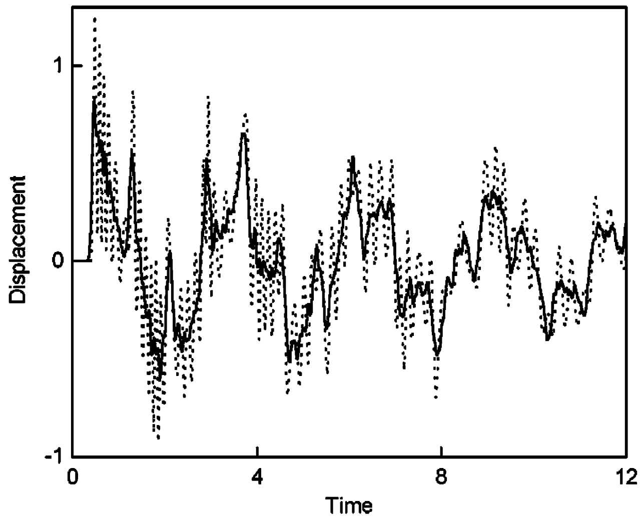

Fig. 13. The displacements of the attachment (dotted line) and the first particle of the chain (solid line) for the case of the nonlinear coupling.

5. Conclusions

The results presented above allow us to conclude that energy pumping may be realized for a two-oscillator system with essential mass asymmetry, provided that the damping and coupling terms of the system are designed properly—according to respective orders of their magnitudes as stated above. Both linear and nonlinear coupling between the oscillators are acceptable; still, the nonlinear coupling allows essentially less mass of the attachment. The multiple-scale expansion allows analytic description of both systems, the validity of the analytic approach is confirmed by comparison with direct numerical simulations. It is demonstrated that for the system with the nonlinear coupling two different mechanisms of coupling (resonant and nonresonant) may be observed. Both mechanisms may occur also in the system consisting of a linear chain with nonlinear attachment.

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