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Short Communication

A uniformly valid multiple scales solution for cut-on cut-off transition of sound in flow ducts

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Abstract

Starting from a multiple-scales approach to model sound transmission through a slowly varying duct of arbitrary cross section, an explicit analytical solution is derived for a mode undergoing cut-on cut-off transition. The solution is a composite one, removing the singularity that appears in the WKB approximation by encompassing the inner boundary-layer solution with the behaviour far upstream and downstream. The solution should not only prove to be a useful benchmark for computational aeroacoustic models, but also enable a designer to continue to use multiple-scales theory to examine sound transmission even when cut-on cut-off transitional modes are present.

1. Introduction

The theory of multiple scales or Wentzel, Kramers and Brillouin (WKB) technique is ideal for use in problems where the typical wavelength of the perturbation (acoustic wave) is much shorter than the typical length scale over which the medium or domain varies considerably. Such a situation occurs in the transmission of sound through aeroengine ducts, which are typically not straight cylinders (see Fig. 1) but have some small degree of variation in diameter and indeed shape. These variations are necessarily gradual over a length scale much larger than typical acoustic wavelengths to preserve the aerodynamics of the mean flow. The theory of multiple scales

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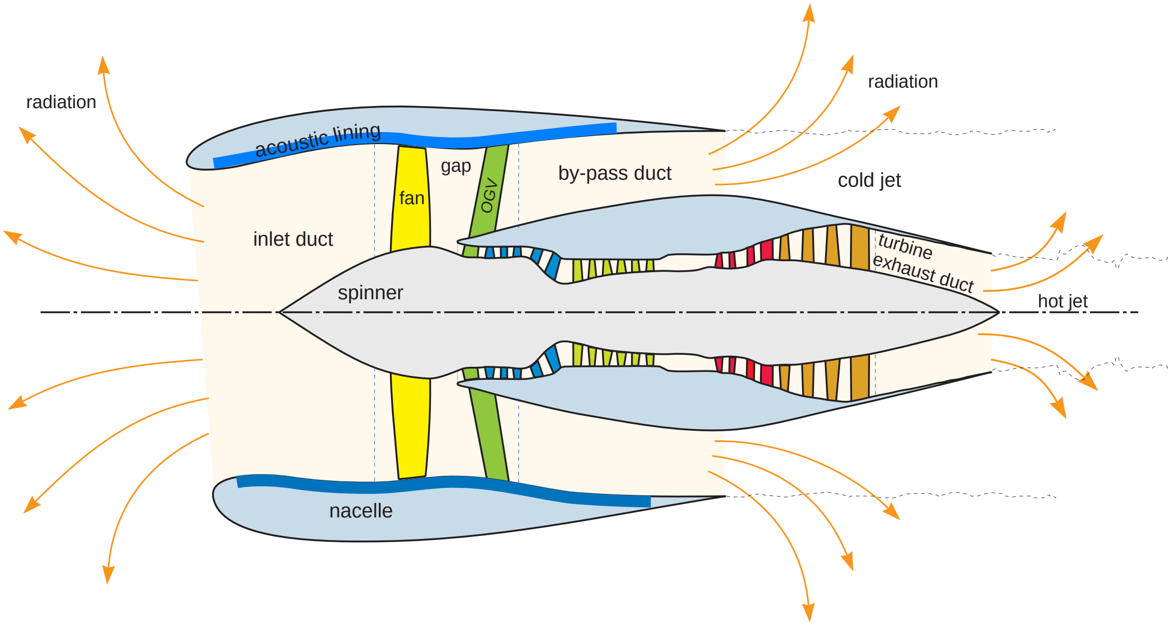

Fig. 1. Sketch of a typical high-bypass ratio turbofan engine with engine ducts labelled.

was first used to model sound propagation in slowly varying axisymmetric ducts without mean flow by Nayfeh and Telionis [1]. This theory was subsequently extended to include cases with mean irrotational flow [2], mean swirling flow [3] and most recently non-axisymmetric ducts (with mean irrotational flow) [4,5].

The use of the multiple-scales approximation is an attractive alternative to a full numerical solution in engine ducts, especially in cylindrical ducts where the Laplace eigenvalue problem can be reduced to a one-dimensional algebraic equation, thus allowing the concept of acoustic modes observed in a straight duct to be retained. Even for more general ducts, the calculation complexities are only marginally more than finding the eigenmodes inside a straight duct of arbitrary cross section. The retention of acoustic modes, along with the ability to model the effects of an irrotational and even swirling mean flow, make the multiple-scales approach a powerful tool in the analysis of engine design. The theory has already proved to be in good agreement with full numerical solutions in realistic engine duct geometries at realistic engine frequencies [6].

As stated above, the multiple-scales approach allows the sound transmission to be represented by a summation of *slowly varying* modes. The amplitudes of these modes are determined via a solvability condition and the multiple-scales approximation breaks down wherever the amplitude of a particular mode becomes singular. These singular transition points are analogous to the turning points observed in solutions to Schrödinger’s equation, where the complete reflection of a cut-on propagating mode and transmission of a cut-off attenuating mode (or vice versa) occur.

The turning-point behaviour of such a mode can be analysed by examining the solution in a boundary-layer region around the singularity. Within this region, the original slowly varying assumption breaks down and a different approximation leads to the non-convective axial variation of the mode satisfying Airy's equation. Such an analysis was performed for both axisymmetric [1,3,7] and non-axisymmetric ducts [4,5] alike in cases of no mean flow, irrotational mean flow and, for axisymmetric ducts, mean swirling flow. In all cases it was shown that the incident cut-on mode is completely reflected in the axial plane with a phase shift of $\pi/2$. In the event that an isolated mode undergoes cut-on cut-off transition, no energy is propagated beyond the transition point.

An understanding of cut-on cut-off behaviour in hard-walled ducts is important for engine design applications. Rotors and stators are usually designed such that at least the first harmonics of all interaction modes are cut-off. Cutting off other acoustic modes by varying the duct geometry could, in principle, result in further reductions to the noise output of an engine. However, reflection of a cut-on mode can result in the mode becoming trapped inside a section of the duct, possibly leading to acoustic resonance and instability. Such a scenario has been investigated previously by Cooper and Peake [8]. Furthermore, similar transitional phenomena has been shown to occur even in lined ducts [9], leading to partial reflection of acoustic modes.

In this paper, an explicit analytical solution of a single mode undergoing cut-on cut-off transition is derived for an arbitrary duct with mean irrotational flow. The analytical solution is a composite solution, encompassing both the inner boundary-layer solution in the neighbourhood of the transition point and the outer slowly varying modal solution far upstream and downstream. It therefore remains uniformly valid to leading order throughout the duct, and can be treated exactly as a normal slowly varying mode, without any need to calculate the size of the transitional boundary layer, nor match the inner and outer solutions at some intermediate interface. Such uniformly valid solutions have been studied in many other branches of physics for many years: these include problems in optics, quantum mechanics and seismology amongst others. The approach is not so well-known in the aeroacoustics community, however. The derivation is based on the method of relevant functions developed by Ludwig and Kravtsov [10,11]. An overview of this method and the use of uniformly valid approximations in other realms of physics can be found in Refs. [12,13].

The solution derived in this article should enable a designer to continue to use multiple-scales theory to accurately predict the flow pressure and noise transmission inside an engine duct, whilst now being able to include the contribution of modes undergoing cut-on cut-off transition. It should also prove to be a useful benchmark for computational aeroacoustical (CAA) codes and enable confirmation of experimental observations of the resultant sound field anywhere inside the duct, including areas in close proximity to the transition region where the outer WKB solution fails.

The outline of the paper is as follows. The basic governing equations for the mean flow and acoustic perturbations are presented in Section 2. Section 3 then introduces the concept of slowly varying modes using the method of multiple scales, as previously described in Ref. [5]. A brief review of the analysis presented in Refs. [5,7] of a mode undergoing cut-on cut-off transition is given in Section 4, where the inner and outer solutions are obtained. The derivation of the uniformly valid solution encompassing both inner and outer solutions is then explained in Section 5. Section 6 presents the results of a number of cases of cut-on cut-off transition, using the uniformly valid solution, inside a typical aeroengine inlet duct geometry. Conclusions and a brief summary follow in Section 7.

2. Problem formulation

Consider a compressible inviscid perfect isentropic irrotational gas flow contained inside a duct of slowly varying cross section. The governing Euler equations and conditions of a perfect gas can be made dimensionless by scaling all spatial dimensions on a typical duct width R_∞, the density $\tilde{\rho}$ on some reference value ρ_∞, velocities $\tilde{\mathbf{v}}$ and sound speed $\tilde{c}$ on a reference sound speed c_∞, time t on R_∞/c_∞ and pressure $\tilde{p}$ on $\rho_\infty c_\infty^2$. Defining γ as the ratio of specific heats, the resulting system of equations to be solved is [14]

$$\frac{\partial\tilde{\rho}}{\partial t} + \nabla\cdot(\tilde{\rho}\tilde{\mathbf{v}}) = 0, \quad \tilde{\rho}\left(\frac{\partial\tilde{\mathbf{v}}}{\partial t} + \tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\right) + \nabla\tilde{p} = 0,$$
$$\gamma\tilde{p} = \tilde{\rho}^\gamma \quad \text{and} \quad \tilde{c}^2 = \frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{\rho}} = \tilde{\rho}^{\gamma-1}. \tag{1}$$

The duct geometry can be described ideally in cylindrical coordinates (x, r, θ), with unit vectors $\mathbf{e}_x$, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ respectively, as the region

$$R_1(X,\theta) \leqslant r \leqslant R_2(X,\theta), \quad X = \varepsilon x. \tag{2}$$

Here, ε is a small parameter distinguishing between slow and fast axial scales, and $R_1(X,\theta)$ and $R_2(X,\theta)$ are the radial positions of the inner and outer duct walls respectively.

We assume that the flow field can be split into a stationary mean flow, which is assumed to be irrotational and nearly uniform with no swirling component, and infinitesimally-small time-harmonic perturbations of non-dimensional frequency (Helmholtz number) ω,

$$[\tilde{\mathbf{v}}, \tilde{\rho}, \tilde{p}, \tilde{c}] = [\mathbf{V}, D, P, C] + [\nabla\phi, \rho, p, c]\mathrm{e}^{\mathrm{i}\omega t}. \tag{3}$$

Here, the mean velocity field, density, pressure and sound speed are given by $\mathbf{V}$, D, P and C respectively. The irrotational velocity field perturbation is the gradient of some velocity potential ϕ, and the density, pressure and sound speed perturbations are given by ρ, p and c respectively.

Assuming that the mean normal velocity vanishes at the duct wall, the mean flow field satisfies the quasi-one-dimensional gas equation and can be expressed to leading-order as functions of the slow axial variable X only. The solution, given in Ref. [5], takes the form

$$\mathbf{V}(X, r, \theta; \varepsilon) = U_0(X)\mathbf{e}_x + \varepsilon\mathbf{V}_{\perp 0}(X, r, \theta) + O(\varepsilon^2), \tag{4}$$

$$D(X, r, \theta; \varepsilon) = D_0(X) + O(\varepsilon^2), \tag{5}$$

$$C(X, r, \theta; \varepsilon) = C_0(X) + O(\varepsilon^2), \tag{6}$$

$$P(X, r, \theta; \varepsilon) = P_0(X) + O(\varepsilon^2), \tag{7}$$

where $\varepsilon\mathbf{V}_{\perp 0}$ represents a small crosswise mean flow component in the $\mathbf{e}_r$ and $\mathbf{e}_\theta$ directions; for the case of an axisymmetric duct, this is purely radial [2]. The acoustic field can be described, after eliminating pressure and density perturbations, as a solution to the general convected wave equation [15]

$$\nabla\cdot(D\nabla\phi) - D(\mathrm{i}\omega + \mathbf{V}\cdot\nabla)\left[\frac{1}{C^2}(\mathrm{i}\omega + \mathbf{V}\cdot\nabla)\phi\right] = 0. \tag{8}$$

Throughout this paper we shall remain concerned only with hard-wall ducts, and so the boundary conditions for the acoustic field are simply that the normal velocity vanishes at both inner and outer walls $(\nabla\phi \cdot \mathbf{n}) = 0$ (where $\mathbf{n}$ is the outer normal to the respective wall).

3. Slowly varying acoustic modes

Using WKB theory, the method of multiple-scales enables the acoustic field in a slowly varying duct to be represented as a summation of slowly varying modes [5], of the form

$$\phi(x, r, \theta; \varepsilon) = N(X)\psi(r, \theta; X)\exp\left(-\frac{\mathrm{i}}{\varepsilon}\int^{X} \mu(\zeta; \varepsilon)\,\mathrm{d}\zeta\right). \tag{9}$$

The function $\psi(r, \theta; X)$ is the solution to the following eigenvalue problem in the cross-sectional plane

$$-\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi = \alpha^2\psi, \tag{10}$$

with hard-wall boundary conditions

$$\frac{\partial\psi}{\partial r} = \frac{1}{R_i}\frac{\partial R_i}{\partial\theta}\frac{\partial\psi}{\partial\theta} \quad \text{at } r = R_i(X, \theta),\ i = 1, 2 \tag{11}$$

and the slow axial variable X acts as a parameter. The eigenvalue α^2 with eigensolution ψ satisfies the dispersion relation

$$\frac{(\omega - \mu U_0)^2}{C_0^2} - \mu^2 = \alpha^2, \tag{12}$$

which, in turn, determines the axial wavenumber $\mu(X; \varepsilon) = \mu(X) + O(\varepsilon^2)$. To perform the analysis in Ref. [5], the eigensolution must be normalised by integrating across the cross-sectional plane at each X-station, ensuring

$$\int_0^{2\pi}\int_{R_1(X,\theta)}^{R_2(X,\theta)} \psi^2(r, \theta; X)r\,\mathrm{d}r\,\mathrm{d}\theta = 1. \tag{13}$$

The slowly varying amplitude $N(X)$ is subsequently determined from a solvability condition [16] to be

$$N(X) = Q\sqrt{\frac{C_0(X)}{\omega\sigma(X)D_0(X)}}, \tag{14}$$

for some constant Q (obtained from the sound source) and where

$$\sigma = \sqrt{1 - (C_0^2 - U_0^2)\frac{\alpha^2}{\omega^2}}, \tag{15}$$

is defined as the reduced axial wavenumber [2].

4. Cut-on cut-off transition

Hard-wall transition points occur in a slowly varying duct when the reduced axial wavenumber (15) becomes zero making the modal amplitude $N(X)$ in Eq. (14) singular. Hence in the neighbourhood of such a point, the slowly varying assumption breaks down and a new approximation to the leading-order governing equations is necessary. The discussion below details briefly the analysis presented in Refs. [5,7], where the neighbourhood and respective inner and outer solutions are obtained.

For an incident cut-on acoustic mode approaching the transition point X_t (where $\sigma = 0$) from $X < X_t$, the outer solution for $|X - X_t| \sim 1$ takes the form

$$\phi = \frac{n(X)}{\sqrt{\sigma(X)}}\psi(r,\theta;X)\mathrm{e}^{\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega U_0/(C_0^2-U_0^2)]\,\mathrm{d}X'}\left[\mathrm{e}^{-\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega C_0\sigma/(C_0^2-U_0^2)]\,\mathrm{d}X'} + \mathscr{R}\mathrm{e}^{+\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega C_0\sigma/(C_0^2-U_0^2)]\,\mathrm{d}X'}\right], \tag{16}$$

where $n(X) = N(X)\sqrt{\sigma(X)}$ from Eq. (14), σ is positive and real, and a reflected component of the wave has been included with unknown reflection coefficient $\mathscr{R}$. Similarly, beyond the transition point $X > X_t$ we assume that a cut-off mode is transmitted with unknown transmission coefficient $\mathscr{T}$, thus

$$\phi = \mathscr{T}\frac{n(X)}{\sqrt{\sigma(X)}}\psi(r,\theta;X)\mathrm{e}^{\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega U_0/(C_0^2-U_0^2)]\,\mathrm{d}X'}\mathrm{e}^{-1/\varepsilon\int_{X_t}^{X}[\omega C_0|\sigma|/(C_0^2-U_0^2)]\,\mathrm{d}X'}. \tag{17}$$

Note that for the cut-off mode $\arg(\sigma) = -\pi/2$ and $\sqrt{\sigma} = \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{|\sigma|}$ is taken.

Examining the asymptotic behaviour of the outer solution near X_t and balancing the terms in the governing equation (8) leads to a boundary-layer region of thickness $O(\varepsilon^{2/3})$ described by the inner axial variable ξ where

$$X = X_t + \varepsilon^{2/3}\lambda^{-1}\xi. \tag{18}$$

The coefficient λ^{-1} is introduced for convenience and is defined in terms of mean flow variables evaluated at the transition point,

$$\lambda^3 = \frac{2\omega^2 C_0^2(X_t)}{(C_0^2(X_t) - U_0^2(X_t))}\left[\frac{C_0(X_t)C_0'(X_t) - U_0(X_t)U_0'(X_t)}{C_0^2(X_t) - U_0^2(X_t)} + \frac{\alpha'(X_t)}{\alpha(X_t)}\right]. \tag{19}$$

For cut-on cut-off transition of a mode incident from $X < X_t$, this factor is positive.

Within the boundary-layer, the modal behaviour is unchanged in the radial direction because the duct is locally parallel. The inner solution thus takes the form

$$\phi_{\text{inner}} = \chi(\xi)\psi(r,\theta;X)\mathrm{e}^{\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega U_0/(C_0^2-U_0^2)]\,\mathrm{d}X'}, \tag{20}$$

and substituting into the original governing equation (8) yields $\chi(\xi)$ as a solution to Airy's equation, $\chi'' - \xi\chi = 0$. By a subsequent matching to the outer solutions (16) and (17) one obtains

$$\chi(\xi) = 2\varepsilon^{-1/6}\pi^{1/2}n(X_t)\left(\frac{\omega C_0(X_t)}{\lambda(C_0^2(X_t) - U_0^2(X_t))}\right)^{1/2}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{Ai}(\xi). \tag{21}$$

The reflection and transmission coefficients predicted from this matching are $\mathscr{R} = \mathrm{i}$ and $\mathscr{T} = 1$ respectively.

5. Deriving the composite solution

The derivation of a composite solution valid to leading order for both $|X - X_t| \sim 1$ and $|X - X_t| \sim \varepsilon^{2/3}$ can be found using a method based on that of Kravtsov and Ludwig [10,11], detailed in the more recent report [12].

The derivation of the composite solution begins where, under the approximation of slowly varying irrotational mean flow (4) to (7), the acoustic governing equation (8) reduces to (see [5])

$$\frac{\partial^2\phi}{\partial x^2} + \nabla_\perp^2\phi - \frac{1}{C_0^2}\left[-\omega^2\phi + 2\mathrm{i}\omega U_0\frac{\partial\phi}{\partial x} + U_0^2\frac{\partial^2\phi}{\partial x^2}\right] = O(\varepsilon). \tag{22}$$

Now,

$$\nabla_\perp^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{23}$$

and we note that a slowly varying mode (9) and both inner (20) and outer solutions (16) and (17) contain the function $\psi(r,\theta;X)$, which is the solution to the cross-sectional eigenvalue problem (10) and (11) for some radial eigenvalue $\alpha^2(X)$. Therefore, the term $\nabla_\perp^2\phi$ in the above equation can be replaced by $-\alpha^2\phi$ so that all the partial derivatives in the reduced governing equation are in x only. Thus,

$$\left(1 - \frac{U_0^2}{C_0^2}\right)\frac{\partial^2\phi}{\partial x^2} - \frac{2\mathrm{i}\omega U_0}{C_0^2}\frac{\partial\phi}{\partial x} + \left(\frac{\omega^2}{C_0^2} - \alpha^2\right)\phi = O(\varepsilon). \tag{24}$$

Guided further by the form of outer (16) and (17) and inner (20) solutions, we split $\phi(X,r,\theta)$ into three parts

$$\phi = \Phi(X,r,\theta)G(s)\mathrm{e}^{\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega U_0/(C_0^2-U_0^2)]\,\mathrm{d}X'}. \tag{25}$$

Here, $\Phi(X,r,\theta)$ represents the slowly varying part in X, the rapidly varying convective part is given by the exponential term and there is a transitional part denoted $G(s)$. The new variable s is assumed to have the form $s = \varepsilon^{-\beta}g(X)$, with $\beta > \frac{1}{2}$ to enable the expansion of $\partial^2\phi/\partial x^2$ to have a term $G''(s)$ which is more significant than $O(\varepsilon)$. Differentiating ϕ leads to

$$\frac{\partial\phi}{\partial x} = \Phi(X,r,\theta)\mathrm{e}^{\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega U_0/(C_0^2-U_0^2)]\,\mathrm{d}X'}\left[\frac{\mathrm{i}\omega U_0}{C_0^2 - U_0^2}G + \varepsilon^{1-\beta}G'g'\right] + O(\varepsilon), \tag{26}$$

and

$$\frac{\partial^2\phi}{\partial x^2} = \Phi(X,r,\theta)\mathrm{e}^{\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega U_0/(C_0^2-U_0^2)]\,\mathrm{d}X'} \times\left[-\frac{\omega^2U_0^2}{(C_0^2-U_0^2)^2}G+\varepsilon^{1-\beta}\frac{2\mathrm{i}\omega U_0}{C_0^2-U_0^2}G'g'+\varepsilon^{2-2\beta}G''(g')^2\right]+O(\varepsilon). \tag{27}$$

Substituting these expressions into Eq. (24) results in the $O(\varepsilon^{1-\beta})$ terms cancelling. Further, by applying the definition of the reduced axial wavenumber (15), one obtains

$$\varepsilon^{2-2\beta}G''(s)(g'(X))^2+\frac{\omega^2C_0^2\sigma^2}{(C_0^2-U_0^2)^2}G(s)=O(\varepsilon), \tag{28}$$

which can be identified with Airy's equation, $G''-sG=0$, to leading order by making

$$s=\varepsilon^{-\beta}g(X)=-\frac{\varepsilon^{2\beta-2}}{(g'(X))^2}\frac{\omega^2C_0^2\sigma^2}{(C_0^2-U_0^2)^2}. \tag{29}$$

Clearly, balancing left- and right-hand sides of this expression requires $\beta=\frac{2}{3}$ (which is greater than $\frac{1}{2}$) and the solution to the ordinary differential equation, with the initial boundary condition $s=0$ at $X=X_t$, leads to

$$s=\left(\frac{3\mathrm{i}}{2\varepsilon}\int_{X_t}^{X}\frac{\omega C_0\sigma}{C_0^2-U_0^2}\,\mathrm{d}X'\right)^{2/3}; \tag{30}$$

the solution branch chosen here is such that s is negative real for $X<X_t$ and positive real for $X>X_t$.

The general solution for a simple turning point is therefore

$$\phi(X,r,\theta)=\mathrm{e}^{\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega U_0/(C_0^2-U_0^2)]\,\mathrm{d}X'}[\Phi_1(X,r,\theta)\mathrm{Ai}(s)+\Phi_2(X,r,\theta)\mathrm{Bi}(s)]. \tag{31}$$

The slowly varying parts Φ_i for $i=1,2$ are not determined from the above analysis and require the solution of the cross-sectional eigenvalue problem (10), (11) and an amplitude part determined from a solvability condition, as well as an initial condition. This could be done directly from the analysis here, but given that the slowly varying form of the incident mode is already known from (16), it is more straightforward to obtain the solution by comparing with the outer solutions (16) and (17) far from the transition point. As there is no incident cut-off mode approaching from $X>X_t$, and thus no exponentially growing term in Eq. (17), we can assume that there is no Bi-term in our solution and so $\Phi_2\equiv 0$. Using the asymptotic forms given in Ref. [17] for Ai(s) we find that well ahead of transition for $X<X_t$ ($s\to-\infty$),

$$\mathrm{Ai}(s)\sim\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{2\sqrt{\pi}}\left|\frac{3\mathrm{i}}{2\varepsilon}\int_{X_t}^{X}\frac{\omega C_0\sigma}{C_0^2-U_0^2}\,\mathrm{d}X'\right|^{-1/6}\left(\mathrm{e}^{-\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega C_0\sigma/(C_0^2-U_0^2)]\,\mathrm{d}X'}+\mathrm{i}\mathrm{e}^{+\mathrm{i}/\varepsilon\int_{X_t}^{X}[\omega C_0\sigma/(C_0^2-U_0^2)]\,\mathrm{d}X'}\right), \tag{32}$$

and well beyond transition for $X > X_t$ ($s \to +\infty$),

$$\mathrm{Ai}(s) \sim \frac{1}{2\sqrt{\pi}} \left| \frac{3\mathrm{i}}{2\varepsilon} \int_{X_t}^{X} \frac{\omega C_0 \sigma}{C_0^2 - U_0^2} \, \mathrm{d}X' \right|^{-1/6} \mathrm{e}^{-1/\varepsilon \int_{X_t}^{X} [\omega C_0 |\sigma| / (C_0^2 - U_0^2)] \, \mathrm{d}X'}. \tag{33}$$

With these asymptotes, ϕ can be made to equal to both outer solutions ahead and beyond the transition point if

$$\Phi_1(X, r, \theta) = 2\sqrt{\pi} \mathrm{e}^{+\mathrm{i}\pi/4} n(X) \psi(r, \theta; X) \left[-\frac{3}{2\varepsilon} \frac{1}{\sigma^3} \int_{X_t}^{X} \frac{\omega C_0 \sigma}{C_0^2 - U_0^2} \, \mathrm{d}X' \right]^{1/6}. \tag{34}$$

The [ ]$^{1/6}$-term in this expression is *not* singular in the limit $\sigma \to 0$ and is always real and positive for all X. This can be demonstrated by examining the solution when the inner variable $\xi \sim 1$, as defined in Eq. (18). From the analysis of Ref. [5] we have

$$\frac{1}{\varepsilon} \int_{X_t}^{X} \frac{\omega C_0 \sigma}{C_0^2 - U_0^2} \, \mathrm{d}X' = \begin{cases} -\frac{2}{3}(-\xi)^{3/2} + \cdots & \text{for } \xi < 0, \\ -\mathrm{i}\frac{2}{3}\xi^{3/2} + \cdots & \text{for } \xi > 0, \end{cases} \tag{35}$$

and

$$\sigma^2(X) = -\varepsilon^{2/3} \left(\frac{C_0^2(X_t) - U_0^2(X_t)}{\omega C_0(X_t)} \right)^2 \lambda^2 \xi + \cdots . \tag{36}$$

From these inner-region expansions, it is straightforward to show that $s = \xi + \cdots$ to leading order and that

$$\left[-\frac{3}{2\varepsilon} \frac{1}{\sigma^3} \int_{X_t}^{X} \frac{\omega C_0 \sigma}{C_0^2 - U_0^2} \, \mathrm{d}X' \right]^{1/6} \sim \varepsilon^{-1/6} \left(\frac{\omega C_0(X_t)}{(C_0^2(X_t) - U_0^2(X_t))\lambda} \right)^{1/2} + \cdots . \tag{37}$$

The inner solution given in Eqs. (20) and (21) can now be recovered from Eqs. (34) and (31).

The main result of this paper is that for any incident cut-on mode travelling in the positive X-direction which undergoes transition at a single axial plane X_t, an explicit analytical multiple-scales solution for this mode can be derived to leading order of the form

$$\begin{aligned} \phi = \bar{Q} \sqrt{\frac{C_0}{\omega D_0}} \psi(r, \theta; X) \left[-\frac{3}{2\varepsilon} \frac{1}{\sigma^3} \int_{X_t}^{X} \frac{\omega C_0 \sigma}{C_0^2 - U_0^2} \, \mathrm{d}X' \right]^{1/6} \\ \times \mathrm{Ai} \left[\left(\frac{3\mathrm{i}}{2\varepsilon} \int_{X_t}^{X} \frac{\omega C_0 \sigma}{C_0^2 - U_0^2} \, \mathrm{d}X' \right)^{2/3} \right] \mathrm{e}^{\mathrm{i}/\varepsilon \int_{X_t}^{X} [\omega U_0 / (C_0^2 - U_0^2)] \, \mathrm{d}X'}, \end{aligned} \tag{38}$$

where $\bar{Q} = 2\sqrt{\pi} \mathrm{e}^{\mathrm{i}\pi/4} Q$ to absorb some constants. The eigensolution $\psi(r, \theta; X)$ and the mean flow field are exactly as those determined for a normal slowly varying mode in Ref. [5].

6. Results

To demonstrate the analytical solution, three examples are presented of modes undergoing cut-on cut-off transition in a typical aeroengine inlet duct, similar to that previously used in Refs. [2,6]. The geometry of the duct is axisymmetric, and so $R_1 = R_1(X)$ and $R_2 = R_2(X)$. Therefore, the eigensolution ψ is a combination of Bessel functions of first and second kinds multiplied by $e^{-im\theta}$, where m is the circumferential wavenumber; see Ref. [2] for more details. To put the axisymmetric case explicitly into the form of Eq. (38) requires

$$\psi(r,\theta;X) = \frac{J_m[\alpha r] - \Upsilon(X) Y_m[\alpha r]}{\sqrt{\dfrac{2}{\pi}\left(\dfrac{R_2^2 - m^2/\alpha^2}{[\alpha R_2 Y_m'(\alpha R_2)]^2} - \dfrac{R_1^2 - m^2/\alpha^2}{[\alpha R_1 Y_m'(\alpha R_1)]^2}\right)}} e^{-im\theta} \quad \text{for } m \neq 0, \tag{39}$$

where $\Upsilon(X)$ and the radial eigenvalue $\alpha(X)$ are determined from the hard-walled boundary condition (11),

$$\frac{J_m'[\alpha(X)R_2(X)]}{Y_m'[\alpha(X)R_2(X)]} = \frac{J_m'[\alpha(X)R_1(X)]}{Y_m'[\alpha(X)R_1(X)]} = \Upsilon(X). \tag{40}$$

For a hollow duct ($R_1 = 0$) these expressions reduce to $\Upsilon(X) = 0$, $\alpha(X)$ determined from the boundary condition $J_m'(\alpha R_2) = 0$, and

$$\psi(r,\theta;X) = \frac{J_m[\alpha r]}{J_m[\alpha R_2]} \sqrt{\frac{2}{\pi}} \left(R_2^2 - \frac{m^2}{\alpha^2}\right)^{-1/2} e^{-im\theta} \quad \text{for } m \neq 0. \tag{41}$$

Results from the three examples are presented in Figs. 2,3,4. In each figure, plot (a) shows absolute pressure contours of each single cut-on cut-off mode inside the duct in (x, r) coordinates. The absolute pressure $|p|$ can be derived from the original Euler momentum equation (1) as

$$p = -i\omega\phi - U_0 \frac{\partial \phi}{\partial x}, \tag{42}$$

noting that only the leading-order terms are taken from the derivative of ϕ. Plot (b) presents a direct comparison of the uniformly valid solution with the outer slowly varying solution across the duct. The non-convective axial variation of the uniformly valid solution is given by

$$2\sqrt{\pi} e^{+i\pi/4} \left[-\frac{3}{2\varepsilon}\frac{1}{\sigma^3}\int_{X_t}^{X} \frac{\omega C_0 \sigma}{C_0^2 - U_0^2}\, dX'\right]^{1/6} \mathrm{Ai}\left[\left(\frac{3i}{2\varepsilon}\int_{X_t}^{X} \frac{\omega C_0 \sigma}{C_0^2 - U_0^2}\, dX'\right)^{2/3}\right],$$

and this is shown by the solid line. The dotted line shows the non-convective axial variation of the original slowly varying (outer) solution, which is

$$\frac{1}{\sqrt{\sigma(X)}}\left[e^{-i/\varepsilon \int_{X_t}^{X} [\omega C_0 \sigma/(C_0^2 - U_0^2)]\, dX'} + i e^{+i/\varepsilon \int_{X_t}^{X} [\omega C_0 \sigma/(C_0^2 - U_0^2)]\, dX'}\right] \quad \text{for } X < X_t,$$

$$\frac{1}{\sqrt{\sigma(X)}} e^{-1/\varepsilon \int_{X_t}^{X} [\omega C_0 |\sigma|/(C_0^2 - U_0^2)]\, dX'} \quad \text{for } X > X_t;$$

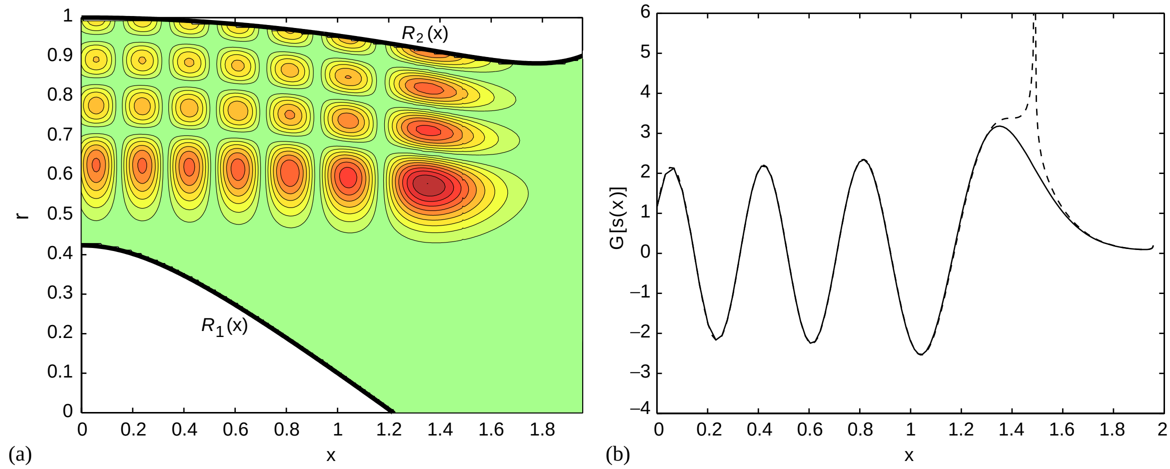

Fig. 2. First test case: no mean flow, $\omega = 41.0$ and $m = 21$. Fourth radial mode cuts off at $X_t = 1.5$. (a) Absolute modal pressure normalised by max $|p|$ with contour levels at intervals of 0.1. (b) Comparison of the non-convective axial variation of the uniformly valid solution (solid line) with the outer WKB solution (dotted line).

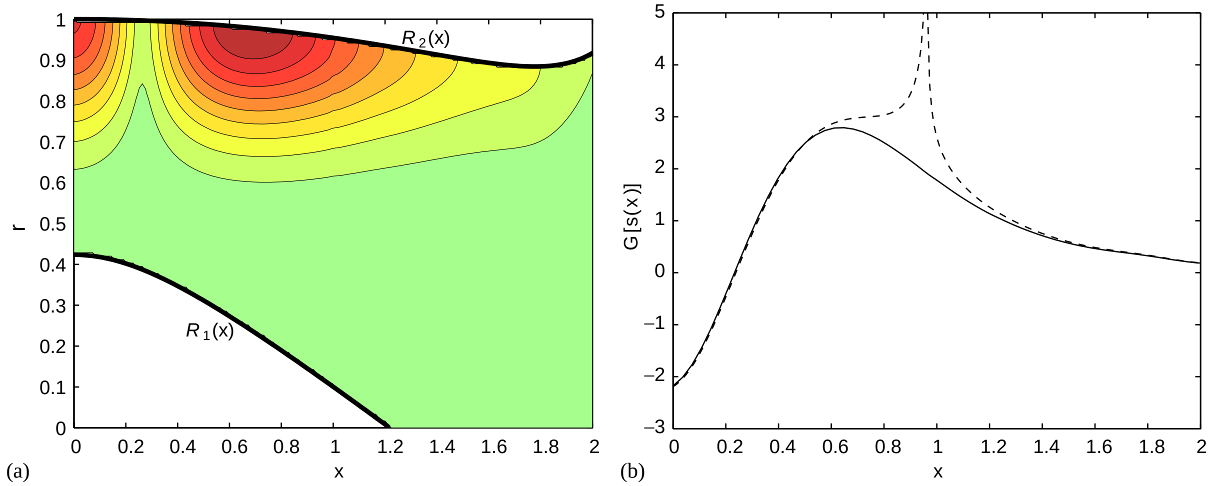

Fig. 3. Second test case: with irrotational mean flow, $\omega = 11.1$ and $m = 10$. First radial mode cuts off at $X_t = 0.95$. (a) Absolute modal pressure normalised by max $|p|$ with contour levels at intervals of 0.1. (b) Comparison of the non-convective axial variation of the uniformly valid solution (solid line) with the outer WKB solution (dotted line).

this is, of course, singular at the transition point X_t. Notice that in all three figures the two approximations agree exactly except in a region neighbouring the singularity.

The first test case (Fig. 2) is for a single mode with no mean flow. The non-dimensional frequency is high, $\omega = 41.0$, and the circumferential wavenumber is $m = 21$. The multiple-scales solution predicts the fourth radial mode to be cut-on initially at $x = 0$ in this case, but to undergo cut-on cut-off transition around $x = 1.5$ in non-dimensional coordinates. The analytical solution

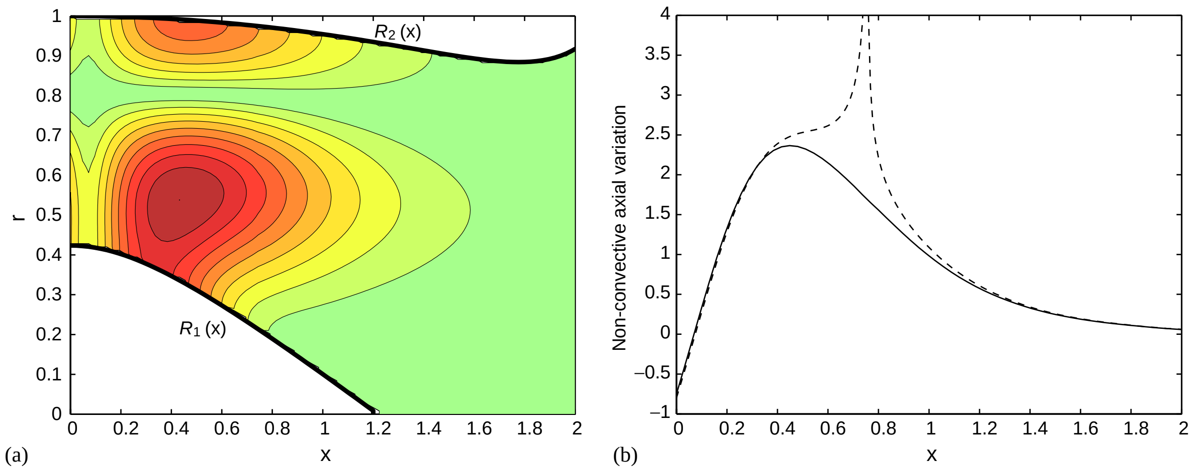

Fig. 4. Third test case: with irrotational mean flow, $\omega = 8.6$ and $m = 4$. Second radial mode cuts off at $X_t = 0.75$. (a) Absolute modal pressure normalised by max $|p|$ with contour levels at intervals of 0.1. (b) Comparison of the non-convective axial variation of the uniformly valid solution (solid line) with the outer WKB solution (dotted line).

shows that a standing wave appears to be set up between the source plane and the transition point. The largest pressure peak occurring just ahead of transition is stretched out axially due to the noticeable exponential decay beyond $x = 1.5$.

The second test case (Fig. 3) has an irrotational mean flow and has been investigated previously by Li et al. [18] using a numerical finite-difference scheme and an almost identical inlet geometry. The axial Mach number at the source plane $x = 0$ is 0.5 and the direction of this mean flow is from right to left. The non-dimensional frequency is $\omega = 11.1$ with circumferential wavenumber $m = 10$. The first radial mode in this case undergoes cut-on cut-off transition at roughly $x = 1.0$. Note that in this case, plot (b) shows the extent of the inner boundary layer to be quite large (spanning well over one-third of the duct length), suggesting that resolving the pressure field accurately using multiple-scales theory throughout the duct would have been very difficult without a composite solution. Indeed, had the transition point occurred closer to the source plane the slowly varying modal solution would not predict the magnitude and phase of the reflected mode accurately, with possible consequences for investigating resonance. The absolute pressure obtained with the composite solution and that obtained from Ref. [18] possess similar, although axially displaced, features. The exponential decay of the cut-off transmitted part does not seem to be so apparent in their numerical solution. Note the appearance of a high fluctuating pressure region again just ahead of the transition point on the outer (engine nacelle) wall. Plot (b) suggests that the exact position and extent of this high-pressure region can only be quantified accurately by the uniformly valid solution. Better quantitative agreement has been obtained with a more recent numerical comparison from Ref. [19].

The third test case has an identical irrotational mean flow to the second case above, but the mode has a lower circumferential wavenumber allowing the effect of the central spinner (the inner wall) to be more significant than observed in the previous two cases. For this case, the non-dimensional frequency $\omega = 8.6$ and the circumferential wavenumber $m = 4$, and the second radial mode undergoes cut-on cut-off transition at around $x = 0.75$. Notice the large (and of course fluctuating) pressure just ahead of the transition point as for the cases above, although this time the focus appears to be concentrated towards the inner wall. Once again here as for the second case, there is a large section of the duct where the slowly varying approximation fails to predict accurately the pressure field. An examination of the scaling for the inner-solution region, given by Eqs. (18) and (19), indicates that it is the lower non-dimensional frequencies in the last two cases that increase the length of this inner-solution region, which scales as $\omega^{-2/3}$, rather than any effect of the mean flow. Such an estimate predicts the second and third cases to have inner-solution regions about 2.5 times the length of the first case, which appears to be very reasonable for these results.

7. Conclusions

We have derived a uniformly valid explicit solution for an acoustic mode undergoing cut-on cut-off transition in a slowly varying hard-walled duct of arbitrary cross section. Computing the solution given by Eq. (38) is no more complicated than for a normal slowly varying mode, so long as the point of transition is known in advance and Airy functions can be evaluated. As a result, the removal of the singularity present in the original multiple scales analysis allows further exploitation of the multiple scales approach as an alternative to full numerical evaluation. The three examples presented demonstrate the applicability of the method to realistic engine duct geometries and realistic engine frequencies. The uniformly valid solution will allow prediction of experimental pressure measurement anywhere inside an engine duct where cut-on cut-off transition is occurring. It should provide a useful benchmark for computational aeroacoustic (CAA) codes in being able to deal with acoustic reflections due to alterations in geometry. As observed in the results, the solution also seems to reveal positions of high fluctuating acoustic pressure that occur slightly ahead of a transition point. Accurate quantification of these regions, impossible from the original multiple-scales analysis without using the uniformly valid solution, may be necessary to judge consequent implications for nacelle structure integrity and structure-borne noise.

Aside from cut-on cut-off transition, the general solution in Eq. (31) allows us to model any isolated transitional phenomena with an incident cut-on and/or an incident cut-off mode (the amplitude of which is represented by Φ_2). A comparison of various problems involving modal transition is currently in preparation with Prof. W. Eversman and Dr. S.W. Rienstra.

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