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Acoustic waves in slender axisymmetric tubes

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Abstract

Acoustic waves in a rigid axisymmetric tube with a variable cross-section are considered. The governing Helmholtz equation is solved using Neumann series (expansions in Bessel functions of various orders) with a stretched radial coordinate, leading to a hierarchy of one-dimensional ordinary differential equations in the longitudinal direction. The lowest approximation for axisymmetric motion includes Webster's horn equation as a special case. Fourth-order differential equations are obtained at the next level of approximation. Good agreement with existing asymptotic theories for waves in slender tubes is found.

1. Introduction

The propagation of sound along rigid tubes is a classical subject with an enormous literature. Early studies are concerned with low-frequency waves, meaning that the acoustic pressure is supposed to be constant over any cross-section of the tube. This assumption leads to an ordinary differential equation,

$$\frac{1}{A}\frac{\mathrm{d}}{\mathrm{d}z}\left(A(z)\frac{\mathrm{d}P}{\mathrm{d}z}\right)+\frac{\omega^2}{c^2}P(z)=0, \tag{1}$$

where z is a coordinate along the tube, $P(z)$ is the pressure at z where the cross-sectional area is $A(z)$, ω is the frequency and c is the speed of sound. Eq. (1) is usually known as *Webster's horn*

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equation, even though it is actually much older; for a thorough review, see Ref. [1]; for a textbook derivation, see Ref. [2, p. 360]. Extensions of Webster's equation to other related problems (such as when there is fluid flow along the tube) have been made; for example, see Refs. [3,4]. Exact solutions of Eq. (1) are also known for various specific functions, $A(z)$; see Refs. [5,6] and references therein for more information on this topic.

The exact problem for time-harmonic waves in a tube can be formulated as a three-dimensional elliptic boundary-value problem for a pressure field, u. However, exact treatments of this boundary-value problem are scarce. The simplest exact solutions concern waves in a cylinder with a circular cross-section; these solutions can be obtained by the method of separation of variables in cylindrical polar coordinates, (r, θ, z), and they involve Bessel functions. Indeed, the lowest axisymmetric mode depends on z only, and it does satisfy Webster's equation. However, all other modes are not governed by Eq. (1).

We are interested in the derivation of other one-dimensional models. Inevitably, these will be approximations, but there is much scope for improvements on Webster's equation. We suppose from the outset that the tube is axisymmetric, with the rigid lateral boundary given by $r = \mathscr{R}(z)$. We then change the independent variables in the governing Helmholtz equation from r and z to ρ and ζ, where ρ is a scaled version of r chosen so that the lateral boundary at $r = \mathscr{R}(z)$ is mapped to $\rho =$ constant, and $\zeta = z$. This leads to a more complicated governing partial differential equation (which we write concisely as $\mathscr{L}u = 0$; see Eq. (9) below), but this is outweighed by moving the lateral boundary condition onto a coordinate surface.

In a previous paper [7], we solved $\mathscr{L}u = 0$ using a power-series expansion in the new 'radial' variable ρ, namely

$$u(\rho, \theta, \zeta) = \sum_{n=0}^{\infty} u_n(\zeta)\rho^{2n+m} \cos m\theta, \tag{2}$$

similar expansions have been used previously by Boström [8] for elastic waves in cylindrical rods of circular cross-section. Substitution of Eq. (2) into $\mathscr{L}u = 0$ leads to a recursive construction of u_n for $n \geqslant 1$ once u_0 has been determined. Substitution of Eq. (2) into the lateral boundary condition leads to a further equation; if the tube is slender (in a sense to be made precise later), this equation can be truncated, resulting in a hierarchy of ordinary differential equations for $u_0(\zeta)$.

The method outlined above is attractive and general: the approximations can be improved at the expense of solving higher-order differential equations, and the method itself can be extended to other governing partial differential equations. However, the lowest-order approximations only work well for low-frequency motions. For this reason, we have modified expansion (2), replacing the power series by a *Neumann series* [9, Chapter 16],

$$u(\rho, \theta, \zeta) = \sum_{n=0}^{\infty} u_n(\zeta) J_{2n+m}(\lambda\rho) \cos m\theta,$$

where J_ν is a Bessel function and λ is a parameter to be selected. The main motivation behind choosing a Neumann series is that we know that all modes for circular cylinders ($\mathscr{R} =$ constant) are given by single Bessel functions of various orders and arguments. Thus, we may expect better results for tubes of slowly varying cross-sections, even when the frequency is not low. In addition, we have previously used Neumann series with success for elastic waves in wooden poles [10].

The problem of waves in a tube of length L is formulated in Section 2; the stretched variables are also used there, leading to $\mathscr{L}u = 0$. The effects of using the Neumann series are explored in Section 3, with ordinary differential equations for u_0 derived in Section 4. At the lowest order of truncation, we find that u_0 solves a second-order equation,

$$u_0'' + D_m^{(1)} u_0' + E_m^{(1)} u_0 = 0, \tag{3}$$

where the coefficients $D_m^{(1)}$ and $E_m^{(1)}$ are explicit functions of the axial coordinate z and the azimuthal mode number m; we recover Webster's equation in an appropriate limit. For tubes of finite length, Eq. (3) is to be solved subject to certain boundary conditions (which are derived below); the resulting problem is a regular Sturm–Liouville problem, for which efficient numerical algorithms are available [11].

At the next order of truncation, we find that u_0 solves a fourth-order equation,

$$u_0^{\text{iv}} + B_m^{(2)} u_0''' + C_m^{(2)} u_0'' + D_m^{(2)} u_0' + E_m^{(2)} u_0 = 0, \tag{4}$$

where, again, the coefficients are known explicit functions of z and m. Higher orders of truncation lead to higher-order differential equations.

In Section 5, the quality of the approximations obtained by solving Eq. (3) or (4) is assessed by comparison with some asymptotic approximations of Ting and Miksis [12] and Geer and Keller [13]. These authors treated the original three-dimensional elliptic boundary-value problem for u, exploiting the slenderness assumption. We find, in particular, that for non-axisymmetric modes with $m = 1$, Eq. (4) gives excellent agreement with the asymptotic approximation in Refs. [12,13], and this agreement is significantly better than that achieved when using the analogous fourth-order equation derived in Ref. [7] from Eq. (2). Consequently, we conclude that similar fourth-order models will offer both simplicity and approximations of good quality: the key is seen to be the use of accurate representations in each cross-sectional plane.

2. Formulation

Consider a tube of circular cross-section and length L; the cross-section's radius can vary along the tube. We define dimensionless variables using L as our length scale. Thus, using (dimensionless) cylindrical polar coordinates, (r, θ, z), the interior of the tube is specified by

$$0 \leqslant r < \varepsilon R(z), \quad 0 \leqslant \theta < 2\pi, \quad 0 < z < 1.$$

Here, $\varepsilon = a/L$ and $0 < R(z) \leqslant 1$, so that $2a$ is the maximum diameter of the tube.

Inside the tube, the acoustic potential $U(r, \theta, z, t)$ satisfies the wave equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = \frac{L^2}{c^2}\frac{\partial^2 U}{\partial t^2}, \tag{5}$$

where c is the speed of sound. On the lateral wall of the tube, the normal derivative of U vanishes:

$$\frac{\partial U}{\partial r} - \varepsilon R'(z)\frac{\partial U}{\partial z} = 0 \quad \text{on } r = \varepsilon R(z), \ 0 < z < 1. \tag{6}$$

We assume that $R(0)$ and $R(1)$ are both positive, and close the two ends of the tube with rigid circular discs, giving

$$\partial U/\partial z = 0 \quad \text{at } z = 0 \text{ and at } z = 1. \tag{7}$$

We seek free vibrations of the compressible fluid within the axisymmetric tube. Thus, we put

$$U(r, \theta, z, t) = u(r, z) \cos m\theta \cos \omega t,$$

where m is a non-negative integer and ω is the frequency. The wave equation becomes

$$r \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) + r^2 \frac{\partial^2 u}{\partial z^2} + (k^2 r^2 - m^2) u = 0, \tag{8}$$

where $k = \omega L / c$ is a dimensionless wavenumber. We are interested in determining *eigen-frequencies* ω so that there is a non-trivial u that satisfies Eq. (8) and the boundary conditions.

It is convenient to make a simple change of the independent variables, from (r, z) to (ρ, ζ), so that the new geometry is a circular cylinder. Thus, define new variables ρ and ζ by

$$\rho = r/R(z) \quad \text{and} \quad \zeta = z$$

so that the tube is mapped onto the circular tube, given by

$$0 \leqslant \rho < \varepsilon, \quad 0 < \zeta < 1.$$

Application of the chain rule shows that Eq. (8) becomes

$$(1 + \rho^2 R'^2)\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho} \right) + \rho^3 (R'^2 - RR'') \frac{\partial u}{\partial \rho} + \rho^2 R^2 \frac{\partial^2 u}{\partial \zeta^2} - 2\rho^3 RR' \frac{\partial^2 u}{\partial \rho \partial \zeta} + [(k\rho R)^2 - m^2] u = 0, \tag{9}$$

the lateral boundary condition (6) becomes

$$(1 + \varepsilon^2 R'^2) \frac{\partial u}{\partial \rho} - \varepsilon R R' \frac{\partial u}{\partial \zeta} = 0 \quad \text{on } \rho = \varepsilon, \; 0 < \zeta < 1 \tag{10}$$

and the end boundary conditions (7) become

$$\frac{\partial u}{\partial \zeta} - \rho \frac{R'}{R} \frac{\partial u}{\partial \rho} = 0 \quad \text{at } \zeta = 0 \text{ and at } \zeta = 1. \tag{11}$$

3. Neumann series

In a previous paper [7], we solved Eq. (9) using a power-series expansion for u; see Eq. (2). Here, motivated by the exact solution for circular tubes, we use a Neumann expansion,

$$u(\rho, \zeta) = \sum_{n=0}^{\infty} J_{2n+\alpha}(\lambda \rho) u_n(\zeta), \tag{12}$$

where α and $u_n(\zeta)$ are to be found, λ is a (non-zero) constant at our disposal and $J_\nu(z)$ is a Bessel function.

When substituting in Eq. (9), we make use of the fact that $w(\rho) = J_\nu(\lambda\rho)$ solves

$$\rho[\rho w']' + (\lambda^2\rho^2 - \nu^2)w = 0.$$

Thus, Eq. (9) becomes

$$\begin{aligned} 0 = &\sum_{n=0}^{\infty} [(2n+\alpha)^2 - m^2] u_n(\zeta) J_{2n+\alpha}(\lambda\rho) \\ &+ \rho^2 \sum_{n=0}^{\infty} \{[(2n+\alpha)^2 R'^2 - \lambda^2 + k^2R^2]u_n + R^2 u_n''\} J_{2n+\alpha}(\lambda\rho) \\ &- \rho^4\lambda^2 R'^2 \sum_{n=0}^{\infty} u_n(\zeta) J_{2n+\alpha}(\lambda\rho) \\ &+ \rho^3\lambda \sum_{n=0}^{\infty} [(R'^2 - RR'')u_n - 2RR'u_n'] J'_{2n+\alpha}(\lambda\rho). \end{aligned} \tag{13}$$

We rewrite this equation in the form

$$\sum_{n=0}^{\infty} \mathcal{G}_n J_{2n+\alpha}(\lambda\rho) = 0, \tag{14}$$

where $\mathcal{G}_n$ is independent of ρ. Then, it follows that $\mathcal{G}_n = 0$ for $n = 0, 1, 2, \ldots$.

In order to obtain Eq. (14), we need the following three expansions:

$$z^2 J_\nu(z) = \sum_{n=1}^{\infty} a_n^{(1)} J_{2n+\nu}(z), \quad z^4 J_\nu(z) = \sum_{n=2}^{\infty} a_n^{(2)} J_{2n+\nu}(z),$$

$$z^3 J'_\nu(z) = \sum_{n=1}^{\infty} a_n^{(3)} J_{2n+\nu}(z).$$

The coefficients $a_n^{(j)}$ can be obtained by manipulating the recurrence relations for Bessel functions. For our purposes, we shall want the coefficients for $n \leqslant 2$. Thus

$$z^2 J_\nu(z) = 4(\nu+2)[(\nu+1)J_{\nu+2}(z) - 2(\nu+4)J_{\nu+4}(z)] + \cdots,$$

$$z^4 J_\nu(z) = 16(\nu+1)(\nu+2)(\nu+3)(\nu+4)J_{\nu+4}(z) + \cdots,$$

$$z^3 J'_\nu(z) = 4(\nu+2)[\nu(\nu+1)J_{\nu+2}(z) - 2(\nu+4)(2\nu+3)J_{\nu+4}(z)] + \cdots.$$

Substitution in Eq. (13) gives Eq. (14) in which

$$\mathcal{G}_0 = (\alpha^2 - m^2)u_0(\zeta),$$

$$\begin{aligned} \mathcal{G}_1 = &[(\alpha+2)^2 - m^2]u_1(\zeta) + 4(\alpha+1)(\alpha+2)\lambda^{-2}\{R^2u_0'' - 2\alpha RR'u_0' \\ &+ [k^2R^2 - \lambda^2 + \alpha(\alpha+1)R'^2 - \alpha RR'']u_0\}, \end{aligned}$$

$$\mathscr{G}_2 = [(\alpha+4)^2 - m^2]u_2(\zeta) + 4(\alpha+3)(\alpha+4)\lambda^{-2}\Omega_1(\alpha) - 8(\alpha+2)(\alpha+4)\lambda^{-2}\Omega_0(\alpha),$$

$$\Omega_0(\alpha) = R^2u_0'' - 2(2\alpha+3)RR'u_0' + [k^2R^2 - \lambda^2 + (3\alpha^2+10\alpha+9)R'^2 - (2\alpha+3)RR'']u_0,$$

$$\Omega_1(\alpha) = R^2u_1'' - 2(\alpha+2)RR'u_1' + [k^2R^2 - \lambda^2 + (\alpha+2)(\alpha+3)R'^2 - (\alpha+2)RR'']u_1.$$

From $\mathscr{G}_0 = 0$, we deduce that $\alpha^2 = m^2$; we take $\alpha = +m \geqslant 0$, as we want u to be bounded at $\rho = 0$. Then, $\mathscr{G}_1 = 0$ yields an expression for u_1 in terms of u_0:

$$-\lambda^2 u_1 = (m+2)\{R^2u_0'' - 2mRR'u_0' + [k^2R^2 - \lambda^2 + m(m+1)R'^2 - mRR'']u_0\}. \qquad (15)$$

Similarly, $\mathscr{G}_2 = 0$ yields an expression for u_2 in terms of u_1 and u_0:

$$-2\lambda^2(m+2)u_2 = (m+4)\{(m+3)\Omega_1(m) - 2(m+2)\Omega_0(m)\}.$$

Eliminating u_1 from this equation, using Eq. (15), gives

$$2\lambda^4 u_2 = (m+3)(m+4)\{\alpha_m u_0^{\mathrm{iv}} + \beta_m u_0''' + \gamma_m u_0'' + \delta_m u_0' + \varepsilon_m u_0\}, \qquad (16)$$

where

$$\alpha_m(z) = R^4, \quad \beta_m(z) = -4mR^3R',$$

$$\gamma_m(z) = 2R^2\{k^2R^2 + 3m[(m+1)R'^2 - RR'']\} - 2\lambda^2R^2(m+2)/(m+3),$$

$$\delta_m(z) = -4mR\{k^2R^2R' + R^2R''' - (m+1)R'[3RR'' - (m+2)R'^2]\} + 4\lambda^2RR'm(m+2)/(m+3),$$

$$\varepsilon_m(z) = k^4R^4 + 2mk^2R^2\{(m+1)R'^2 - RR''\} - mR^3R^{\mathrm{iv}} + m(m+1)\{R^2(4R'R''' + 3R''^2) - (m+2)R'^2[6RR'' - (m+3)R'^2]\} + \lambda^4\frac{m+1}{m+3} - 2\lambda^2\frac{m+2}{m+3}[k^2R^2 + m(m+1)R'^2 - mRR''].$$

Proceeding in a similar way, we could determine all the functions $u_n(\zeta)$ in Eq. (12) in terms of derivatives of $u_0(\zeta)$. Evidently, hand calculation of u_n with $n>2$ would be tedious, but the calculation could be mechanized if desired.

3.1. Lateral boundary condition

Substituting Eq. (12) in the lateral boundary condition (10) gives

$$\sum_{n=0}^{\infty} \lambda(1+\varepsilon^2R'^2)u_n(\zeta)J'_{2n+m}(\lambda\varepsilon) - \sum_{n=0}^{\infty} \varepsilon RR'u_n'(\zeta)J_{2n+m}(\lambda\varepsilon) = 0, \qquad (17)$$

where we have used $\alpha = m$. Later, we shall want to truncate this equation, assuming that $\varepsilon \ll 1$.

With this in mind, we rewrite Eq. (17) as

$$\sum_{n=0}^{\infty} \mathscr{H}_n J'_{2n+m}(\lambda\varepsilon) = 0, \tag{18}$$

using the following two expansions:

$$\begin{aligned} z^2 \mathrm{J}'_\nu(z) &= \sum_{n=1}^{\infty} a_n^{(4)} J'_{2n+\nu}(z) \\ &= 4\nu(\nu+1)J'_{\nu+2}(z) - 8[(\nu+2)(\nu+4)-2]J'_{\nu+4}(z) + \cdots, \end{aligned}$$

$$\begin{aligned} z\mathrm{J}_\nu(z) &= \sum_{n=1}^{\infty} a_n^{(5)} J'_{2n+\nu}(z) \\ &= 4(\nu+1)J'_{\nu+2}(z) - 8J'_{\nu+4}(z) + \cdots. \end{aligned}$$

Thus, we find that the first three $\mathscr{H}_n$ are

$$\mathscr{H}_0 = \lambda^2 u_0,$$

$$\mathscr{H}_1 = \lambda^2 u_1 + 4(m+1)(mR'^2 u_0 - RR'u'_0),$$

$$\begin{aligned} \mathscr{H}_2 = \lambda^2 u_2 &+ 4(m+3)\{(m+2)R'^2 u_1 - RR'u'_1\} \\ &- 8\{[(m+2)(m+4)-2]R'^2 u_0 - RR'u'_0\}. \end{aligned}$$

Eliminating u_1, using (15), gives

$$\mathscr{H}_1(z) = -(m+2)R^2\{u''_0(z) + D_m^{(1)}(z)u'_0(z) + E_m^{(1)}(z)u_0(z)\}, \tag{19}$$

where

$$D_m^{(1)} = \frac{2(2-m^2)}{m+2}\mathscr{S}_1, \quad E_m^{(1)}(z) = k^2 - \frac{\lambda^2}{R^2} - m\mathscr{S}_2 + \frac{m(m+1)(m-2)}{m+2}\mathscr{S}_1^2 \tag{20}$$

and, here and below, we use the following shorthand notation:

$$\mathscr{S}_1 = R'/R, \quad \mathscr{S}_2 = R''/R, \quad \mathscr{S}_3 = R'''/R, \quad \mathscr{S}_4 = R^{\mathrm{iv}}/R.$$

A similar calculation yields

$$2\lambda^2\mathscr{H}_2(z) = (m+3)(m+4)R^4\{u_0^{\mathrm{iv}} + \mathscr{B}u'''_0 + \mathscr{C}u''_0 + \mathscr{D}u'_0 + \mathscr{E}u_0\}, \tag{21}$$

where

$$\mathscr{B} = 4\mathscr{S}_1(4-2m-m^2)/(m+4),$$

$$\mathscr{C} = 2k^2 - 2\frac{\lambda^2}{R^2}\frac{m+2}{m+3} - 6m\left\{\mathscr{S}_2 + \frac{4-m-m^2}{m+4}\mathscr{S}_1^2\right\},$$

$$\mathscr{D} = 4k^2\mathscr{S}_1 \frac{4-2m-m^2}{m+4} + 4\mathscr{S}_1 \frac{\lambda^2}{R^2}\frac{m^2-2}{m+3} - 4m\mathscr{S}_3 + \frac{4m\mathscr{S}_1}{m+4}\{(m+1)(4-m^2)\mathscr{S}_1^2 + 3m(m+3)\mathscr{S}_2\},$$

$$\begin{aligned}\mathscr{E} = k^4 &+ \frac{\lambda^4}{R^4}\frac{m+1}{m+3} - m\mathscr{S}_4 + 3m(m+1)\mathscr{S}_2^2 \\ &+ 2mk^2\left\{\frac{m^2+m-4}{m+4}\mathscr{S}_1^2 - \mathscr{S}_2\right\} + \frac{4m^2(m+3)}{m+4}\mathscr{S}_1\mathscr{S}_3 \\ &- \frac{m(m+1)(m+2)}{m+4}\mathscr{S}_1^2\{6m\mathscr{S}_2 + (4+m-m^2)\mathscr{S}_1^2\} \\ &- 2\frac{\lambda^2}{R^2}\left\{\frac{m+2}{m+3}(k^2 - m\mathscr{S}_2) + \frac{m(m+1)(m-2)}{m+3}\mathscr{S}_1^2\right\}.\end{aligned}$$

3.2. End boundary conditions

Substituting Eq. (12) in the end boundary conditions (11) gives

$$\sum_{n=0}^{\infty} u_n'(\zeta) J_{2n+m}(\lambda\rho) - \lambda\rho\mathscr{S}_1 \sum_{n=0}^{\infty} u_n(\zeta) J_{2n+m}'(\lambda\rho) = 0 \tag{22}$$

at $\zeta = 0$ and at $\zeta = 1$, where we have used $\alpha = m$. Using the expansion

$$zJ_v'(z) = vJ_v(z) + 2\sum_{l=1}^{\infty}(-1)^l(v+2l)\,J_{v+2l}(z)$$

and then changing the order of summation, we can write Eq. (22) as

$$\sum_{n=0}^{\infty} \mathscr{M}_n(\zeta) J_{2n+m}(\lambda\rho) = 0 \quad \text{at } \zeta = 0 \text{ and at } \zeta = 1,$$

where $\mathscr{M}_0(\zeta) = u_0' - m\mathscr{S}_1 u_0$ and

$$\mathscr{M}_n(\zeta) = u_n' - (2n+m)\mathscr{S}_1\left\{u_n + 2\sum_{l=0}^{n-1}(-1)^{n+l}u_l\right\}$$

for $n = 1, 2, \ldots$. Hence, we immediately obtain

$$\mathscr{M}_n(0) = \mathscr{M}_n(1) = 0 \quad \text{for } n = 0, 1, 2, \ldots.$$

In particular, we have

$$u_0'(z) - m\mathscr{S}_1(z)\,u_0(z) = 0 \quad \text{at } z = 0 \text{ and at } z = 1 \tag{23}$$

and

$$u_1'(z) - (m+2)\mathscr{S}_1(z)\{u_1(z) - 2u_0(z)\} = 0 \quad \text{at } z = 0 \text{ and at } z = 1. \tag{24}$$

Eliminating u_1 using Eq. (15), and then using Eq. (23), we can write Eq. (24) as

$$u_0''' - 3m\mathscr{S}_1 u_0'' + \{2(m^2-1)\mathscr{S}_1^2 + 3\mathscr{S}_2\}u_0' - m\mathscr{S}_3 u_0 = 0 \tag{25}$$

at the two ends of the tube. This form of the boundary condition does not involve k^2.

4. Ordinary differential equations

In order to obtain ordinary differential equations for $u_0(z)$, we truncate Eq. (18). (Some comments on this procedure are given in Section 6.) The term with $n = 0$ gives

$$\lambda^2 \mathrm{J}_m'(\lambda\varepsilon)u_0(z) = 0$$

and so we take

$$\lambda = \frac{j_{m,\ell}'}{\varepsilon} = \lambda_{m,\ell}, \tag{26}$$

say, where ℓ is an integer and $j_{m,\ell}'$ is the ℓth zero of J_m': $J_m'(j_{m,\ell}') = 0$ for $\ell = 1, 2, \ldots$. Note that $j_{0,1}' = 0$.

4.1. First approximation

Retaining the $n = 1$ term in Eq. (18), we obtain

$$\mathscr{H}_1 J_{m+2}'(j_{m,\ell}') = 0.$$

Hence, our first approximation is $\mathscr{H}_1 = 0$, so that Eq. (19) gives

$$u_0''(z) + D_m^{(1)}(z)u_0'(z) + E_m^{(1)}(z)\,u_0(z) = 0, \tag{27}$$

where $D_m^{(1)}$ and $E_m^{(1)}$ are defined by Eq. (20) wherein λ is given by Eq. (26). Eq. (27) is to be solved subject to Eq. (23).

When $R(z) \equiv 1$, Eq. (27) reduces to

$$u_0''(z) + (k^2 - \lambda_{m,\ell}^2)u_0(z) = 0, \tag{28}$$

which yields the exact solution for circular tubes. (Note that we also obtain $u_n(z) \equiv 0$ for $n = 1, 2, \ldots$.)

When $m = 0$, Eq. (27) reduces to

$$(R^2 u_0')' + (k^2R^2 - \lambda_{0,\ell}^2)u_0 = 0. \tag{29}$$

This is a generalization of Webster's horn equation (for which $\lambda_{0,\ell} = 0$). It is to be solved subject to $u_0'(0) = u_0'(1) = 0$. Note that exact solutions of Eq. (29) can be found for specific functions, $R(z)$; for example, if $R(z) = \mu z$ (where μ is a constant), solutions for u_0 can be constructed in terms of Bessel functions.

Eq. (27) can be transformed so as to eliminate the first-derivative term. Thus, put

$$u_0(z) = R^\gamma U_0(z) \quad \text{with } \gamma = (m^2-2)/(m+2). \tag{30}$$

Then, we find that $U_0(z)$ solves

$$U_0''(z) + [k^2 - K(z)]U_0(z) = 0, \tag{31}$$

where

$$K(z) = \frac{2(m+1)}{m+2}\left\{\frac{m\mathscr{S}_1^2}{m+2} + \mathscr{S}_2\right\} + \frac{\lambda_{m,\ell}^2}{R^2}$$

subject to $(m+2)U_0' = 2(m+1)\mathscr{S}_1 U_0$ at $z=0$ and at $z=1$. Notice that Eq. (31) has oscillatory solutions when $k^2 > K$ but exponential solutions when $k^2 < K$.

Eq. (27) and its associated boundary conditions can also be written as a regular Sturm–Liouville problem. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(p(z)\frac{\mathrm{d}u_0}{\mathrm{d}z}\right) + [q(z) + k^2 w(z)]u_0(z) = 0,$$

where $p = w = R^{-2\gamma}$, γ is defined by Eq. (30), and

$$q(z) = R^{-2\gamma}\left\{\frac{m(m+1)(m-2)}{m+2}\,\mathscr{S}_1^2 - m\mathscr{S}_2 - \frac{\lambda_{m,\ell}^2}{R^2}\right\}.$$

Software for solving Sturm–Liouville problems is available [11].

4.2. Second approximation

If we retain both the $n=1$ and the $n=2$ terms in Eq. (18), we obtain

$$\mathscr{H}_1 J'_{m+2}(j'_{m,\ell}) + \mathscr{H}_2 J'_{m+4}(j'_{m,\ell}) = 0.$$

Then, Eq. (19) and (21) give

$$u_0^{\mathrm{iv}}(z) + B_m^{(2)}(z)\,u_0'''(z) + C_m^{(2)}(z)u_0''(z) + D_m^{(2)}(z)u_0'(z) + E_m^{(2)}(z)\,u_0(z) = 0, \tag{32}$$

where

$$B_m^{(2)} = \mathscr{B}, \quad C_m^{(2)} = \mathscr{C} + (\varepsilon R)^{-2}\tau_{m,\ell},$$

$$D_m^{(2)} = \mathscr{D} + (\varepsilon R)^{-2}\tau_{m,\ell}D_m^{(1)}, \quad E_m^{(2)} = \mathscr{E} + (\varepsilon R)^{-2}\tau_{m,\ell}E_m^{(1)}$$

and

$$\tau_{m,\ell} = -\frac{2(m+2)[j'_{m,\ell}]^2}{(m+3)(m+4)}\,\frac{J'_{m+2}(j'_{m,\ell})}{J'_{m+4}(j'_{m,\ell})}.$$

Elementary calculations, using the recurrence relations for J_ν together with $J'_m(j'_{m,\ell}) = 0$, give

$$\tau_{m,\ell} = \frac{(m+1)j^4[j^2 - m(m+2)]}{(m+3)(m+4)[j^4 - 3(m+2)^2 j^2 + 2m(m+1)(m+3)(m+4)]} \tag{33}$$

with $j \equiv j'_{m,l}$. Formula (33) is not valid when $j = 0$. This case can arise only for $\tau_{0,1}$: as $[J'_2(w)]/[J'_4(w)] \sim 24/w^2$ as $w \to 0$, we have

$$\tau_{0,1} = -8. \tag{34}$$

Eq. (32) is to be solved subject to Eqs. (23) and (25).

When $R(z) \equiv 1$, Eq. (32) reduces to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + k^2 - \frac{m+1}{m+3}\,\lambda_{m,\ell}^2 + \frac{\tau_{m,\ell}}{\varepsilon^2}\right)\left(\frac{\mathrm{d}^2 u_0}{\mathrm{d}z^2} + (k^2 - \lambda_{m,\ell}^2)u_0\right) = 0,$$

which is satisfied by any solution of Eq. (28).

When $m = 0$, Eq. (32) reduces to

$$\begin{aligned} u_0^{\mathrm{iv}} &+ \frac{4R'}{R}\,u_0''' + 2\left(k^2 - \frac{v_{1,\ell}}{\varepsilon^2 R^2}\right)u_0'' \\ &+ \frac{4R'}{R}\left(k^2 - \frac{v_{1,\ell}}{\varepsilon^2 R^2}\right)u_0' + \left(k^4 - \frac{2k^2 v_{1,\ell}}{\varepsilon^2 R^2} + \frac{v_{2,\ell}}{\varepsilon^4 R^4}\right)u_0 = 0, \end{aligned} \tag{35}$$

where

$$v_{1,\ell} = \frac{2}{3}\,[j'_{0,\ell}]^2 - \frac{1}{2}\,\tau_{0,\ell} \quad \text{and} \quad v_{2,\ell} = \frac{1}{3}\,[j'_{0,\ell}]^4 - [j'_{0,\ell}]^2\tau_{0,\ell}.$$

The boundary conditions (23) and (25) reduce to $u_0'(0) = u_0'''(0) = u_0'(1) = u_0'''(1) = 0$.

5. Asymptotic approximations

Asymptotic approximations for waves in slender tubes have been derived in Refs. [12,13]. Two kinds of approximation are available. First, there are those for which the wavelength is comparable to the length of the tube, so that $k = O(1)$ as $\varepsilon \to 0$. It can be shown that all solutions of this kind are axisymmetric.

Thus, consider the fourth-order equation (35) with $m = 0$ and $\ell = 1$. Eq. (34) gives $v_{1,1} = 4$ and $v_{2,1} = 0$, and then Eq. (35) reduces to an equation studied previously; see Section IV.A of Ref. [7]. Moreover, this reduced equation was shown to yield the two-term approximation $k \simeq k_0 + \varepsilon^2 k_2$, with a formula for k_2 that agrees precisely with that obtained in Ref. [13] using a completely different approach.

All other solutions have shorter wavelengths. For wavelengths comparable to the tube diameter, we have $k = O(\varepsilon^{-1})$ as $\varepsilon \to 0$. For such solutions, the comparisons made in Ref. [7] were disappointing; indeed, these comparisons motivated the present work. Let us examine how our new approximations perform. Thus, in the differential equations (27) and (32), put

$$k = \frac{\kappa}{\varepsilon} \quad \text{and} \quad u_0(z) = E(z)w(z) \quad \text{with } E(z) = \exp\left\{\frac{\mathrm{i}}{\varepsilon}\int_{z_0}^{z} \Phi(t)\,\mathrm{d}t\right\}, \tag{36}$$

where $\Phi(t)$ and $w(z)$ are to be determined, and z_0 is a constant. Suppose further that

$$w(z) = w_0(z) + \varepsilon w_1(z) + \varepsilon^2 w_2(z) + \cdots.$$

Then, we obtain the following approximations:

$$u_0/E = w_0 + \varepsilon w_1 + O(\varepsilon^2), \quad u_0'/E = \varepsilon^{-1}\mathrm{i}\Phi w_0 + O(1),$$

$$u_0''/E = -\varepsilon^{-2}\Phi^2 w_0 + \varepsilon^{-1}(2\mathrm{i}\Phi w_0' + \mathrm{i}\Phi' w_0 - \Phi^2 w_1) + O(1),$$

$$u_0'''/E = -\varepsilon^{-3}\mathrm{i}\Phi^3 w_0 + O(\varepsilon^{-2}),$$

$$u_0^{\mathrm{iv}}/E = \varepsilon^{-4}\Phi^4 w_0 + \varepsilon^{-3}\Phi^2\{\Phi^2 w_1 - 2\mathrm{i}(3\Phi' w_0 + 2\Phi w_0')\} + O(\varepsilon^{-2})$$

as $\varepsilon \to 0$.

Let us begin with the second-order equation (27). We note that

$$D_m^{(1)} = O(1) \quad \text{and} \quad E_m^{(1)} = \varepsilon^{-2}(\kappa^2 - j^2/R^2) + O(1) \quad \text{as } \varepsilon \to 0,$$

where $j \equiv j'_{m,\ell}$. Then, substituting in Eq. (27) and collecting powers of ε, we find that the terms in ε^{-2} give

$$[\Phi(z)]^2 = \kappa^2 - j^2/R^2. \tag{37}$$

This agrees with the asymptotic solution obtained by Ting and Miksis [12] and by Geer and Keller [13].

The terms in ε^{-1} give

$$2\Phi w_0' + \{\Phi' + \Phi D_m^{(1)}\} w_0 = 0,$$

which is a first-order differential equation for w_0. Rearranging gives

$$\frac{(w_0\Phi^{1/2})'}{w_0\Phi^{1/2}} = \frac{w_0'}{w_0} + \frac{\Phi'}{2\Phi} = -\frac{D_m^{(1)}}{2} = -q_m^{(1)}\frac{R'}{R},$$

where $q_m^{(1)} = (2 - m^2)/(m + 2)$. An integration gives

$$w_0^2 \Phi R^{2q} = \text{constant}, \tag{38}$$

where w_0, Φ and R are all functions of z only, and $q \equiv q_m^{(1)}$.

Next, consider the fourth-order equation (32). Substituting as before, we find that

$$B_m^{(2)} = O(1), \quad C_m^{(2)} = \varepsilon^{-2}\mathscr{C}_m^{(2)} + O(1),$$

$$D_m^{(2)} = \varepsilon^{-2}\mathscr{D}_m^{(2)} + O(1) \quad \text{and} \quad E_m^{(2)} = \varepsilon^{-4}\mathscr{E}_m^{(2)} + O(\varepsilon^{-2})$$

as $\varepsilon \to 0$, where

$$\mathscr{C}_m^{(2)} = 2\kappa^2 + \frac{1}{R^2}\left(\tau_{m,\ell} - 2j^2\frac{m+2}{m+3}\right),$$

$$\mathscr{D}_m^{(2)} = 2\mathscr{S}_1\left\{2\kappa^2\frac{4 - 2m - m^2}{m+4} + \frac{2 - m^2}{(m+2)R^2}\left(\tau_{m,\ell} - 2j^2\frac{m+2}{m+3}\right)\right\},$$

$$\mathscr{E}_m^{(2)} = \left(\kappa^2 - \frac{j^2}{R^2}\right)\left\{\kappa^2 + \frac{1}{R^2}\left(\tau_{m,\ell} - j^2\frac{m+1}{m+3}\right)\right\}.$$

Then, we see that the terms in ε^{-4} give

$$\Phi^4 - \Phi^2 \mathscr{C}_m^{(2)} + \mathscr{E}_m^{(2)} = 0, \tag{39}$$

which can be rewritten as

$$\left(\Phi^2 - \left[\kappa^2 - \frac{j^2}{R^2}\right]\right)\left(\Phi^2 - \left[\kappa^2 - \frac{m+1}{m+3}\frac{j^2}{R^2} + \frac{\tau_{m,\ell}}{R^2}\right]\right) = 0.$$

This equation is satisfied by Eq. (37).

The terms in ε^{-3} give

$$\begin{aligned}0 = \{\Phi^4 - \Phi^2 \mathscr{C}_m^{(2)} + \mathscr{E}_m^{(2)}\}w_1 + 2\mathrm{i}\Phi w_0'\{\mathscr{C}_m^{(2)} - 2\Phi^2\} \\ + \mathrm{i}w_0\{\Phi \mathscr{D}_m^{(2)} + \Phi' \mathscr{C}_m^{(2)} - \Phi^3 B_m^{(2)} - 6\Phi^2\Phi'\}.\end{aligned}$$

The factor multiplying w_1 vanishes, due to Eq. (39), leaving a first-order differential equation for $w_0(z)$:

$$\frac{w_0'}{w_0} + \frac{\Phi^2 \mathscr{D}_m^{(2)} + \Phi\Phi' \mathscr{C}_m^{(2)} - \Phi^4 B_m^{(2)} - 6\Phi^3\Phi'}{2\Phi^2[\mathscr{C}_m^{(2)} - 2\Phi^2]} = 0.$$

Rearranging this equation gives

$$\frac{(w_0\Phi^{1/2})'}{w_0\Phi^{1/2}} = \frac{4\Phi\Phi' - \mathscr{D}_m^{(2)} + \Phi^2 B_m^{(2)}}{2[\mathscr{C}_m^{(2)} - 2\Phi^2]} = -q_{m,\ell}^{(2)}\frac{R'}{R},$$

where

$$q_{m,\ell}^{(2)} = \frac{(2-m^2)(m+3)(m+4)\tau_{m,\ell} - 2j^2(m+2)(2m^2+11m+8)}{(m+2)(m+4)[(m+3)\tau_{m,\ell} + 2j^2]}, \tag{40}$$

$j \equiv j'_{m,\ell}$ and we have used Eq. (37). Hence, an integration gives Eq. (38) again, but with $q \equiv q_{m,\ell}^{(2)}$.

The version of Eq. (38) derived in Refs. [12,13] has $q = 1$ for all m and for all ℓ. In general, our approximate solutions for q differ from unity. For example, we have $q_0^{(1)} = 1$ but $q_1^{(1)} = \frac{1}{3}$. For $m = 0$, our second approximation becomes

$$q_{0,\ell}^{(2)} = \frac{3\tau_{0,\ell} - 4[j'_{0,\ell}]^2}{3\tau_{0,\ell} + 2[j'_{0,\ell}]^2}$$

for the lowest axisymmetric mode, this gives $q_{0,1}^{(2)} = 1$. For the lowest non-axisymmetric mode ($m = \ell = 1$), we have $j = j'_{1,1} \simeq 1.841$, so that Eq. (33) gives $\tau_{1,1} \simeq -12.22$ and then Eq. (40) gives $q_{1,1}^{(2)} \simeq 1.06$; this is a significant improvement over the value 1.43 obtained from the fourth-order differential equation derived in Ref. [7] using the power-series expansion (2).

6. Conclusions

The fourth-order ordinary differential equation (32) is seen to be worthy of further study. We have begun to develop the model for elastic waves in slender solid axisymmetric bodies [14]. (A good example of such a body is a wooden baseball bat.) The underlying method is sufficiently general that cylindrical orthotropy can be included without much extra effort; see Ref. [10] for information on the modelling of wooden poles.

One mathematical point remains. Recall that ordinary differential equations were obtained by truncating an equation that was derived by imposition of the lateral boundary condition. The relevant equation is Eq. (18). We truncated this equation by truncating the sum over n. It is difficult to give a rigorous justification of this step because, although ε is small, we typically take $\lambda\varepsilon$ as being independent of ε (see Eq. (26)); in addition, the terms $\mathscr{H}_n$ also depend on ε. Nevertheless, the basic strategy is sound: we always generate an exact solution of the underlying partial differential equation via Eq. (12), regardless of the choice of u_0; then, good approximations of the lateral boundary condition should lead to good approximations for u_0.

Much remains to be done with the method developed above, even for axisymmetric, long-wavelength motions. For such motions, the textbook approach is to use Webster's horn equation, which is derived by assuming that the pressure is constant at any station z. We have shown that Webster's equation can be generalized and refined; we can represent *any* (unknown) cross-sectional variation via Eq. (12). Notice that $u_1, u_2, u_3, \ldots$ are all determined from u_0: in general, we always have an infinite series for u. The function u_0 itself is found by imposing the boundary condition on the rigid wall of the tube. We anticipate that a variety of exact analytical solutions of the simplest generalization of Webster's equation, namely Eq. (29), will be found for specific geometries, $R(z)$. Moreover, various numerical treatments are envisaged, but all this remains for future investigations.

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