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Short Communication

Damping in exact analysis of tapered beams

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Abstract

In this paper the free vibration frequencies of an Euler–Bernoulli tapered beam are deduced, in the presence of a concentrated mass at the end, and of a concentrated linear damper at an arbitrary section along the beam. The frequency equation is deduced and analytically solved in terms of Bessel functions.

1. Introduction

Quite recently, the exact free vibration analysis of Euler–Bernoulli beams in the presence of a linear dashpot has been performed by several authors. More particularly, a cantilever beam with a tip mass and a single dashpot at an arbitrary location has been studied in Ref. [1], and the resulting eigenvalue problem leads to complex solutions. Subsequently, the authors have examined [2] a generally restrained beam with a concentrated mass and a single dashpot placed at arbitrary sections, using an exact approach and employing the symbolic software Mathematica [3].

In this paper a beam with a power law variation of cross-section area is introduced, with concentrated mass at the left end and a concentrated linear dashpot at an arbitrary location. In fact, using some results from Ref. [4], the solution of the differential equation of motion is

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obtained in terms of Bessel functions, and subsequently various boundary conditions are dealt with.

2. The equation of motion

The equations of motion for the tapered beam in Fig. 1 are given by

$$\frac{\partial^2}{\partial z^2}\left\{EI(z)\,\frac{\partial^2 v_1}{\partial z^2}\right\} + \rho A(z)\,\frac{\partial^2 v_1}{\partial t^2} = 0, \quad 0<z<z_c, \tag{1}$$

$$\frac{\partial^2}{\partial z^2}\left\{EI(z)\,\frac{\partial^2 v_2}{\partial z^2}\right\} + \rho A(z)\,\frac{\partial^2 v_2}{\partial t^2} = 0, \quad z_c<z<L, \tag{2}$$

where

$$A(z) = A_0\left(1 + c\,\frac{z}{L}\right)^n, \quad I(z) = I_0\left(1 + c\,\frac{z}{L}\right)^{(n+2)} \tag{3}$$

and A_0 and I_0 are the cross-sectional area and moment of inertia of the section at left, respectively, z is the abscissa such that $-L/c \leqslant z \leqslant L$, L is the span of the beam, E is the Young modulus, ρ is the mass density, c is the taper ratio of the cross-section (with $c > -1$), t is the time, z_c is the variable location of the dashpot and n is a numerical exponent which will be usually equal to 1 or 2, as already described in Ref. [4].

The cross-sectional area $A(z)$ and the moment of inertia $I(z)$, as given in Eq. (3), can be introduced into Eqs. (1), (2), the following variable $\zeta = (1 + cz/L)$ can be introduced, and finally the variables can be separated as follows:

$$v_h(\zeta, t) = V_h(\zeta)\mathrm{e}^{\lambda t} \tag{4}$$

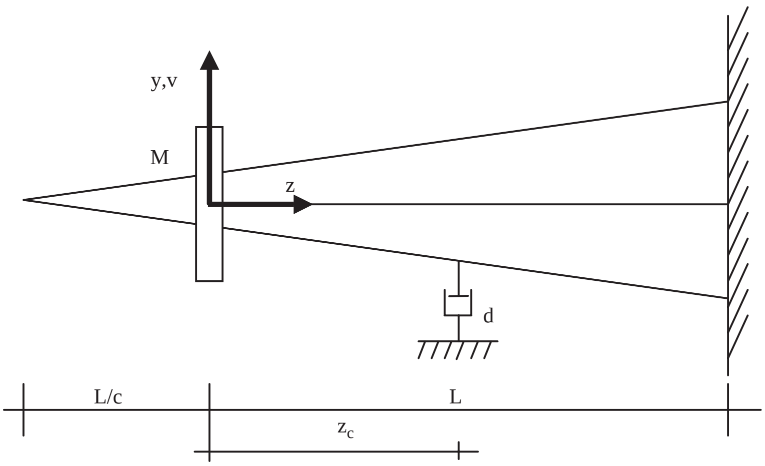

Fig. 1. Structural system.

with $h = 1, 2$. The equations of motion become:

$$\zeta^2 \frac{d^4 V_h}{d\zeta^4} + 2(n+2)\zeta \frac{d^3 V_h}{d\zeta^3} + (n+2)(n+1) \frac{d^2 V_h}{d\zeta^2} - \frac{\beta^4}{c^4} V_h = 0 \tag{5}$$

with

$$\beta^4 = -\frac{\lambda^2 L^4 \rho A_0}{EI_0}. \tag{6}$$

The general solutions of these equations are given by

$$V_1(\phi) = \frac{1}{\phi^n} (C_1 J_n(\phi) + C_2 Y_n(\phi) + C_3 I_n(\phi) + C_4 K_n(\phi)), \tag{7}$$

$$V_2(\phi) = \frac{1}{\phi^n} (C_5 J_n(\phi) + C_6 Y_n(\phi) + C_7 I_n(\phi) + C_8 K_n(\phi)) \tag{8}$$

with

$$\phi = \left(\frac{2\beta}{c} \sqrt{\zeta}\right), \tag{9}$$

where J, Y, I and K are the Bessel functions of the first and second kinds and their modified versions.

From Eqs. (7), (8) rotations, bending moments and shear stresses can be deduced, using the recurrence formulae for the Bessel functions (see Ref. [5]):

$$\psi_1(\zeta) = -\frac{\beta}{L\phi^n\sqrt{\zeta}} (C_1 J_{n+1}(\phi) + C_2 Y_{n+1}(\phi) - C_3 I_{n+1}(\phi) + C_4 K_{n+1}(\phi)), \tag{10}$$

$$\psi_2(\zeta) = -\frac{\beta}{L\phi^n\sqrt{\zeta}} (C_5 J_{n+1}(\phi) + C_6 Y_{n+1}(\phi) - C_7 I_{n+1}(\phi) + C_9 K_{n+1}(\phi)), \tag{11}$$

$$M_1(\zeta) = \frac{EI_0 \beta^2 \zeta^{(n+1)}}{L^2 \phi^n} (C_1 J_{n+2}(\phi) + C_2 Y_{n+2}(\phi) + C_3 I_{n+2}(\phi) + C_4 K_{n+2}(\phi)), \tag{12}$$

$$M_2(\zeta) = \frac{EI_0 \beta^2 \zeta^{(n+1)}}{L^2 \phi^n} (C_5 J_{n+2}(\phi) + C_6 Y_{n+2}(\phi) + C_7 I_{n+2}(\phi) + C_8 K_{n+2}(\phi)), \tag{13}$$

$$T_1(\zeta) = \frac{EI_0 \beta^3 \zeta^{(n+1/2)}}{L^3 \phi^n} (C_1 J_{n+1}(\phi) + C_2 Y_{n+1}(\phi) + C_3 I_{n+1}(\phi) - C_4 K_{n+1}(\phi)), \tag{14}$$

$$T_2(\zeta) = \frac{EI_0 \beta^3 \zeta^{(n+1/2)}}{L^3 \phi^n} (C_5 J_{n+1}(\phi) + C_6 Y_{n+1}(\phi) + C_7 I_{n+1}(\phi) - C_8 K_{n+1}(\phi)). \tag{15}$$

The boundary conditions can be written as follows:
At $z = 0$, and therefore at $\zeta = 1$ we have

Clamped end:

$$V_1(1) = 0, \quad \psi_1(1) = 0. \tag{16}$$

Supported end:

$$V_1(1) = 0, \quad M_1(1) = 0. \tag{17}$$

Free end:

$$M_1(1) = 0, \quad T_1(1) = 0. \tag{18}$$

Sliding end:

$$\psi_1(1) = 0, \quad T_1(1) = 0. \tag{19}$$

Concentrated mass M at free end:

$$T_1(1) - \mu EI_0\beta^4 V_1(1) = 0, \quad M_1(1) = 0 \tag{20}$$

with $\mu = M/\rho A_0 L$.

At the variable dashpot section ζ_c with constant damping coefficient d we have:

$$V_1(\zeta_c) - V_2(\zeta_c) = 0, \tag{21}$$

$$\psi_1(\zeta_c) - \psi_2(\zeta_c) = 0, \tag{22}$$

$$M_1(\zeta_c) - M_2(\zeta_c) = 0, \tag{23}$$

$$T_1(\zeta_c) - T_2(\zeta_c) + \frac{\mathrm{i}d\beta^2}{L^2}\sqrt{\frac{EI_0}{\rho A_0}}V_1(\zeta_c) = 0, \tag{24}$$

where $\mathrm{i} = \sqrt{-1}$.

At the right end, where $z = L$ and $\zeta = 1 + c$ the boundary conditions are given by

Clamped end:

$$V_2(1 + c) = 0, \quad \psi_2(1 + c) = 0. \tag{25}$$

Supported end:

$$V_2(1 + c) = 0, \quad M_2(1 + c) = 0. \tag{26}$$

Free end:

$$M_2(1 + c) = 0, \quad T_2(1 + c) = 0. \tag{27}$$

Sliding end:

$$\psi_2(1 + c) = 0, \quad T_2(1 + c) = 0. \tag{28}$$

Eight boundary conditions must be chosen, in order to correctly define the resulting boundary value problem, whose solution creates no difficulties, except for the presence of complex conjugate eigensolutions.

Table 1
First three free vibration frequencies against the dashpot location, with different values of c and d

		ζ_c	λ_1	λ_2	λ_3
$c = 0.5$	$d = 5$				
		0	$-14.7137 \pm i38.4510$	$-15.6995 \pm i214.139$	$-16.1472 \pm i575.504$
		0.1	$-10.4803 \pm i39.3739$	$-4.08247 \pm i215.938$	$-0.60720 \pm i577.014$
		0.2	$-7.17209 \pm i40.0362$	$-0.04962 \pm i216.376$	$-2.77372 \pm i576.996$
		0.3	$-4.58664 \pm i40.4248$	$-1.51057 \pm i216.323$	$-5.86407 \pm i576.908$
		0.4	$-2.67433 \pm i40.6005$	$-4.60770 \pm i216.236$	$-2.08133 \pm i576.995$
		0.5	$-1.37897 \pm i40.6563$	$-6.04713 \pm i216.232$	$-0.16969 \pm i577.041$
		0.6	$-0.59843 \pm i40.6656$	$-4.96275 \pm i216.329$	$-3.88082 \pm i576.958$
		0.7	$-0.19896 \pm i40.6648$	$-2.62883 \pm i216.386$	$-5.93104 \pm i576.977$
		0.8	$-0.04098 \pm i40.6641$	$-0.77024 \pm i216.384$	$-3.11415 \pm i577.055$
		0.9	$-0.00265 \pm i40.6640$	$-0.06544 \pm i216.379$	$-0.37864 \pm i577.045$
	$d = 10$				
		0	$-28.6738 \pm i31.8704$	$-30.7039 \pm i206.880$	$-31.8927 \pm i570.778$
		0.1	$-21.2133 \pm i35.0677$	$-7.93406 \pm i214.645$	$-1.20537 \pm i576.922$
		0.2	$-14.3045 \pm i38.0833$	$-0.09813 \pm i216.367$	$-5.54592 \pm i576.852$
		0.3	$-9.20076 \pm i39.6923$	$-3.01243 \pm i216.154$	$-11.7204 \pm i576.497$
		0.4	$-5.36742 \pm i40.4076$	$-9.21135 \pm i215.802$	$-4.15768 \pm i576.848$
		0.5	$-2.76352 \pm i40.6332$	$-12.0973 \pm i215.788$	$-0.33895 \pm i577.030$
		0.6	$-1.19759 \pm i40.6706$	$-9.93663 \pm i216.176$	$-7.75552 \pm i576.699$
		0.7	$-0.39796 \pm i40.6672$	$-5.26114 \pm i216.406$	$-11.8668 \pm i576.775$
		0.8	$-0.08197 \pm i40.6643$	$-1.54051 \pm i216.396$	$-6.2301 \pm i577.088$
		0.9	$-0.00531 \pm i40.6640$	$-0.13088 \pm i216.380$	$-0.7573 \pm i577.049$
$c = 1$	$d = 5$				
		0	$-14.0982 \pm i54.8858$	$-16.4451 \pm i267.469$	$-17.2567 \pm i693.375$
		0.1	$-9.92666 \pm i55.3916$	$-4.11797 \pm i268.896$	$-0.49360 \pm i694.708$
		0.2	$-6.64324 \pm i55.8192$	$-0.03788 \pm i269.237$	$-3.03346 \pm i694.687$
		0.3	$-4.12952 \pm i56.0682$	$-1.46748 \pm i269.199$	$-2.25533 \pm i694.633$
		0.4	$-2.33320 \pm i56.1729$	$-4.10258 \pm i269.147$	$-1.36673 \pm i694.705$
		0.5	$-1.16386 \pm i56.2022$	$-4.99186 \pm i269.158$	$-0.33926 \pm i694.722$
		0.6	$-0.48812 \pm i56.2057$	$-3.81123 \pm i269.217$	$-3.47160 \pm i694.675$
		0.7	$-0.15677 \pm i56.2048$	$-1.88550 \pm i269.243$	$-4.44549 \pm i694.700$
		0.8	$-0.03120 \pm i56.2044$	$-0.51822 \pm i269.240$	$-2.08037 \pm i694.733$
		0.9	$-0.00195 \pm i56.2043$	$-0.04149 \pm i269.238$	$-0.23147 \pm i694.728$
	$d = 10$				
		0	$-29.6739 \pm i50.0365$	$-32.7195 \pm i261.804$	$-34.2892 \pm i689.230$
		0.1	$-20.0041 \pm i52.8212$	$-8.09978 \pm i267.882$	$-0.98257 \pm i694.648$
		0.2	$-13.2986 \pm i54.6436$	$-0.07531 \pm i269.232$	$-6.06643 \pm i694.566$
		0.3	$-8.27300 \pm i55.6558$	$-2.93080 \pm i269.080$	$-10.5062 \pm i694.348$
		0.4	$-4.67452 \pm i56.0782$	$-8.20329 \pm i268.872$	$-2.73177 \pm i694.637$
		0.5	$-2.32979 \pm i56.1958$	$-9.98538 \pm i268.916$	$-0.67809 \pm i694.707$
		0.6	$-0.97647 \pm i56.2098$	$-7.62647 \pm i269.153$	$-6.94092 \pm i694.521$
		0.7	$-0.31355 \pm i56.2062$	$-3.77192 \pm i269.256$	$-8.89288 \pm i694.618$
		0.8	$-0.06240 \pm i56.2045$	$-1.03644 \pm i269.245$	$-4.16113 \pm i694.749$
		0.9	$-0.00390 \pm i56.2043$	$-0.08298 \pm i269.238$	$-0.46294 \pm i694.729$

3. Numerical examples

All the numerical examples will assume the cross-section variation law as given in Eq. (3), with the following numerical data: span $L = 1\,\mathrm{m}$, Young modulus $E = 7 \times 10^{10}\,\mathrm{N\,m^{-2}}$, moment of inertia of the initial cross-section $I_0 = 5.20833 \times 10^{-10}\,\mathrm{m}^4$, mass of the initial cross-section $\rho A_0 = 0.675\,\mathrm{kg\,m^{-1}}$ and $n = 1$.

As a first example, a free–clamped cantilever beam is examined, with a concentrated dashpot at an intermediate section ζ_c (Table 1). The damping coefficient is assumed to be equal to $d = 5\,\mathrm{N\,m^{-1}\,s}$, and the first three free frequency vibration frequencies λ_i are given, as a function of the abscissa ζ_c. As can be observed, the (negative) real part goes to zero as ζ_c increases from 0 to 0.9, and then it becomes zero, reproducing the classical case of tapered beam without damping, for which the first frequency is equal to i40.664.

In Table 2, the clamped–free beam is subjected to a concentrated mass at the free end, the dashpot is supposed to be placed at the mid-span $\zeta = 0.5$, the taper ratio is equal to $c = 0.5$. The first three free vibration frequencies are given against the damping coefficient d and the nondimensional concentrated mass value.

At $\mu = 0$ and $d = 0$ the classical cantilever beam is recovered, and the eigenvalues become purely imaginary. The imaginary part of the eigenvalues decreases with increasing values of the ratio between the concentrated mass and the beam mass.

Table 2
First three free vibration frequencies as a function of the ratio μ, with $d = 0, 5, 10\,\mathrm{N\,m^{-1}\,s}$

d		$\mu = 0$	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$
0	λ_1	±i40.644	±i34.4537	±i30.378	±i27.457	±i25.237
	λ_2	±i216.379	±i188.349	±i176.419	±i169.897	±i165.799
	λ_3	±i577.044	±i514.955	±i495.954	±i486.960	±i481.741
5	λ_1	−1.3790 ± i40.656	−0.9444 ± i34.452	−0.7125 ± i30.378	−0.5703 ± i27.458	−0.4747 ± i25.238
	λ_2	−6.0471 ± i216.232	−5.8389 ± i188.219	−5.8085 ± i176.298	−5.8123 ± i169.781	−5.8230 ± i165.687
	λ_3	−0.1697 ± i577.041	−0.7861 ± i514.937	−1.0690 ± i495.929	−1.2201 ± i486.932	−1.3132 ± i481.710
10	λ_1	−2.7635 ± i40.633	−1.8919 ± i34.449	−1.4268 ± i30.380	−1.1418 ± i27.461	−0.9504 ± i25.241
	λ_2	−12.097 ± i215.788	−11.685 ± i187.828	−11.626 ± i175.934	−11.635 ± i169.433	−11.657 ± i165.350
	λ_3	0.3389 ± i577.030	−1.5700 ± i514.883	−2.1353 ± i495.855	−2.4372 ± i486.847	−2.6231 ± i481.618

d		$\mu = 0.5$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$
0	λ_1	±i23.477	±i22.039	±i20.835	±i19.8091	±i18.920
	λ_2	±i162.989	±i160.944	±i159.390	±i158.169	±i157.184
	λ_3	±i478.336	±i475.942	±i474.167	±i472.799	±i471.713
5	λ_1	−0.4063 ± i23.478i	−0.3550 ± i22.040	−0.3151 ± i20.836	−0.2833 ± i19.810	−0.2572 ± i18.921
	λ_2	−5.8341 ± i162.880	−5.8422 ± i160.837	−5.8531 ± i159.284	−5.8607 ± i158.064	−5.8670 ± i157.075
	λ_3	−1.3761 ± i478.304	−1.4213 ± i475.909	−1.4555 ± i474.133	−1.4821 ± i472.765	−1.5035 ± i471.677
10	λ_1	−0.8133 ± i23.481	−0.7106 ± i22.043	−0.6307 ± i20.839	−0.5669 ± i19.812	−0.5147 ± i18.923
	λ_2	−11.680 ± i162.551	−11.700 ± i160.514	−11.718 ± i158.966	−11.734 ± i157.750	−11.747 ± i156.769
	λ_3	−2.7487 ± i478.207	−2.8319 ± i475.809	−2.9073 ± i474.031	−2.9606 ± i472.660	−3.0033 ± i471.571

4. Conclusions

The free vibrations of an Euler–Bernoulli beam have been examined, in the presence of a concentrated dashpot at an intermediate variable section and of a concentrated mass at the tip. The analysis can be considered exact, in the sense that the differential equations of motion are solved in terms of Bessel functions, and the resulting boundary value problem is solved using the symbolic package Mathematica. The paper ends with some numerical examples, in which the complex conjugate eigensolutions are given as functions of the dashpot location or of the damping coefficient.

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