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A general approach to formulating the frequency equation for a beam carrying miscellaneous attachments

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Abstract

Beams carrying various lumped attachments have been studied extensively over the years. In this paper, a simple approach is proposed that can be used to readily determine the eigenvalues of an arbitrarily supported single-span or multi-span beam carrying any combination of lumped mass, rotary inertia, grounded translational or torsional spring, grounded translational or torsional viscous damper, an undamped or damped oscillator with or without a rigid body degree of freedom. Rather than solving a generalized eigenvalue problem to obtain the eigenvalues of the system, a frequency equation is formulated instead whose solution can be easily solved either numerically or graphically. The proposed scheme is easy to code, and can be easily modified to accommodate beams with arbitrary supports and carrying any number of miscellaneous lumped attachments.

1. Introduction

Frequency analysis of combined dynamical systems consisting of beams carrying lumped attachments has been studied extensively over the years, and hence only a few selected recent references are given here [1–25]. Commonly used analytical approaches include the assumed-modes method [21,25], the Lagrange multipliers formalism [9,16,18,20], dynamic Green's function approach [10,17,19], Laplace transform with respect to the spatial variable approach [8,24], and the analytical-and-numerical-combined method [12,22].

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In most of the previous work, the lumped attachments are often of the same type, i.e., all point masses, undamped oscillators or damped spring–mass systems. Moreover, most authors considered only beams with a single-span. Posiadała [18] analyzed the free vibration of a beam with various translational and torsional lumped attachments. He derived the frequency equation for the combined system by means of the Lagrange multiplier approach. This method is based on using the spatial functions of the unconstrained structure in a Rayleigh–Ritz analysis with the constraint conditions enforced by means of Lagrange multipliers. Using this particular approach, S Lagrange multipliers and S constraint equations are introduced in the analysis, where S corresponds to the number of attachments the beam is carrying. Manipulating the equations of motion, the eigenvalues must satisfy the zeros of the S constraint equations in matrix form. While the final results obtained by the Lagrange multiplier approach are usually concise, the scheme is rather laborious to apply, because Lagrange multipliers are required and constraint equations are imposed. Due to its complexity, the method of Lagrange multipliers seems to have been used less for free vibration than other methods. While Posiadała included various elements in his analysis, he did not include any damping elements nor did he generalize his findings to a beam carrying different combinations of attachments.

In this paper, the discretized governing equations for an arbitrarily supported single-span or multi-span beam structure carrying various attachments, including point masses, inertia elements, translational/torsional springs and dampers, undamped and damped oscillators with and without rigid body degree of freedom (dof), etc., are first obtained by using the common assumed-modes method. It will be shown that the characteristic determinants governing the free vibration of beams carrying miscellaneous attachments all have the same form. With proper algebraic manipulations, each characteristic determinant can be reduced to one of a smaller size, thus providing an alternative means to solve for the eigenvalues of the combined system. A look-up table will be provided that can be used to help code the proposed algorithm, and a graphical procedure will be outlined to assist with the estimation of the solution. The benefits of the proposed scheme will be discussed and highlighted, and numerous numerical examples will be provided to illustrate the utility of the new formalism.

2. Theory

Consider the free vibration of an arbitrarily supported beam carrying a grounded translational spring of stiffness k at x_1, a lumped mass m at x_2, a grounded torsional spring of stiffness k_t at x_3, a grounded viscous damper of coefficient c at x_4, a damped oscillator with a rigid body dof with parameters m_1, c_1 and k_1 at x_5, a grounded torsional viscous damper of coefficient c_t at x_6, and an element with rotary inertia J at x_7, as shown in Fig. 1. Using assumed-modes method [26], the lateral displacement of the combined system at point x can be expressed in the form of a finite series as

$$w(x,t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t), \tag{1}$$

where N represents the number of modes used in the expansion, $\phi_i(x)$ are the eigenfunctions of the unconstrained beam (i.e., the beam without any attachment), that serve as the basis functions for this

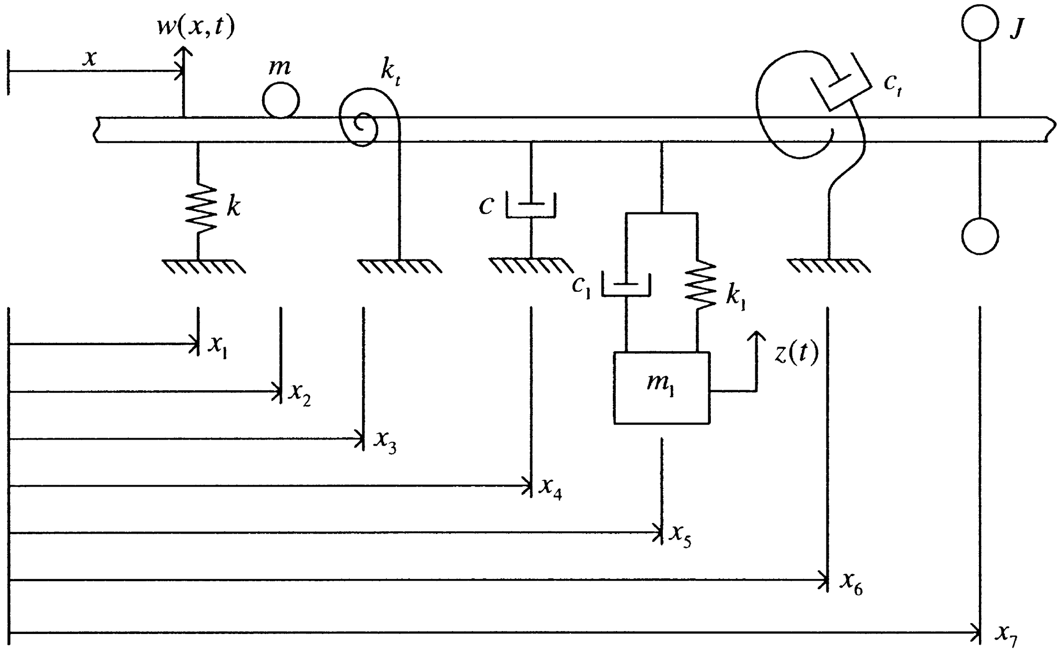

Fig. 1. An arbitrarily supported beam carrying various lumped elements.

approximate solution, and $\eta_i(t)$ are the generalized coordinates. The total kinetic energy of the system is

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}m\dot{w}^2(x_2,t) + \frac{1}{2}m_1\dot{z}^2(t) + \frac{1}{2}J\dot{\theta}^2(x_7,t), \tag{2}$$

where M_i are the generalized masses of the unconstrained beam, an overdot denotes a derivative with respect to t, $z(t)$ denotes the lateral displacement of the damped oscillator, and $\theta(x,t)$ represents the rotational displacement of the beam, and is given by

$$\theta(x,t) = \frac{\partial w}{\partial x}(x,t) = \sum_{i=1}^{N} \phi_i'(x)\eta_i(t), \tag{3}$$

where the prime denotes a derivative with respect to x. The total potential energy of the system is

$$V = \frac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2(t) + \frac{1}{2}kw^2(x_1,t) + \frac{1}{2}k_t\theta^2(x_3,t) + \frac{1}{2}k_1[z(t) - w(x_5,t)]^2, \tag{4}$$

where K_i are the generalized stiffnesses of the unconstrained beam. Finally, the Rayleigh's dissipation function is

$$R = \frac{1}{2}c\dot{w}^2(x_4,t) + \frac{1}{2}c_1[\dot{z}(t) - \dot{w}(x_5,t)]^2 + \frac{1}{2}c_t\dot{\theta}^2(x_6,t). \tag{5}$$

Substituting Eqs. (1) and (3) into Eqs. (2), (4) and (5), and applying Lagrange's equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\eta}_i}\right) - \frac{\partial T}{\partial \eta_i} + \frac{\partial V}{\partial \eta_i} + \frac{\partial R}{\partial \dot{\eta}_i} = 0, \quad i = 1, 2, \ldots, N \tag{6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} + \frac{\partial V}{\partial z} + \frac{\partial R}{\partial \dot{z}} = 0, \tag{7}$$

the following equations of motion are obtained:

$$\begin{bmatrix} [\mathscr{M}] & \underline{0} \\ \underline{0}^{\mathrm{T}} & m_1 \end{bmatrix} \begin{bmatrix} \ddot{\underline{\eta}} \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} [\mathscr{C}] & -c_1\underline{\phi}_5 \\ -c_1\underline{\phi}_5^{\mathrm{T}} & c_1 \end{bmatrix} \begin{bmatrix} \dot{\underline{\eta}} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} [\mathscr{K}] & -k_1\underline{\phi}_5 \\ -k_1\underline{\phi}_5^{\mathrm{T}} & k_1 \end{bmatrix} \begin{bmatrix} \underline{\eta} \\ z \end{bmatrix} = \begin{bmatrix} \underline{0} \\ 0 \end{bmatrix}, \tag{8}$$

where $\underline{\phi}_i$ is a vector of the eigenfunctions evaluated at the attachment location, x_i, as follows:

$$\underline{\phi}_i = [\phi_1(x_i), \ldots, \phi_j(x_i), \ldots, \phi_N(x_i)]^{\mathrm{T}} \tag{9}$$

and

$$\begin{aligned} &[\mathscr{M}] = [M^d] + m\underline{\phi}_2\underline{\phi}_2^{\mathrm{T}} + J\underline{\phi}_7'\underline{\phi}_7'^{\mathrm{T}}, \quad [\mathscr{K}] = [K^d] + k\underline{\phi}_1\underline{\phi}_1^{\mathrm{T}} + k_t\underline{\phi}_3'\underline{\phi}_3'^{\mathrm{T}}, + k_1\underline{\phi}_5\underline{\phi}_5^{\mathrm{T}} \\ &[\mathscr{C}] = c\underline{\phi}_4\underline{\phi}_4^{\mathrm{T}} + c_t\underline{\phi}_6'\underline{\phi}_6'^{\mathrm{T}} + c_1\underline{\phi}_5\underline{\phi}_5^{\mathrm{T}}. \end{aligned} \tag{10}$$

Matrices $[K^d]$ and $[M^d]$ are both diagonal, whose ith elements are given by K_i and M_i, respectively. Eq. (8) can be expressed as

$$[\mathscr{M}_s]\,\ddot{\underline{q}} + [\mathscr{C}_s]\,\dot{\underline{q}} + [\mathscr{K}_s]\,\underline{q} = \underline{0}, \tag{11}$$

where $[\mathscr{M}_s]$, $[\mathscr{C}_s]$ and $[\mathscr{K}_s]$ are the mass, damping and stiffness matrices, respectively, of the system, and $\underline{q} = [\underline{\eta}^{\mathrm{T}}\ z]^{\mathrm{T}}$. The system matrices are all of size $N_s \times N_s$ and the vector $\underline{q}$ is of length N_s, where $N_s = N + 1$. The free response behavior of system (11) can determined by using a state matrix approach, which effectively replaces the N_s coupled second-order differential equations by $2N_s$ coupled first-order ordinary differential equations as follows [26]. A state vector of length $2N_s$ is introduced,

$$\underline{y} = \begin{bmatrix} \dot{\underline{q}} \\ \underline{q} \end{bmatrix}, \tag{12}$$

such that Eq. (11) can be rewritten in a form that consists of $2N_s$ simultaneous first-order ordinary differential equations:

$$[A]\,\dot{\underline{y}} - [B]\,\underline{y} = \underline{0}, \tag{13}$$

where matrices $[A]$ and $[B]$ are both symmetric and are given by

$$[A] = \begin{bmatrix} [0] & [\mathscr{M}_s] \\ [\mathscr{M}_s] & [\mathscr{C}_s] \end{bmatrix} \quad \text{and} \quad [B] = \begin{bmatrix} [\mathscr{M}_s] & [0] \\ [0] & -[\mathscr{K}_s] \end{bmatrix}. \tag{14}$$

Because Eq. (13) is homogeneous, its solution is given by

$$\underline{y} = \underline{\bar{y}}\ \mathrm{e}^{\lambda t}, \tag{15}$$

where the constant exponent λ is also known as the eigenvalue of the system. Substituting Eq. (15) into Eq. (13) yields the $2N_s \times 2N_s$ generalized eigenvalue problem

$$[B]\underline{\bar{y}} = \lambda[A]\underline{\bar{y}}, \tag{16}$$

where λ corresponds to the exponent or the eigenvalue of the system. Eq. (16) can be readily solved by using any existing prepackaged code such as *rsg* in EISPACK or *eig* in MATLAB.

Alternatively, an exponential solution can be assumed from the outset, in which case

$$\begin{bmatrix} \underline{\eta} \\ z \end{bmatrix} = \begin{bmatrix} \underline{\bar{\eta}} \\ \bar{z} \end{bmatrix} \mathrm{e}^{\lambda t} \tag{17}$$

and Eq. (8) becomes

$$\left(\lambda^2 \begin{bmatrix} [\mathscr{M}] & \underline{0} \\ \underline{0}^{\mathrm{T}} & m_1 \end{bmatrix} + \lambda \begin{bmatrix} [\mathscr{C}] & -c_1\underline{\phi}_5 \\ -c_1\underline{\phi}_5^{\mathrm{T}} & c_1 \end{bmatrix} + \begin{bmatrix} [\mathscr{K}] & -k_1\underline{\phi}_5 \\ -k_1\underline{\phi}_5^{\mathrm{T}} & k_1 \end{bmatrix}\right) \begin{bmatrix} \underline{\bar{\eta}} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ 0 \end{bmatrix}. \tag{18}$$

Using the last equation of Eq. (18) to obtain an expression for $\bar{z}$ in terms of $\underline{\bar{\eta}}$ yields

$$\bar{z} = \frac{k_1 + c_1\lambda}{k_1 + c_1\lambda + m_1\lambda^2}\underline{\phi}_5^{\mathrm{T}}\underline{\bar{\eta}}. \tag{19}$$

Substituting Eq. (19) into the top equation of Eq. (18) leads to

$$\left(\lambda^2[\mathscr{M}] + \lambda[\mathscr{C}] + [\mathscr{K}] - \frac{(k_1 + c_1\lambda)^2}{k_1 + c_1\lambda + m_1\lambda^2}\underline{\phi}_5\underline{\phi}_5^{\mathrm{T}}\right)\underline{\bar{\eta}} = \underline{0}. \tag{20}$$

Finally, substituting Eq. (10) into Eq. (20) gives

$$\left(\lambda^2[M^d] + [K^d] + \sum_{i=1}^{7} \sigma_i \underline{u}_i \underline{u}_i^{\mathrm{T}}\right)\underline{\bar{\eta}} = \underline{0}, \tag{21}$$

where

$$\sigma_1 = k, \quad \sigma_2 = m\lambda^2, \quad \sigma_3 = k_t, \quad \sigma_4 = c\lambda, \quad \sigma_5 = \frac{(k_1 + c_1\lambda)m_1\lambda^2}{k_1 + c_1\lambda + m_1\lambda^2}, \quad \sigma_6 = c_t\lambda, \quad \sigma_7 = J\lambda^2 \tag{22}$$

and $\underline{u}_i$ is a vector of length N, given by

$$\underline{u}_i = \underline{\phi}(x_i) \quad \text{for } i = 1, 2, 4, 5, \quad \underline{u}_i = \underline{\phi}'(x_i) \quad \text{for } i = 3, 6, 7. \tag{23}$$

Note that the coefficient matrix of Eq. (21) consists of a diagonal matrix $(\lambda^2[M^d] + [K^d])$ modified by a series of rank one matrices. For a nontrivial solution, the eigenvalue λ must satisfy

$$\det\left(\lambda^2[M^d] + [K^d] + \sum_{i=1}^{7} \sigma_i \underline{u}_i \underline{u}_i^{\mathrm{T}}\right) = 0. \tag{24}$$

For an undamped system, the eigenvalue λ is purely imaginary, implying that the system executes simple harmonic motion, consistent with physical intuition. In this case, $\lambda = \mathrm{j}\omega$, where $\mathrm{j} = \sqrt{-1}$ and ω represents the undamped natural frequency of the system. When damping is present, the eigenvalues may now be complex of the form

$$\lambda = \lambda_r + \mathrm{j}\lambda_i, \tag{25}$$

where λ_r and λ_i correspond to the real and imaginary parts of λ, respectively.

2.1. Single attachment

Suppose the beam is only carrying one lumped attachment at x_1, in which case Eq. (24) simplifies to

$$\det([K^d] + \lambda^2[M^d] + \sigma \underline{u}_1 \underline{u}_1^{\mathrm{T}}) = 0, \tag{26}$$

where σ and $\underline{u}_1$ depend on the element type. Table 1 summarizes the expressions for σ and $\underline{u}_1$ for various lumped attachments. Because the matrix of Eq. (26) consists of a diagonal matrix modified by a simple rank one matrix, it can be reduced to a simple secular equation as follows:

$$\begin{aligned}\det(\lambda^2[M^d] + [K^d] + \sigma \underline{u}_1 \underline{u}_1^{\mathrm{T}}) &= \det(\lambda^2[M^d] + [K^d]) \det([I] + \sigma(\lambda^2[M^d] + [K^d])^{-1} \underline{u}_1 \underline{u}_1^{\mathrm{T}}) \\ &= \left(\prod_{i=1}^{N} (\lambda^2 M_i + K_i)\right)\left(1 + \sigma \sum_{i=1}^{N} \frac{u_i^2(x_1)}{\lambda^2 M_i + K_i}\right) = 0. \end{aligned} \tag{27}$$

The eigenvalues correspond to the zeros of Eq. (27), which can be determined either graphically or numerically using any standard root solvers routine such as *fsolve* in MATLAB. The product terms in Eq. (27) are significant because they serve as a reminder that when the attachment location for the lumped element coincides with the node of any component modes, $u_i(x)$, of the unconstrained beam, then some of the eigenvalues of the combined system will be identical to the

Table 1
Expression for σ and $\underline{u}_1$ for any lumped attachment at x_1

Lumped attachment	σ	$\underline{u}_1 = \underline{u}(x_1)$
Point mass	$m\lambda^2$	$\underline{\phi}_1$
Rotary inertia	$J\lambda^2$	$\underline{\phi}_1'$
Grounded translational spring	k	$\underline{\phi}_1$
Grounded torsional spring	k_t	$\underline{\phi}_1'$
Grounded translational viscous damper	$c\lambda$	$\underline{\phi}_1$
Grounded torsional viscous damper	$c_t\lambda$	$\underline{\phi}_1'$
Undamped oscillator with no rigid dof	$k + m\lambda^2$	$\underline{\phi}_1$
Undamped oscillator with rigid dof	$\frac{km\lambda^2}{k+m\lambda^2}$	$\underline{\phi}_1$
Damped oscillator with no rigid dof	$k + c\lambda + m\lambda^2$	$\underline{\phi}_1$
Damped oscillator with rigid dof	$\frac{(k+c\lambda)m\lambda^2}{k+c\lambda+m\lambda^2}$	$\underline{\phi}_1$
In-span simple support	$k \to \infty$	$\underline{\phi}_1$

dof stands for degree of freedom.

natural frequencies of the beam without any attachment. Moreover, when the attachment location is in the vicinity of a node of the beam's normal mode, then one of the eigenvalues of the combined system will be a perturbation of the beam's natural frequency corresponding to that particular mode. Finally, when the attachment location x_1 does not coincide with a node of any of the normal modes of the beam, then $\lambda^2 M_i + K_i \neq 0$, and Eq. (27) reduces to

$$1 + \sigma \sum_{i=1}^{N} \frac{u_i^2(x_1)}{\lambda^2 M_i + K_i} = 0. \tag{28}$$

To demonstrate how Table 1 can be used to formulate the appropriate frequency equation of a beam carrying a single attachment, consider the case where the lumped element consists of an undamped oscillator of parameters m and k with a rigid body dof (see Fig. 2). From Table 1, one immediately finds

$$\sigma = \frac{km\lambda^2}{k + m\lambda^2} \quad \text{and} \quad \underline{u}_1 = \underline{\phi}_1. \tag{29}$$

Substituting Eq. (29) into Eq. (27), one obtains the frequency equation of Fig. 2. For this particular case, note that as $k \to \infty$, Eq. (29) reduces to $\sigma = m\lambda^2$ (a beam carrying a point mass), and as $m \to \infty$, Eq. (29) simplifies to $\sigma = k$ (a beam carrying a grounded translational spring). Both limiting cases are in complete agreement with the results of Table 1. Interestingly, Eq. (27) can also be extended to find the eigenvalues of a two span beam (see Fig. 3), where the in-span simple support can be modeled as a grounded translational spring whose stiffness tends to infinity. Thus, for an in-span simple support,

$$\sigma = k \to \infty \quad \text{and} \quad \underline{u}_1 = \underline{\phi}_1. \tag{30}$$

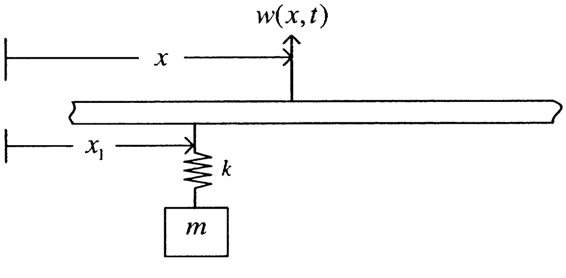

Fig. 2. An arbitrarily supported beam carrying an undamped oscillator with a rigid body dof x_1.

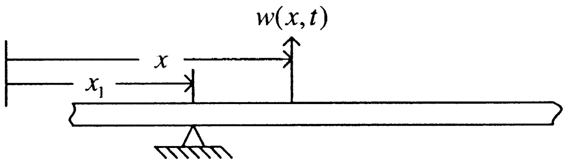

Fig. 3. An arbitrarily supported two span beam with a simple support at x_1.

2.2. Multiple attachments

Expressions for σ and $\underline{u}_1$ depend on the lumped attachments, and are given in Table 1, for $i = 1$. The results of Table 1, however, can be easily extended to an arbitrarily supported beam carrying S lumped attachments at locations x_i ($i = 1, \ldots, S$), where the corresponding σ_i and $\underline{u}_i$ are obtained directly from Table 1. For this general case, the eigenvalues are given by the solution of the characteristic determinant

$$\det\left(\lambda^2[M^d] + [K^d] + \sum_{i=1}^{S} \sigma_i \underline{u}_i \underline{u}_i^{\mathrm{T}}\right) = 0, \quad (31)$$

where the vector $\underline{u}_i$ is evaluated at x_i. Eq. (31) can be shown [21,27] to be identical to

$$\det(\lambda^2[M^d] + [K^d]) \det[B] = \left(\prod_{i=1}^{N} (\lambda^2 M_i + K_i)\right) \det[B] = 0, \quad (32)$$

where the (i,j)th element of $[B]$, of size $S \times S$, is given by

$$b_{ij} = \sum_{r=1}^{N} \frac{u_r(x_i)u_r(x_j)}{\lambda^2 M_r + K_r} + \frac{1}{\sigma_i}\,\delta_i^j, \quad i,j = 1,2,\ldots,S \quad (33)$$

and δ_i^j represents the Kronecker delta. The eigenvalues of the system with multiple attachments are given by the roots of Eq. (32), which can be readily solved using any existing prepackaged code such as *fsolve* in MATLAB. Finally, when $\lambda^2 M_i + K_i \neq 0$, i.e., when the attachment locations do not coincide with the nodes of any normal modes of the unconstrained beam, then Eq. (32) reduces to

$$\det[B] = 0. \quad (34)$$

Incidentally, many different approaches can be used to arrive at the frequency equations given by Eqs. (28) and (32). Weissenburger [2] generated similar characteristic equations using the method of localized modifications. Jacquot and Gibson [5] found a expression nearly identical to Eq. (28) for an undamped simply supported beam with rotational end restraint and a cantilever beam with tip mass and stiffness. Their approach consisted of expanding the time-dependent deflection curve in terms of the eigenfunctions of the beam and then solving a set of linear equations for the natural frequencies. Using component mode analysis, Dowell [6] obtained the frequency equation for a system consisting of a beam or a plate on spring supports by means of Lagrange multipliers. Gürgöze [20] obtained Eq. (32) for a beam carrying multiple spring–mass systems in-span using the Lagrange multipliers formalism. Clearly, expressions similar to Eqs. (28) and (32) can be derived using other methods. In this paper, they are obtained by the direct manipulation of the characteristic determinant associated with a diagonal matrix modified by one or multiple rank one matrices.

The proposed scheme of determining the eigenvalues of a beam carrying an assortment of lumped elements is easy to apply with the aid of Table 1. To illustrate how to properly use this table, consider the system of Fig. 4, which consists of an arbitrarily supported beam carrying a grounded translational spring of stiffness k_1 at x_1, a lumped mass m_1 at x_2, a damped oscillator with a rigid dof with parameters m_2, c and k_2 at x_3, and a grounded torsional spring of stiffness k_t

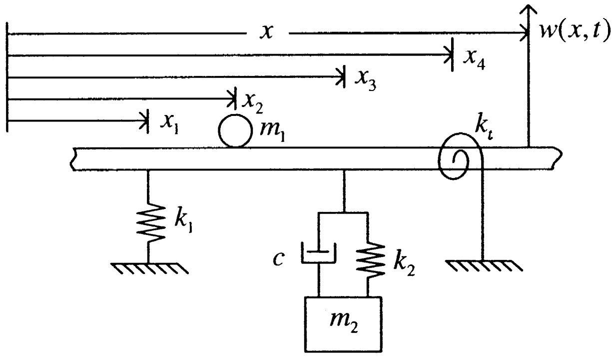

Fig. 4. An arbitrarily supported beam carrying a grounded spring at x_1, a lumped mass at x_2, a damped oscillator with a rigid dof at x_3, and a grounded torsional spring at x_4.

at x_4. For this beam system, $S = 4$ and mapping each element to the appropriate σ_i and $\underline{u}_i$ using Table 1, one finds

$$\sigma_1 = k_1, \quad \sigma_2 = m_1\lambda^2, \quad \sigma_3 = \frac{(k_2 + c\lambda)m_2\lambda^2}{m_2\lambda^2 + c\lambda + k_2}, \quad \sigma_4 = k_t \tag{35}$$

and

$$\underline{u}_i = \underline{\phi}(x_i) \quad \text{for } i = 1, 2, 3, \quad \underline{u}_4 = \underline{\phi}'(x_4). \tag{36}$$

Substituting Eqs. (35) and (36) into Eq. (32) and expanding the resulting characteristic determinant yields the frequency equation for the system of Fig. 4, which can be easily solved graphically or numerically.

3. Results

The proposed scheme of calculating the eigenvalues of a beam carrying one or multiple lumped elements offers numerous advantages. Firstly, Eqs. (27) and (32) are simple to code. Given the eigenfunctions, $\phi_i(x)$, of the arbitrarily supported beam, the parameters for the lumped attachments and their locations, x_i, Eqs. (27) and (32) can be easily programmed and solved either graphically or numerically using any existing root solvers. Secondly, the proposed approach can be extended to determine the eigenvalues of a beam with any arbitrary boundary conditions by simply using the appropriate generalized masses, stiffnesses and eigenfunctions. Finally, Eq. (32) can be easily modified to analyze a beam carrying any combination of lumped attachments.

To show the versatility of the proposed scheme, the eigenvalues of a uniform fixed-free (or cantilever) and simply supported Euler–Bernoulli beam carrying various lumped attachments are computed by solving Eq. (27) or (32), and the results are compared to those obtained by using the finite element method. Instead of solving the generalized eigenvalue problem (16), the finite element method was chosen because it offers a completely different approach of verifying the solutions. In all of the subsequent numerical examples, when the finite element method was used, the beam was discretized into 100 finite elements. For the proposed scheme, the MATLAB routine *fsolve* was used to compute the eigenvalues of Eq. (27) or (32), depending on the number of attachments the beam is carrying.

The MATLAB routine *fsolve* requires an initial guess of the unknown eigenvalue to be provided. When the beam is not carrying any damped attachments, an estimation of the eigenvalues can be established by plotting Eq. (27) or (32) as a function of the eigenvalue λ. The zeros of the curve are the system's eigenvalues, or they can be used as the required initial guesses for *fsolve* if more accuracy is desired. When the system is damped, an estimation of the solution can be obtained by means of simultaneous contour plots of the real and imaginary parts of the system's characteristic equation, obtained by expanding Eq. (27) or (32). These approximate locations can be used as an estimation of the eigenvalues of the combined system, or if greater accuracy is desired, they can be used as inputs to *fsolve*. Finally, this graphical procedure of determining the approximate solutions can be used to locate the eigenvalues within any desired range.

The eigenfunctions used in the assumed-modes method depend on the boundary conditions of the beam with no attachments. For a uniform fixed-free beam, its normalized (with respect to the mass per unit length, ρ, of the beam) eigenfunctions are given by

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}}\left(\cos \beta_i x - \cosh \beta_i x + \frac{\sin \beta_i L - \sinh \beta_i L}{\cos \beta_i L + \cosh \beta_i L}(\sin \beta_i x - \sinh \beta_i x)\right) \tag{37}$$

such that the generalized masses and stiffnesses of the beam are

$$M_i = 1 \quad \text{and} \quad K_i = (\beta_i L)^4 EI/(\rho L^4), \tag{38}$$

where E is the Young's modulus, I is the moment of inertia of the cross-section of the beam, and $\beta_i L$ satisfies the following transcendental equation:

$$\cos \beta_i L \cosh \beta_i L = -1. \tag{39}$$

For a uniform simply supported beam, its normalized eigenfunctions are given by

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}} \sin \frac{i\pi x}{L} \tag{40}$$

such that the generalized masses and stiffnesses of the beam become

$$M_i = 1 \quad \text{and} \quad K_i = (i\pi)^4 EI/(\rho L^4). \tag{41}$$

Eqs. (27) (for one lumped attachment) and (32) (for multiple attachments) can be used to solve for the eigenvalues of the system regardless of the attachment locations. However, if the attachment locations do not coincide with the nodes of any normal modes, then Eqs. (28) and (34) are recommended because they are much simpler to solve. Thus, it is imperative to first compare the attachment locations with the node locations of the unconstrained beam, and whenever possible, solve Eqs. (28) and (34) instead. Table 2 shows the node locations for the second to the fifth component modes of a fixed-free and a simply supported Euler–Bernoulli beam.

As the first example, consider a simply supported, uniform Euler–Bernoulli beam carrying an undamped oscillator with no rigid body dof at $x_1 = 0.30L$ (see Fig. 5). When the system is undamped, the eigenvalues correspond to the natural frequencies of the system. The oscillator parameters are $m = 1.0\rho L$ and $k = 3.0EI/L^3$. When the beam is simply supported, $\phi_i(x)$ is given

Table 2
Location of nodes for the component modes of a fixed-free (ff) and a simply supported (ss) Euler–Bernoulli beam

Mode number	x_n^{ff}				x_n^{ss}			
2	$0.7834L$				$0.5000L$			
3	$0.5035L$	$0.8677L$			$0.3333L$	$0.6667L$		
4	$0.3583L$	$0.6441L$	$0.9056L$		$0.2500L$	$0.5000L$	$0.7500L$	
5	$0.2788L$	$0.4999L$	$0.7232L$	$0.9265L$	$0.2000L$	$0.4000L$	$0.6000L$	$0.8000L$

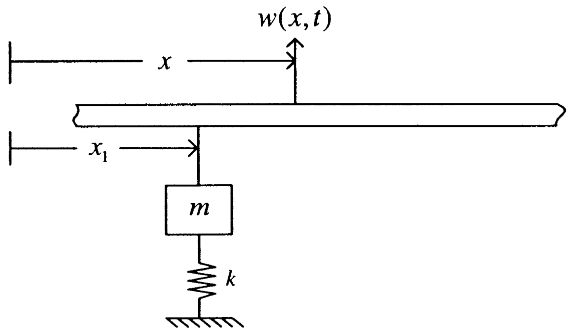

Fig. 5. An arbitrarily supported beam carrying an undamped oscillator with no rigid body dof at x_1.

by Eq. (40), and the K_i and M_i are given by Eq. (41). Using the proposed scheme, the number of modes used in the analysis is $N = 20$ to ensure sufficient convergence of the results. When the beam is discretized into 100 elements, the finite element approach requires solving a generalized eigenvalue problem of size 200×200. Table 3 compares the first five natural frequencies of the system obtained by the proposed formalism and the finite element method. Because the attachment location does not coincide with a node of any of the first five normal modes of a simply supported beam (see Table 2), Eq. (28) was used to compute the natural frequencies using the proposed scheme, where the initial guesses for *fsolve* are obtained by simply plotting the characteristic equation as a function of λ. Note the excellent agreement between the proposed and the finite element results. Finally, note that because the attachment location is near a node of the third normal mode of a simply supported beam (see Table 2), the third eigenvalue of the combined system is merely a perturbation of the third natural frequency of the simply supported beam.

Table 4 displays the first five natural frequencies for a system consisting of an undamped oscillator with a rigid body dof attached to a fixed-free beam at $x_1 = 0.90L$ (see Fig. 2). The spring–mass parameters are $m = 5.0\rho L$ and $k = 4.0EI/L^3$. Because the beam is cantilevered, $\phi_i(x)$ is given by Eq. (37), and the K_i and M_i are given by Eq. (38). The number of modes used in the analysis is $N = 14$ for the proposed approach. Using the finite element method, a generalized eigenvalue problem of size 201×201 needs to be solved. Because the attachment location, x_1, does not coincide with any nodes of the eigenfunctions of the fixed-free beam (see Table 2), Eq. (28) becomes

$$1 + \frac{km\lambda^2}{k + m\lambda^2} \sum_{i=1}^{N} \frac{\phi_i^2(x_1)}{\lambda^2 M_i + K_i} = 0, \tag{42}$$

Table 3
The first five natural frequencies of a simply supported, uniform Euler–Bernoulli beam carrying an undamped oscillator with no rigid body dof at $x_1 = 0.30L$

Nat. freq.	Beam only	FEM	Eq. (28) ($N = 20$)
1	9.869604	6.532205	6.532235
2	39.478418	29.759095	29.760059
3	88.826440	86.729607	86.731226
4	157.913671	143.226510	143.257055
5	246.740110	209.374306	209.463777

The oscillator parameters are $m = 1.0\rho L$ and $k = 3.0EI/L^3$. The first five natural frequencies of a simply supported beam are also given. The natural frequencies are nondimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 4
The first five natural frequencies of a fixed-free, uniform Euler–Bernoulli beam carrying an undamped oscillator with a rigid body dof at $x_1 = 0.90L$

Nat. freq.	Beam only	FEM	Eq. (28) ($N = 14$)
1	3.516015	0.631804	0.631805
2	22.034492	4.953629	4.953637
3	61.697214	22.136523	22.136523
4	120.901916	61.704017	61.704015
5	199.859530	120.902108	120.902095

The oscillator parameters are $m = 5.0\rho L$ and $k = 4.0EI/L^3$. The first natural frequencies of a fixed-free beam are also shown.

which is identical to Eq. (7a) that Dowell derived in [9]. Dowell [9] noted that if an undamped oscillator with a rigid body dof is attached to a beam, a new natural frequency appears between the original pair of beam frequencies nearest the oscillator frequency. Thus as expected, for the spring–mass parameters chosen, a new natural frequency appears before the first natural frequency of the cantilever beam. The first five natural frequencies of a cantilever beam are also shown. From Table 4, note the excellent agreement between the finite element results and the solution of Eq. (42). For this system, the attachment location is near the node of the fourth normal mode of a cantilever beam (see Table 2). Thus, the fifth natural frequency of the combined system is nearly identical to the fourth natural frequency of the fixed-free beam. Finally, the chosen set of lumped parameters only affects the lower natural frequencies. Specifically, it can be shown that the ith natural frequency of the combined system, for $i \geqslant 3$, is merely a perturbation of the $(i-1)$th natural frequency of the fixed-free beam only. Table 4 confirms the previous observation. In particular, note that the third and fourth natural frequencies of the combined system deviate slightly from the second and third natural frequencies of the cantilever beam, respectively.

Table 5 shows the first five natural frequencies of a two span cantilever beam (see Fig. 3). The in-span simple support is located at $x_1 = 0.65L$. For the proposed approach, Eq. (28) was used to determine the natural frequencies of the two span beam. Because the stiffness cannot be set to

Table 5
The first five natural frequencies of a two span, fixed-free, uniform Euler–Bernoulli beam

Nat. freq.	FEM	Eq. (28) ($N = 14$)	Eq. (43) ($N = 14$)
1	16.156817	16.157823	16.157910
2	46.910570	46.921507	46.922455
3	120.724293	120.725348	120.725440
4	162.408975	162.656676	162.677995
5	267.456193	267.767897	267.794470

The in-span simple support is located at $x_1 = 0.65L$. Using Eq. (28), $k = 1.0 \times 10^7 EI/L^3$ to model the simple support.

Table 6
The first five natural frequencies of simply supported, uniform beam carrying a grounded torsional spring $k_t = 10.0EI/L$ at $x_1 = 0.75L$

Nat. freq.	FEM	Eq. (28) ($N = 20$)
1	12.866415	12.912249
2	39.478418	39.478418
3	92.341021	92.402763
4	166.058365	166.223907
5	251.321762	251.427177

infinity, to solve Eq. (28) numerically, $k = 1.0 \times 10^7 EI/L^3$ to model the simple support. Alternatively, by letting $k \to \infty$, Eq. (28) can be approximated by

$$\sum_{i=1}^{N} \frac{\phi_i^2(x_1)}{\lambda^2 M_i + K_i} \approx 0. \tag{43}$$

Table 5 shows the results obtained by solving Eqs. (28) and (43). Note how well they track one another and the finite element results.

Table 6 lists the first five natural frequencies of a simply supported beam carrying a grounded torsional spring at $x_1 = 0.75L$ (see Fig. 6). The chosen attachment point coincides with a location of zero angular displacement for the second eigenfunction of a simply supported beam. Thus, the grounded torsional spring does not alter the second natural frequency of the simply supported beam, and as expected, the second natural frequency of the combined system is exactly $4\pi^2\sqrt{EI/(\rho L^4)}$. Incidentally, it should be noted that Eq. (28) fails to produce the second natural frequency of the system because the restoring torque at this point is exactly zero. In this case, sound engineering judgment must be applied to identify the "missing" natural frequency. Alternatively, Eq. (27) can be used and it will yield all of the natural frequencies of the combined system. From Table 6, note the excellent agree between the finite element results and the proposed solution scheme.

Consider now the system of Fig. 7, where the beam is simply supported and the oscillator parameters are $m = 1.0\rho L$, $c = 0.5\sqrt{EI\rho/L^2}$, and $k = 5.0EI/L^3$. The attachment location is at $x_1 = 0.23L$. When damping is present, the finite element approach requires the solution of a

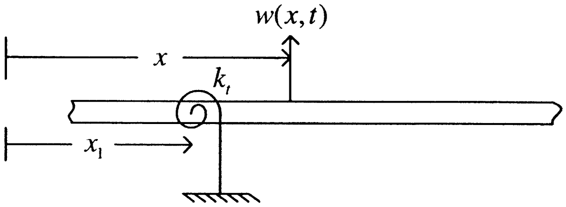

Fig. 6. An arbitrarily supported beam carrying a grounded torsional spring at x_1.

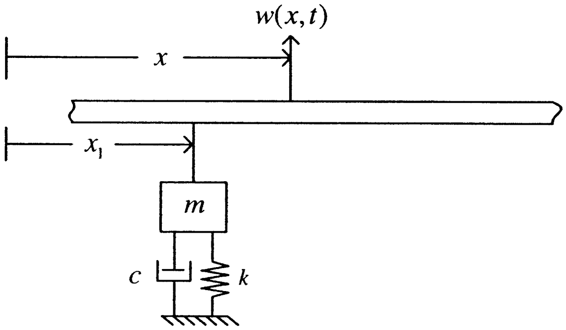

Fig. 7. An arbitrarily supported beam carrying a damped oscillator with no rigid body dof at x_1.

generalized eigenvalue problem of size 400×400 (when the beam is discretized into 100 elements). For the proposed scheme, one only needs to solve a secular equation consisting of the sum of N terms, where N is the number of terms used in the assumed modes analysis. Note that when the system is damped, the number of terms within the summation remains the same as that of undamped system of Fig. 4. Table 7 shows the first five eigenvalues for the system of Fig. 7. Note the excellent agreement between the two completely different approaches.

Table 8 displays the first five eigenvalues for a cantilever beam carrying a damped oscillator with a rigid body dof at $x_1 = 1.0L$ (see Fig. 8). The oscillator parameters are $m = 1.0\rho L$, $c = 0.2\sqrt{EI\rho/L^2}$, and $k = 0.5EI/L^3$. Like before, the two results track one another very well.

Consider now a beam carrying multiple lumped elements. Fig. 4 consists of a simply supported beam carrying an assorted attachments, including a grounded translational spring $k_1 = 5.0EI/L^3$ at $x_1 = 0.2L$, a lumped mass $m_1 = 1.75\rho L$ at $x_2 = 0.35L$, a damped oscillator with a rigid body dof and system parameters $m_2 = 5.0\rho L$, $c = 2.0\sqrt{EI\rho/L^2}$, $k_2 = 4.0EI/L^3$ at $x_3 = 0.5L$, and a grounded torsional spring $k_t = 10.0EI/L$ at $x_4 = 0.75L$. This problem can be solved by using many different methods. Lagrange multipliers approach also leads to Eq. (34), but it can be laborious to apply, because one needs to introduce S Lagrange multipliers and to formulate S constraint equations. The assumed-modes can also be employed to obtain the system mass, damping and stiffness matrices. To find the eigenvalues of the system, one needs to solve a generalized eigenvalue problem of size $2(N+1) \times 2(N+1)$, where N is the number of modes used in the analysis. The finite element method can also be used, in which case one needs to solve an eigenvalue problem of size $(2N_e + 1) \times (2N_e + 1)$, where N_e corresponds to the number of finite

Table 7
The first five eigenvalues of a simply supported, uniform Euler–Bernoulli beam carrying a damped oscillator with no rigid body dof at $x_1 = 0.23L$

Eigenvalues	FEM	Eq. (27) ($N = 20$)
1	−0.125894 + 7.249135j	−0.125892 + 7.249162j
2	−0.064985 + 27.430152j	−0.064977 + 27.431101j
3	−0.013154 + 77.364868j	−0.013147 + 77.369087j
4	−0.001212 + 155.982214j	−0.001206 + 155.985457j
5	−0.006637 + 233.490158j	−0.006582 + 233.548199j

The oscillator parameters are $m = 1.0\rho L$, $c = 0.5\sqrt{EI\rho/L^2}$, and $k = 5.0EI/L^3$. The eigenvalues are nondimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 8
The first five eigenvalues of a fixed-free, uniform Euler–Bernoulli beam carrying a damped oscillator with a rigid body dof at $x_1 = 1.0L$

Eigenvalues	FEM	Eq. (27) ($N = 20$)
1	−0.072937 + 0.651957j	−0.072938 + 0.651958j
2	−0.423076 + 3.760568j	−0.423079 + 3.760566j
3	−0.403063 + 22.060788j	−0.403065 + 22.060772j
4	−0.400606 + 61.702601j	−0.400605 + 61.702557j
5	−0.400197 + 120.902586j	−0.400200 + 120.902502j

The oscillator parameters are $m = 1.0\rho L$, $c = 0.2\sqrt{EI\rho/L^2}$, and $k = 0.5EI/L^3$.

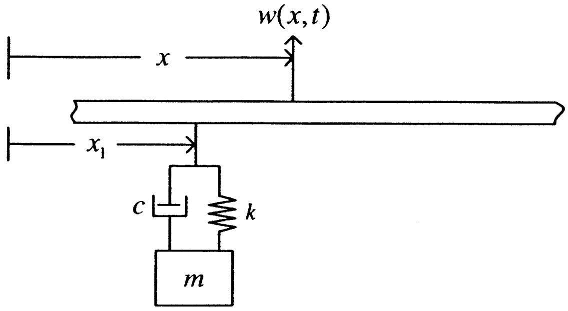

Fig. 8. An arbitrarily supported beam carrying an damped oscillator with a rigid body dof at x_1.

elements used. Clearly, as N_e becomes large, the resulting generalized eigenvalue problem becomes prohibitive to solve.

Consider now the approach proposed in this paper. For this system with multiple attachments, Eq. (34) was used to find the eigenvalues of the system, because the attachment locations do not coincide simultaneously to the nodes of the first five normal modes of a simply supported beam (see Table 2). The MATLAB routine *fsolve* requires an initial guess to be provided. For a beam

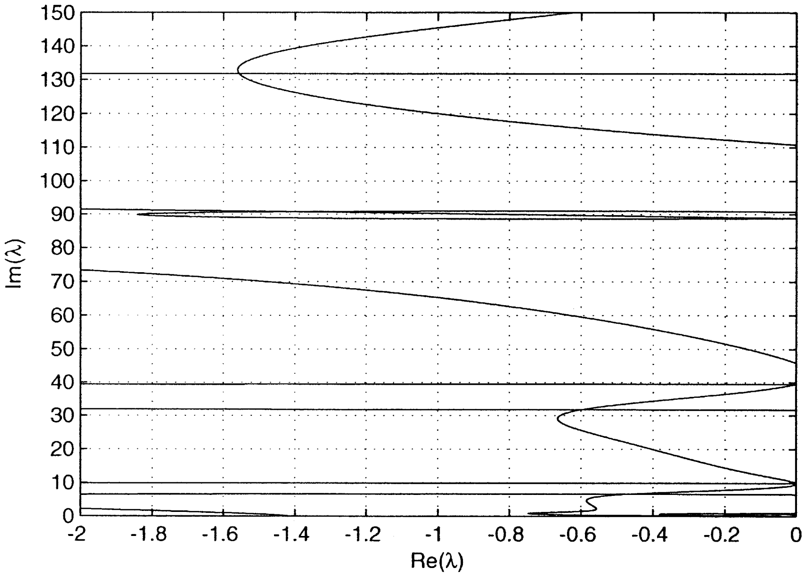

Fig. 9. Contour plots for the system of Fig. 4. The system parameters are given in Table 9.

carrying multiple attachments, an approximation of the eigenvalues can be established by means of simultaneous contour plots of the real and imaginary parts of the characteristic equation obtained by expanding Eq. (34). Once the approximate eigenvalues are known, *fsolve* can be used to quickly converge to the desired results.

When damping is present, the characteristic determinant of matrix $[B]$ yields a frequency equation $f(\lambda)$ that is complex,

$$\det[B] = f(\lambda) = f(\lambda_r + j\lambda_i) = f_r + jf_i = 0, \tag{44}$$

where λ denotes a complex eigenvalue of the system, and f_r and f_i are the real and imaginary parts of the function $f(\lambda)$, respectively. To find the eigenvalues graphically, the contour plots for $f_r = 0$ and $f_i = 0$ are generated using the MATLAB command *contour*, with the level set to zero. Fig. 9 shows the two-dimensional contour plots for the system of Fig. 4 with the given set of system parameters. The intersections of the curves are the solution to the simultaneous equations $f_r = 0$ and $f_i = 0$, and they correspond to the approximate eigenvalues of the system, which can then be used as the initial guesses for *fsolve*. Table 9 shows the first five eigenvalues of the system. Note the excellent results between the proposed scheme and the finite element results.

The proposed approach is highly versatile and can be used to find the eigenvalues of an arbitrarily supported beam carrying any number of lumped attachments. Because the assumed-modes method was used, the approach can be easily extended to find the eigenvalues of a bar in longitudinal vibration carrying any number of lumped elements by simply modifying the eigenfunctions and the generalized masses and stiffnesses [28]. It can also be used to find the eigenvalues of any linear structure carrying a chain of oscillators [29], as well as a plate with beam

Table 9
The first five eigenvalues of a simply supported, uniform Euler–Bernoulli beam carrying a grounded spring $k_1 = 5.0EI/L^3$ at $x_1 = 0.2L$, a lumped mass $m_1 = 1.75\rho L$ at $x_2 = 0.35L$, a damped oscillator with a rigid body dof at $x_3 = 0.5L$, and a grounded torsional spring $k_t = 10.0EI/L$ at $x_4 = 0.75L$

Eigenvalues	FEM	Eq. (32) ($N = 40$)
1	−0.183674 + 0.858271j	−0.183792 + 0.858374j
2	−0.466497 + 6.596888j	−0.465642 + 6.617391j
3	−0.599650 + 31.748922j	−0.599389 + 31.755864j
4	−1.402122 + 90.725528j	−1.400479 + 90.766378j
5	−1.544674 + 131.646158j	−1.556103 + 131.778924j

The damped oscillator parameters are $m_2 = 5.0\rho L$, $c = 2.0\sqrt{EI\rho/L^2}$, $k_2 = 4.0EI/L^3$.

stiffeners [6] and a plate carrying lumped masses and grounded translational springs [30]. The approach can also be used to investigate the sensitivity of the eigenvalues of a combined system on the attachment parameters [31]. Finally, the proposed scheme can be used to easily solve an inverse problem of imposing nodes at specified locations for any normal mode of a linear structure [32].

4. Conclusions

An alternative formulation is proposed that can be used to determine the eigenvalues of any arbitrarily supported beam carrying any number of lumped attachments, including point masses, rotary inertias, grounded translational or torsional springs, grounded translational or torsional viscous dampers, undamped and damped oscillators with no rigid body degree of freedom, and undamped and damped oscillators with a rigid body degree of freedom. The proposed scheme is versatile and leads to several noticeable advantages. Specifically, the proposed approach is simple to code, and leads to a frequency equation that can be solved either graphically or numerically; it can be easily extended to accommodate any beam with any boundary conditions; it can be easily modified to analyze a beam carrying any combination of miscellaneous attachments; and finally, it can be used to analyze multi-span beams with lumped attachments. Numerical experiments were performed to validate the proposed approach, and excellent agreements were found between the proposed scheme and the finite element solutions.

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