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Short Communication

# Simplified continuous finite element method for a class of nonlinear oscillating equations

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## 1. Introduction

The finite element method is one important technique for approximating the solutions of differential equations, and it is widely used for science and engineering algorithms. Aziz and Monk [1] solved the heat equation with continuous finite element method. Pan and Chen [2] studied the superconvergence of continuous finite element method for initial value problem of ordinary differential equation. For nonlinear parabolic problem, Zlamal [3] proposed a simplified finite element method—the interpolated coefficient finite element.

The main purpose of this paper is to use the simplified quadratic continuous finite element method for computing numerical oscillatory solutions of nonlinear equations:

$$\ddot{x} = f(\omega_0^2, \varepsilon, x), \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

where overdots denote differential with respect to time $t$ and $\varepsilon$ is a positive parameter, and where $f(x) = f(\omega_0^2, \varepsilon, x)$ is an odd function for any $x \in \mathbf{R} = (-\infty, +\infty)$.

Mickens [4] systematically studied oscillatory problems. Lim and Wu [5–7] present some new approaches to solving the nonlinear oscillators. Recently Hu [8] studied the Duffing equation by a classical perturbation technique which is valid for large parameters. He pointed out that the maximal relative error of the second approximate frequency with respect to the exact solution is

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less than 0.03%. This paper will apply the simplified continuous finite element method which is also valid for large parameters. We can obtain the better approximate frequency whose maximal relative error is less than 0.00001% for any $\varepsilon$.

For introducing the finite element method, we change Eq. (1) into first-order ordinary differential equations

$$\begin{aligned} \dot{x} &= y, & x(0) &= A, \\ \dot{y} &= f(\omega_0^2, \varepsilon, x), & y(0) &= 0. \end{aligned} \tag{2}$$

Setting $\mathbf{u} = (x, y)^{\mathrm{T}}$ and $\mathbf{F}(\mathbf{u}) = (y, f(\omega_0^2, \varepsilon, x))^{\mathrm{T}}$, Eq. (2) also becomes

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}), \quad \mathbf{u}(0) = (A, 0)^{\mathrm{T}}. \tag{3}$$

## 2. Simplified quadratic continuous finite element method

Taking a suitable large number $T^* > 0$, the interval $[0, T^*]$ is partitioned uniformly into $N$ elements. Let $h = T^*/(2N)$ denote half-step size of this partition and $J_n = [t_{n-1}, t_n]$ an element. Denote integer nodes by $t_n = 2nh$ and half-integer nodes by $t_{n-1/2} = (2n-1)h$. We define the following nodal basis:

$$\varphi_0(t) = \begin{cases} \dfrac{1}{2h^2}(t - t_1/2)(t - t_1), & t_0 \leqslant t \leqslant t_1, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

$$\varphi_0(t) = \begin{cases} \dfrac{1}{2h^2}(t - t_{N-1/2})(t - t_{N-1}), & t_{N-1} \leqslant t \leqslant t_N, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

$$\varphi_i(t) = \begin{cases} \dfrac{1}{2h^2}(t - t_{i-1/2})(t - t_{i-1}), & t_{i-1} \leqslant t \leqslant t_i, \\ \dfrac{1}{2h^2}(t - t_{i+1/2})(t - t_{i+1}), & t_i \leqslant t \leqslant t_{i+1}, \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

$$\varphi_{i-1/2} = \begin{cases} -\dfrac{1}{h^2}(t - t_{i-1})(t - t_i), & t_{i-1} \leqslant t \leqslant t_i, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, N, \tag{7}$$

then the quadratic continuous finite element subspace is defined by

$$S_0^h = \left\{ \mathbf{u} = \sum_{i=0}^{N} \varphi_i \alpha_i + \sum_{i=0}^{N} \varphi_{i-1/2} \alpha_{i-1/2}, \alpha_i, \alpha_{i-1/2} \in \mathbf{R}^2, u_1(0) = A, u_2(0) = 0 \right\}.$$

Now let $\mathbf{U} \in S_0^h$ be the quadratic continuous finite element approximation of the exact solution $\mathbf{u}$ of Eq. (3) by

$$\int_{J_n} \eta \dot{\mathbf{U}}\,\mathrm{d}t = \int_{J_n} \eta \mathbf{F}(\mathbf{U})\,\mathrm{d}t \tag{8}$$

for any $\eta \in \mathbf{P}_1(J_n)$ where $\mathbf{P}_1$ denotes the space of all linear, one-variable polynomial. Substituting the nodal basis, the above formula becomes

$$\begin{aligned}&\int_{J_n} [\dot{\varphi}_{n-1}\mathbf{U}_{n-1} + \dot{\varphi}_{n-1/2}\mathbf{U}_{n-1/2} + \dot{\varphi}_n \mathbf{U}_n]\eta\,\mathrm{d}t \\ &\quad = \int_{J_n} \mathbf{F}(\varphi_{n-1}\mathbf{U}_{n-1} + \varphi_{n-1/2}\mathbf{U}_{n-1/2} + \varphi_n \mathbf{U}_n)\eta\,\mathrm{d}t\end{aligned} \tag{9}$$

for any $\eta \in \mathbf{P}_1(J_n)$.

For the sake of simplicity, we take interpolation $\mathbf{I}_h\mathbf{F}(\mathbf{U}) = \varphi_{n-1}\mathbf{F}(\mathbf{U}_{n-1}) + \varphi_{n-1/2}\mathbf{F}(\mathbf{U}_{n-1/2}) + \varphi_n\mathbf{F}(\mathbf{U}_n) \in S_0^h$ in the place of $\mathbf{F}(\varphi_{n-1}\mathbf{U}_{n-1} + \varphi_{n-1/2}\mathbf{U}_{n-1/2} + \varphi_n\mathbf{U}_n)$ where $\mathbf{I}_h$ denotes interpolating operator. So we define the simplified quadratic continuous finite element solution $\mathbf{U}$ satisfying

$$\begin{aligned}&\int_{J_n} [\dot{\varphi}_{n-1}\mathbf{U}_{n-1} + \dot{\varphi}_{n-1/2}\mathbf{U}_{n-1/2} + \dot{\varphi}_n \mathbf{U}_n]\eta\,\mathrm{d}t \\ &\quad = \int_{J_n} [\varphi_{n-1}\mathbf{F}(\mathbf{U}_{n-1}) + \varphi_{n-1/2}\mathbf{F}(\mathbf{U}_{n-1/2}) + \varphi_n\mathbf{F}(\mathbf{U}_n)]\eta\,\mathrm{d}t\end{aligned} \tag{10}$$

for any $\eta \in \mathbf{P}_1(J_n)$.

The test functions $\eta(t)$ are taken, respectively, by $\dot{\varphi}_{n-1/2}(t)$ and $\dot{\varphi}_n(t)$ which are linearly independent linear polynomials. By rearranging the computing schemes,

$$\begin{bmatrix} \frac{8}{3}\mathbf{I}_2 & -\frac{4}{3}\mathbf{I}_2 \\ -\frac{4}{3}\mathbf{I}_2 & \frac{7}{6}\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{n-1/2} \\ \mathbf{U_n} \end{bmatrix} = h \begin{bmatrix} \mathbf{O}_2 & -\frac{2}{3}\mathbf{I}_2 \\ \frac{2}{3}\mathbf{I}_2 & \frac{1}{2}\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}(\mathbf{U}_{n-1/2}) \\ \mathbf{F}(\mathbf{U}_n) \end{bmatrix} + \begin{bmatrix} \frac{4}{3}\mathbf{U}_{n-1} + \frac{2}{3}h\mathbf{F}(\mathbf{U}_{n-1}) \\ -\frac{1}{6}\mathbf{U}_{n-1} - \frac{1}{6}h\mathbf{F}(\mathbf{U}_{n-1}) \end{bmatrix} \tag{11}$$

is gained, where $\mathbf{I}_2$ denotes two-order unit matrix, $\mathbf{O}_2$ denotes two-order zero matrix.

Letting

$$\mathbf{A} = \begin{bmatrix} \frac{8}{3}\mathbf{I}_2 & -\frac{4}{3}\mathbf{I}_2 \\ -\frac{4}{3}\mathbf{I}_2 & \frac{7}{6}\mathbf{I}_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{O} & -\frac{2}{3}\mathbf{I}_2 \\ \frac{2}{3}\mathbf{I} & \frac{1}{2}\mathbf{I} \end{bmatrix}$$

and

$$\mathbf{W}_n = \begin{bmatrix} \mathbf{U}_{n-1/2} \\ \mathbf{U}_n \end{bmatrix}, \quad \mathbf{G}(\mathbf{W}_n) = \begin{bmatrix} \mathbf{F}(\mathbf{U}_{n-1/2}) \\ \mathbf{F}(\mathbf{U}_n) \end{bmatrix}, \quad \mathbf{E}_{n-1} = \begin{bmatrix} \frac{4}{3}\mathbf{U}_{n-1} + \frac{2}{3}h\mathbf{F}(\mathbf{U}_{n-1}) \\ -\frac{1}{6}\mathbf{U}_{n-1} - \frac{1}{6}h\mathbf{F}(\mathbf{U}_{n-1}) \end{bmatrix}$$

then we have the following simplified quadratic continuous finite element method scheme:

$$\mathbf{H}(\mathbf{W}_n) = \mathbf{A}\mathbf{W}_n - h\mathbf{B}\mathbf{G}(\mathbf{W}_n) - \mathbf{E}_{n-1} = 0. \tag{12}$$

From (12), we obtain corresponding Newton iterative algorithm scheme

$$\mathbf{W}_n^{k+1} = \mathbf{W}_n^k - [\mathbf{DH}(\mathbf{W}_n^k)]^{-1}\mathbf{H}(\mathbf{W}_{n-1}), \quad k = 0, 1, \ldots, \tag{13}$$

where **D** denotes differential with respect to vector **W**.

## 3. Continuous finite element solution of the Duffing equation

In this section, in order to analyze the quadratic finite element solution, we take the Duffing equation

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0 \tag{14}$$

for example. For computing the approximate frequency and solution in one period, the calculation procedure is as follows:

(1) When the parameter $\varepsilon$, the basic frequency $\omega_0$, and the initial value $x_0 = A$, $y_0 = 0$ are given, an initial guess period is determined by [8]

$$T^* = 2\pi \Big/ \sqrt{\omega_0^2 + \tfrac{3}{4}\varepsilon A^2}. \tag{15}$$

(2) After the half-time step size $h$ being given, in the light of the Newton iterative algorithm scheme (13), we can compute continuous finite element solution $x_n = \mathbf{W}_n(3)$ and $x_{n-1/2} = \mathbf{W}_n(1)$ on the integer nodes and half-integer nodes in the time interval $[0, 3T^*]$.

(3) If $fx_n > 0,\ 0 \leqslant n < n_k$ and $x_{n_k} \leqslant 0$, then $T_h/4 \in J_n = [t_{n_k-1}, t_{n_k}]$ holds where $T_h$ is approximate period. In the element $J_n$ the approximate analytic formula of $x(t)$ satisfies

$$\begin{aligned} x(t) &= x_{n_k}\varphi_{n_k}(t) + x_{n_k+1/2}\varphi_{n_k+1/2}(t) + x_{n_k+1}\varphi_{n_k+1}(t) \\ &= \frac{1}{2h^2} x_{n_k}(t - t_{n_k+1/2})(t - t_{n_k}) - \frac{1}{h^2} x_{n_k+1/2}(t - t_{n_k+1})(t - t_{n_k}) \\ &\quad + \frac{1}{2h^2} x_{n_k+1}(t - t_{n_k+1/2})(t - t_{n_k+1}). \end{aligned} \tag{16}$$

The polynomial (16) must have a root which is a quarter of approximate period. So high accuracy approximate period $T_h$ and frequency $\omega_h$ are solved.

(4) Letting $N = T_h/(2h),\ (t_n, x_n),\ n = 0, 1, 2, \ldots, N$ and $(t_{n-1/2}, x_{n-1/2}), n = 1, 2, \ldots, N$ are plotted on $t$–$x$ plane.

To compare the present results with exact results, we take $\omega_0^2 = 1$. When half-step size $h \approx 0.025T^*$, we compute continuous finite element solution of the Duffing equation (14), respectively, for (1) $A = 1,\ \varepsilon = 1$, (2) $A = 10,\ \varepsilon = 10$, and plotted in Figs. 1 and 2.

The exact frequency of the periodic motion of the Duffing equation is given by [9]

$$\omega_e = \frac{\pi\sqrt{1 + \varepsilon A^2}}{2}\left(\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}\right)^{-1}, \quad m = \frac{\varepsilon A^2}{2(1 + \varepsilon A^2)}. \tag{17}$$

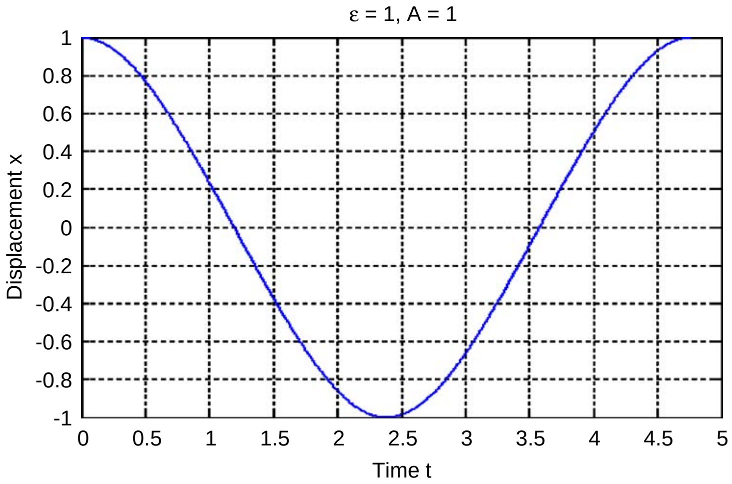


Fig. 1. The continuous finite element solution for $A = 1$, $\varepsilon = 1$.

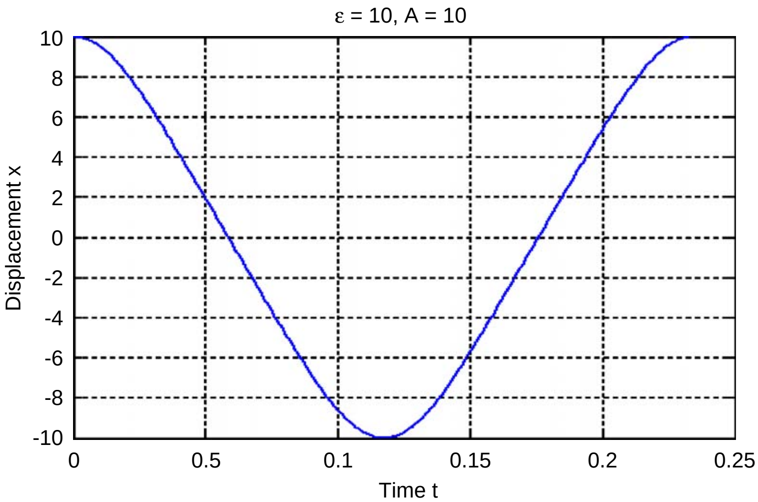


Fig. 2. The continuous finite element solution for $A = 10$, $\varepsilon = 10$.

Table 1
Comparison of the continuous finite element approximate frequency with the exact frequency for the Duffing equation

| $\varepsilon A^2$ | $\omega_e$ by Eq. (17) [9] | $\omega_h$ by CFEM | Relative error |
|---|---|---|---|
| 0.2 | 1.072000173865327 | 1.072000172990982 | 8.1562E − 10 |
| 1 | 1.317776083580348 | 1.317776073728296 | 7.4763E − 9 |
| 10 | 2.866640331675818 | 2.866640136434981 | 6.8108E − 8 |
| 100 | 8.533586188528274 | 8.533586181667106 | 2.1640E − 9 |
| 1000 | 26.81073816784440 | 26.81073845294955 | 1.0634E − 8 |
| 10 000 | 84.72747890121053 | 84.72747996079603 | 1.2506E − 8 |

For comparison, the exact frequency $\omega_e$ obtained by integrating Eq. (17) and the approximate frequency $\omega_h$ computed by simplified quadratic continuous finite element method (CFEM) are listed in Table 1.

## 4. Continuous finite element solution of $\ddot{x} + x^{(2m+1)/(2n+1)} = 0$

Similar to the algorithm of the Duffing equation, we compute continuous finite element solution of $\ddot{x} + x^{(2m+1)/(2n+1)} = 0$, $x(0) = 1$, $\dot{x}(0) = 0$ for (1) $m = 0$, $n = 1$, (2) $m = 2$, $n = 1$ and picture them in period in Figs. 3 and 4.

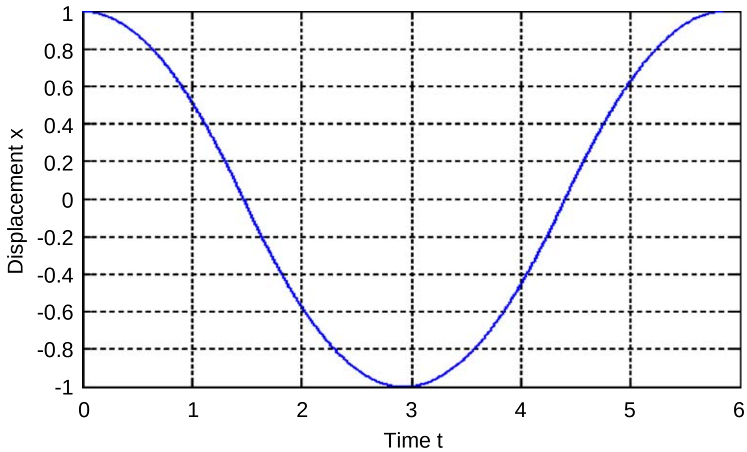


Fig. 3. The continuous finite element solution for $\ddot{x} + x^{1/3} = 0$.

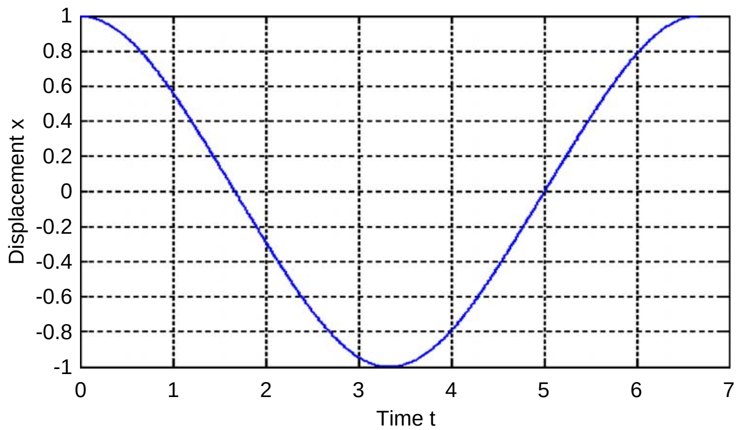


Fig. 4. The continuous finite element solution for $\ddot{x} + x^{5/3} = 0$.

For $m = 0$ and $n = 1$, the approximate frequency $\omega_h = 1.070450419702578$ is obtained by continuous finite element method. Gottlieb [10] obtained the exact frequency $\omega_{\mathrm{EX}} = 1.070451$. This means that we obtain the better approximate frequency whose relative error is less than 0.00006%.

## 5. Concluding remarks

The simplified quadratic continuous finite element method for a class of nonlinear oscillating equations is introduced. Chen [11] pointed out that the quadratic finite element solution have four-order high accuracy for ordinary differential equations with initial problem. When half-time step-size is taken for suitable small, the algorithm scheme (13) can give excellent numerical frequencies of the Duffing equation for both small and large parameters and oscillation amplitude.

Now finite element method is an effective technique for solving differential equations. For the semilinear differential equations with nonlinear term $f(u)$, the simplified finite element method, which requests that $f(u_h)$ is replaced by interpolation $I_h f(u_h)$ in numerical computation, can simplify calculation. In summary, simplified continuous finite element method may be a better numerical method for complex nonlinear oscillating equations.

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