Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Sound and Vibration 287 (2005) 459–479
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Responses of discretized systems under narrow band nonstationary random excitations. Part 2: nonlinear problems

Z. Chen, C.W.S. To*

Department of Mechanical Engineering, University of Nebraska, 104N Walter Scott Engineering Center, P.O. Box 880656, Lincoln, NE 68588-0656, USA

Received 26 April 2004; received in revised form 20 October 2004; accepted 11 November 2004
Available online 18 January 2005

Abstract

In a companion paper, the extended stochastic central difference (ESCD) method was presented for computation of responses of linear multi-degrees-of-freedom (mdof) systems under narrow band stationary and nonstationary random excitations. The present paper is concerned with the generalization of the ESCD method for computation of nonlinear mdof systems. The generalization is based on a combination of the ESCD method with statistical linearization (SL) technique, modified adaptive time scheme (ATS), and time coordinate transformation (TCT). Unlike the conventional SL technique, the generalized ESCD method is applicable to mdof systems with large nonlinearities. Comparison is made of the computed results applying the generalized ESCD method to those obtained by the Monte Carlo simulation (MCS). Excellent agreements were obtained. It was observed that the proposed method is very efficient compared with the MCS.

1. Introduction

In the companion paper [1] the extended stochastic central difference (ESCD) method was proposed as a viable alternative to large-scale computation of responses of multi-degrees-of-

*Corresponding author. Tel.: +1 402 472 1678; fax: +1 402 472 1465.
E-mail address: cto2@unl.edu (C.W.S. To).

0022-460X/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2004.11.021

freedom (mdof) linear systems under narrow band random excitations. It was found that the proposed method has several advantageous features over existing techniques. This paper is a sequel to the companion paper [1]. It presents a generalization of the ESCD method for computation of responses of nonlinear systems under wide band and narrow band random excitations. The novel feature of this generalized ESCD method is its combined use of the ESCD method [1], statistical linearization (SL) technique [2], the modified adaptive time scheme (ATS) [3], and the time coordinate transformation (TCT) for stiff systems [4]. For conciseness, in the following the above novel approach is simply referred to as the generalized ESCD method.

In the next section the application of the SL technique for single degree-of-freedom (dof) nonlinear systems excited by narrow band random disturbances is made. Numerical results are presented to demonstrate the efficiency and accuracy of the generalized ESCD method. Once, the superior features of the latter are established, it is applied to the response computation of mdof nonlinear systems in Section 3. For tractability and without loss of generality, 2dof nonlinear systems are considered. Section 4 includes concluding remarks.

2. Generalization of extended stochastic central difference for single degree-of-freedom nonlinear oscillators

The lowest order symmetric nonlinear power series representation of an elastic system is the so-called Duffing oscillator which has been applied to the analysis of many engineering systems [5]. In the latter reference, the excitations are wide band stationary and nonstationary random excitations. Davies and Nandlall [6] applied the Fokker–Plank–Kolmogorov (FPK) equation approach to a Duffing oscillator excited by a narrow band stationary random excitation. It was shown that multiple values of the response could occur at some frequencies for very narrow excitation bandwidths, but those values reduced to a single one as the bandwidth was increased.

In this section the ESCD method for narrow band random excitations presented in the companion paper [1] for linear systems is applied in conjunction with the SL technique, modified ATS and TCT to calculate the variance of response of a Duffing oscillator. In the approach presented here, after applying the SL technique to the original governing equation an equivalent linear governing equation is established. The present approach is different, from those applying the conventional SL technique, in that the latter applies to systems with small nonlinearities, while the presently proposed approach is applicable to systems with large nonlinearities. This is made possible by the fact that the system nonlinear parameters are updated at every time step during the computation. The following subsections include a brief description of the system inputs and modified ATS, and an outline of the SL technique followed by numerical results in Section 2.4.

2.1. Interpolation of system inputs

Because the time step size depends on the natural frequency and the natural frequencies of the filter and system are normally different, especially with nonlinear problems, the time step size of the system is different from that of the filter in general. Thus, an output from the filter cannot be directly used as an input to the system of interest since the time steps of the filter and the system will not match. To resolve this problem, one can adopt the interpolation or extrapolation to adjust the time step of the output of the filter so that it can be applied to the computation of the response of the system. Interpolation is computationally stable whereas it is possible for extrapolation running into computational instability if over shooting occurs. Therefore, in this investigation an interpolation strategy is applied. It will be demonstrated in the Section 2.4 that this strategy can provide excellent results.

2.2. Modified adaptive time scheme in the generalized ESCD method

As mentioned earlier, the time step size depends on the natural frequency of the excited system. In a nonlinear system, the natural frequency varies with the response, and therefore the time step size has to be changed accordingly. The so-called ATS has been applied to deal with this situation in systems under wide band random excitations [3]. In other words, while the equivalent natural frequency is being updated at every time step, the time step size also has to be adjusted. Moreover, what needs to be modified is that all terms in the ESCD method presented in Ref. [1] computed at time steps prior to the current step and to be used in the current step also have to be adjusted in order to suit the current time step. The terms are $R_y(s-1), G(s-2), H(s-2), R_f(s-2), N_{1y}, N_{2y}$ and N_{3y}. This becomes more critical in the generalized ESCD method for narrow band excitations because the error coming from not only the system but also from the responses of the filter if the adjustment was not conducted. The interpolation scheme is employed for this modification to the original ATS and is referred to as the modified ATS.

It will be shown in Figs. 1 and 2, to be presented in Section 2.4, that the modification to the original ATS is important in terms of accuracy and computational stability.

2.3. The statistical linearization

The SL technique is included in this subsection for completeness. Consider a class of general nonlinear systems which can be described by the following equation of motion:

$$\ddot{y} + 2\zeta\omega\dot{y} + \omega^2(1 + \varepsilon y^{2q})y = r, \tag{1}$$

where $q = 1, 2, 3, \ldots, r = p/m$, ε is the strength of nonlinearity, ω is the corresponding linear natural frequency, ζ is the damping ratio and m is the mass. Another form of Eq. (1) is

$$\ddot{y} + 2\zeta\omega\dot{y} + \omega^2\left[1 + \varepsilon\left(\frac{y}{\sigma}\right)^{2q}\right]y = r, \tag{2}$$

in which $\sigma^2 = \pi S/(2\zeta\omega^3)$ with S being the spectral density of the zero-mean Gaussian white noise excitation process. If the excitation is nonstationary random then the σ^2 is replaced by $(\sigma_y^2)_{\max}$ which is the maximum value of the variance of response for the corresponding linear system. In general, the nonlinear single dof (sdof) oscillator can be described by

$$\ddot{y} + 2\zeta\omega\dot{y} + g(y, \dot{y}, \varepsilon, t) = r, \tag{3}$$

where g, with the arguments being disregarded for conciseness, is the nonlinearity function of the oscillator. In particular, with $q = 1$ Eq. (2) becomes

$$\ddot{y} + 2\zeta\omega\dot{y} + g(y) = r, \tag{4}$$

where

$$g(y) = \omega^2 \left[1 + \varepsilon \left(\frac{y}{\sigma} \right)^2 \right] y. \tag{5}$$

Applying the SL technique [2], Eq. (4) is approximated by an equivalent linear equation

$$\ddot{y} + 2\zeta\omega\dot{y} + \omega_{\text{eq}}^2 y = r, \tag{6}$$

where ω_{eq} is the equivalent natural frequency of the system.

It can be shown that [2] the equivalent natural frequency ω_{eq} is given by

$$\omega_{\text{eq}}^2 = \omega^2 \left[1 + 3\varepsilon \left(\frac{\sigma_y^2}{\sigma^2} \right) \right]. \tag{7}$$

Applying the SCD method [5] to Eq. (7), at time step t_s, yields

$$\omega_{\text{eq}}^2(s) = \omega^2 \left\{ 1 + 3\varepsilon \left[\frac{\sigma_y^2(s)}{\sigma^2} \right] \right\}, \tag{8}$$

where $\sigma_y^2(s)$ is the variance of discretized response of the system while σ, to be replaced by $(\sigma_y^2)_{\max}$, is the maximum variance of response of the corresponding linear system. Eq. (8) was employed in the SCD method [5] for the computation of responses of nonlinear oscillators. The equivalent natural frequency ω_{eq} is updated at every time step [5] and consequently, it provides accurate solution compared with that of the original nonlinear equation.

2.4. Numerical results

Results of two cases are presented. One is applied to assess the accuracy of the generalized ESCD method for nonlinear systems under narrow band random excitations by way of comparing results obtained by the generalized ESCD method with those of MCS. Results for the other case are employed for the study of effects of band widths of the excitation processes on the responses of the nonlinear systems. It may be appropriate to mention that at every time step various terms in Section 2 of the companion paper [1] have to be updated due to the nonlinear behavior of the system. These terms are $R_y(s-1), G(s-2), H(s-2), R_f(s-2), N_{1y}, N_{2y}$ and N_{3y}. With reference to Eq. (3) of the companion paper [1] the variances and covariances of responses considered here and subsequently in this paper are those with zero time lag.

2.4.1. Comparison of generalized ESCD method and MCS

This subsection aims at assessing the importance and relevance of the modified ATS in the computation of the nonlinear response. The governing equation of the Duffing oscillator is Eq. (2) with $q = 1.0$. The parameters for these tests are: mass of the system, $M_s = m = 1.0\,\text{kg}$, natural frequency corresponding to the linear system, $\omega_s = 1.0\,\text{rad/s}$, mass of the sdof filter, $M_f = 1.0\,\text{kg}$, $\omega_f = 1.0\,\text{rad/s}$, $\zeta_s = 1.25\%$, $\zeta_f = 1.0\%$, and σ^2 in Eq. (2) is replaced by $(\sigma_y^2)_{\max} = 22105\,\text{m}^2$, which is the maximum value of the variance of displacement response of corresponding linear oscillator.

The inputs to the filter are modulated zero-mean Gaussian white noise defined by

$$p = e(t)w(t), \quad e(t) = 4.0(\mathrm{e}^{-0.05t} - \mathrm{e}^{-0.1t}), \quad \langle w^2(0) \rangle = 2\pi S_0, \tag{9}$$

where $S_0 = 1.0$ and $(\sigma_y^2)_{\max}$ is employed to normalize the nonlinear coefficient.

The responses of the filter are used as inputs to the system directly. For the MCS, 200 realizations, each of which has 25,600 points, were considered. The time taken by the MCS is 40 min on average while the generalized ESCD method takes less than a second. Results were obtained with the SGI machine mentioned in the companion paper [1]. The computed results are presented in Figs. 1–3. The nonlinear coefficients or nonlinear intensity parameters ε of Figs. 1–3 are 5.0, 0.05 and 80.0, respectively. Figs. 1 and 2 present results computed by the modified and original ATS, respectively. The parameters of filter and system are the same. Note that in these and subsequent figures MCS represents results from MCS while SCD designates data computed by using the generalized ESCD method with modified ATS. Figs. 1 and 2 clearly demonstrated the

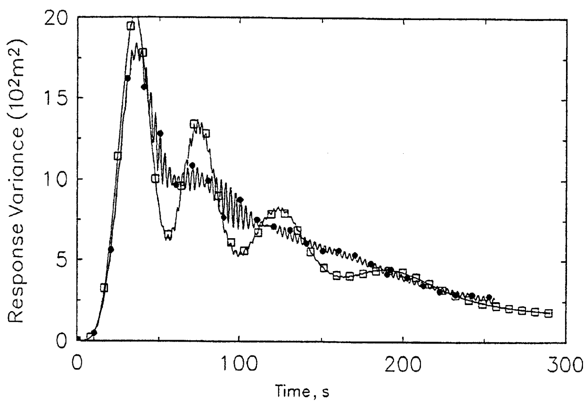

Fig. 1. Response variance of Duffing oscillator under narrow band nonstationary random excitation with $\omega_f = 1.0$ rad/s, $\zeta_f = 0.01$, $\omega_s = 1.0$ rad/s, $\zeta_s = 0.01$, $\varepsilon = 5.0$; MCS (•), and SCD (□).

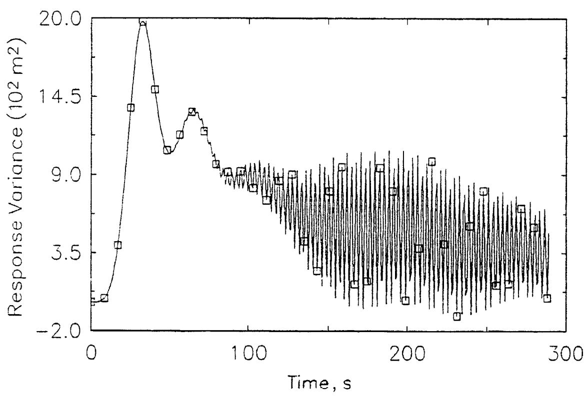

Fig. 2. Computational instability of response variance of Duffing oscillator with $\omega_f = 1.0$ rad/s, $\zeta_f = 0.01$, $\omega_s = 1.0$ rad/s, $\zeta_s = 0.01$, $\varepsilon = 0.05$. SCD (□).

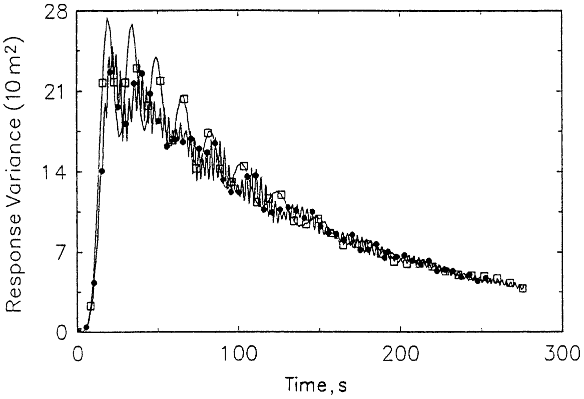

Fig. 3. Response variance of a Duffing oscillator with $\omega_f = 1.0$ rad/s, $\zeta_f = 0.01$, $\omega_s = 1.0$ rad/s, $\zeta_s = 0.01$, $\varepsilon = 80.0$; MCS (•), and SCD (□).

importance of employing the generalized ESCD method with modified ATS. The latter gives accurate results and does not give rise to computational instability which can occur in results obtained by applying the generalized ESCD with the unmodified or original ATS. Figs. 1 and 3 show very good agreement between results by the generalized ESCD method with modified ATS, and those by the MCS. Note that the amplitudes of responses decrease with increasing nonlinear coefficient. This is because the equivalent natural frequency increases with increasing nonlinear coefficient. It is also observed that the number of multiple peaks increases with the increase of nonlinear coefficient. An explanation is that when the value of the nonlinear coefficient increases, the natural frequency of the system shifts away from resonance at a higher rate, and therefore, the amplitude of response reduces temporally. Once the amplitude is reduced, the natural frequency of the system returns to resonance again and so another peak appears. This will continue until the input to the system vanishes. Note that in Fig. 3 the nonlinear coefficient is $\varepsilon = 80$. This large nonlinear coefficient is, of course, not realistic. However, such a large nonlinear coefficient is applied to test the capability of the ESCD method with the modified ATS. The foregoing computed results show that the proposed approach of employing the generalized ESCD method with modified ATS can provide very accurate solution for large nonlinear problems without encountering computational instability.

In closing, Fig. 2 demonstrates the importance of the modified ATS. When the normalized nonlinearity reaches $\varepsilon = 5.0$, the computed solution becomes unstable if the original ATS was applied. This is because once the nonlinearity becomes larger, the difference between two adjacent time step sizes also increases and so does the error due to the lack of the adjustment of the time steps. During the numerical experiments it was observed that with the adjustment the normalized nonlinearity ε can reach as large as 100 and yet the solution was still stable.

2.4.2. *Effect of bandwidth of excitation process on responses*

This case is concerned with the study of the effect of bandwidth of the excitation process on responses. A secondary objective of the study is the effect of nonlinearity on responses. The narrow band excitation to the system is defined by Eqs. (9), whereas the white noise excitation to the system is given by

$$p = e(t)\sqrt{I}, \quad e(t) = 4.0(e^{-0.05t} - e^{-0.1t}), \quad I = 2\pi. \tag{10}$$

The time step size evaluated by Eq. (10) in the companion paper [1] is incorporated in the source code of the digital computer program developed for the present investigation so that the time step size is calculated according to the natural frequency at every time step. Computed results are presented in Figs. 4–8, in which WB denotes wide band results while NB refers to solution due to narrow band random excitations. The pertinent parameters are listed in Table 1. In the latter the subscript f denotes the filter, that is, the narrow band process, while the subscript s represents the system. From these figures, one notices that all responses to the narrow band process have multiple peaks. The explanation of this phenomenon has been provided in the last subsection. Comparing Fig. 4 with Fig. 5, and Fig. 6 with Fig. 8, it is observed that the difference between

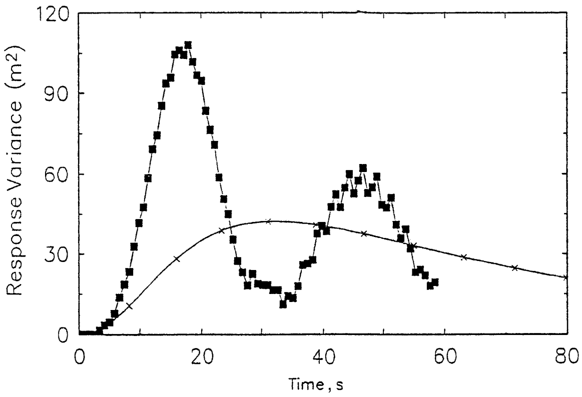

Fig. 4. Effect of bandwidth on response variance of a nonlinear oscillator with $\omega_f = 1.0$ rad/s, $\zeta_f = 0.01$, $\omega_s = 1.0$ rad/s, $\zeta_s = 0.01$, $\varepsilon = 5.0$; wide band (×), and narrow band (■).

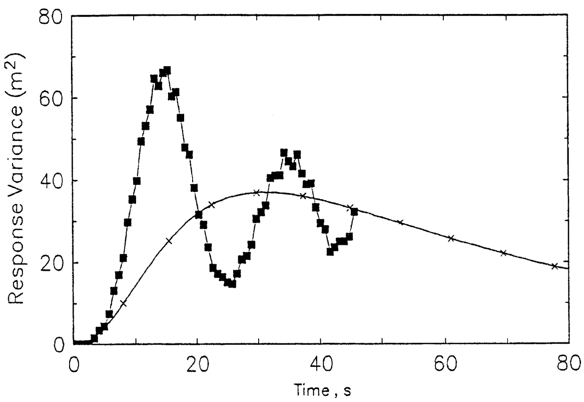

Fig. 5. Effect of bandwidth on response variance of a nonlinear oscillator with $\omega_f = 1.0$ rad/s, $\zeta_f = 0.01$, $\omega_s = 1.0$ rad/s, $\zeta_s = 0.01$, $\varepsilon = 10.0$; wide band (×), and narrow band (■).

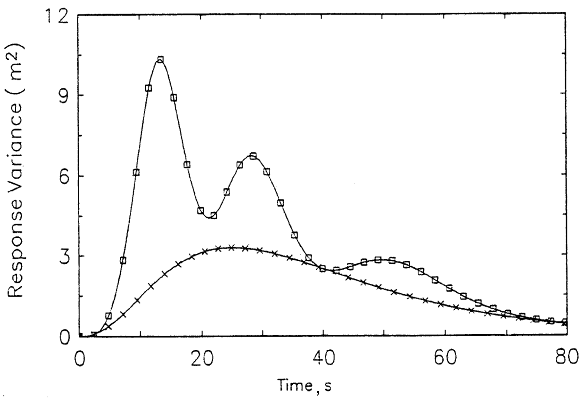

Fig. 6. Effect of bandwidth on response variance of a nonlinear oscillator with $\omega_f = 3.0$ rad/s, $\zeta_f = 0.01$, $\omega_s = 3.0$ rad/s, $\zeta_s = 0.01$, $\varepsilon = 1.0$; wide band ($\times$), and narrow band ($\square$).

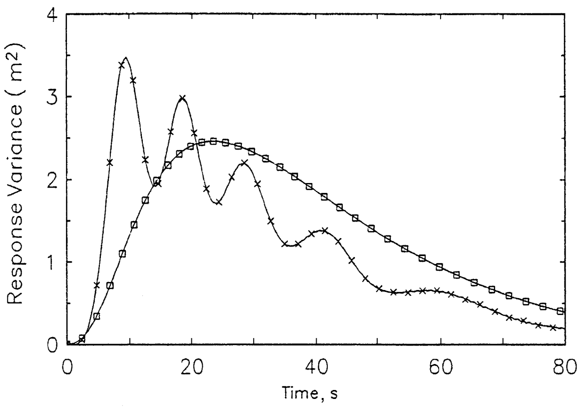

Fig. 7. Effect of bandwidth on response variance of a nonlinear oscillator with $\omega_f = 3.0$ rad/s, $\zeta_f = 0.01$, $\omega_s = 3.0$ rad/s, $\zeta_s = 0.01$, $\varepsilon = 5.0$; narrow band ($\times$), and wide band ($\square$).

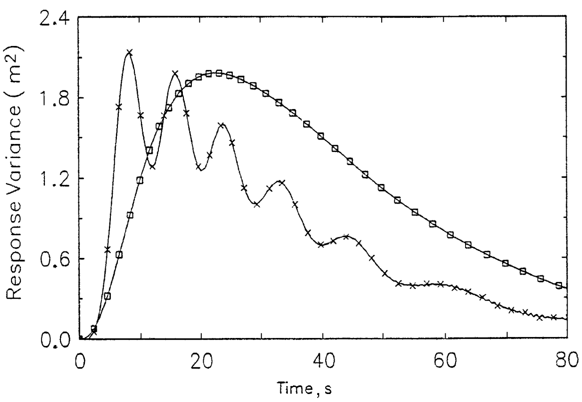

Fig. 8. Effect of bandwidth on response variance of a nonlinear oscillator with $\omega_f = 3.0$ rad/s, $\zeta_f = 0.01$, $\omega_s = 3.0$ rad/s, $\zeta_s = 0.01$, $\varepsilon = 10.0$; narrow band ($\times$), and wide band ($\square$).

Table 1
Parameters of systems for comparison

Parameters	ε
$M_s = 1.0\,\text{kg}$, $\omega_s = 1.0\,\text{rad/s}$, $\zeta_s = 1.0\%$, $S_0 = 1.0$, $I = 1.0$, $\omega_f = 1.0\,\text{rads/s}$, $\zeta_f = 1.0\%$, $(\sigma_y^2)_{\max} = 1488\,\text{m}^2$	5.0
	10.0
$M_s = 1.0\,\text{kg}$, $\omega_s = 3.0\,\text{rad/s}$, $\zeta_s = 1.0\%$, $S_0 = 1.0$, $I = 1.0$, $\omega_f = 3.0\,\text{rad/s}$, $\zeta_f = 1.0\%$, $(\sigma_y^2)_{\max} = 46.15\,\text{m}^2$	1.0
	5.0
	10.0

responses to narrow band and to wide band excitations decreases with increasing linear natural frequency as well as increasing nonlinear stiffness coefficient. Furthermore, comparing Figs. 6–8, one notices that these results are consistent with the finding in the last subsection. That is, the higher the nonlinear coefficient is, the more peaks the response has to narrow band random excitation.

3. Generalization of extended stochastic central difference for multi-degrees-of-freedom nonlinear systems

Having established the accuracy and usefulness of the generalized ESCD method incorporating SL technique and modified ATS for sdof nonlinear systems, in this section the proposed approach is applied to mdof nonlinear systems. Examples of the latter are nonlinear structural systems approximated by the FEM. The mdof nonlinear system has been investigated and reported in Ref. [7] in which the excitations are modulated Gaussian white noise processes. The focus of this section is on the application of the generalized ESCD method to the nonlinear mdof system under narrow band nonstationary random excitations.

By employing the SCD method, recursive expressions of responses of the mdof nonlinear system under modulated Gaussian white noise excitations have been derived [7]. In the latter reference, several schemes of obtaining the equivalent linear terms by applying the SL technique have been studied and discussed. Various techniques of dealing with the nonzero time-dependent mean of responses were investigated and Scheme IV of Ref. [7] has proved to be the most appropriate algorithm in that it requires less algebraic manipulation and also less computational time with excellent accuracy. Therefore, Scheme IV of Ref. [7] is adopted in this section.

3.1. Statistical linearization for multi-degrees-of-freedom nonlinear systems

For tractability and without loss of generality, the 2dof system shown in Fig. 9 is considered. The system is subjected to a narrow band nonstationary random excitation at its base and the restoring force of the spring connecting the two masses M_1 and M_2 has quadratic and cubic nonlinear terms associated with, respectively, parameters η' and ε'. This 2dof system under a modulated white noise excitation was studied and reported in Refs. [3,7,8]. Practical examples that

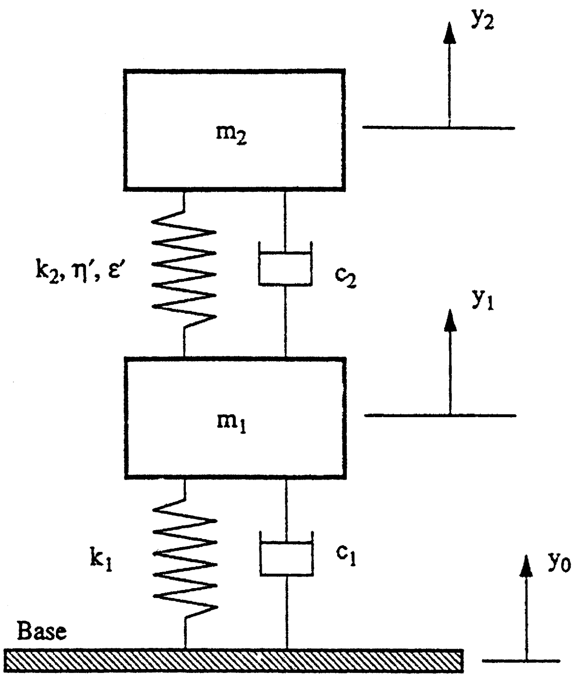

Fig. 9. 2dof nonsymmetric nonlinear system.

can be modeled by the system in Fig. 9 are soil–structure interaction system, and a primary building structure with a secondary system representing installed equipment under an earthquake excitation.

If one introduces relative displacements such that $Y_1 = y_1 - y_0$ and $Y_2 = y_2 - y_1$, where y_0, y_1 and y_2 are, respectively, the absolute displacements with respect to the base, the matrix equation of motion can be expressed as [3,7,8]

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{Y}_1 \\ \ddot{Y}_2 \end{Bmatrix} + \begin{bmatrix} 2\zeta_1 W & -2\mu\zeta_2 \\ -2\zeta_1 W & 2(1+\mu)\zeta_2 \end{bmatrix} \begin{Bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{Bmatrix} + \begin{bmatrix} W^2 & -\mu \\ -W^2 & 1+\mu \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}$$
$$+ \begin{Bmatrix} \mu\eta Y_2^2 - \mu\varepsilon Y_2^3 \\ (1-\mu)\eta Y_2^2 + (1+\mu)\varepsilon Y_2^3 \end{Bmatrix} = \begin{Bmatrix} -\ddot{y}_0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} e(\tau)\sqrt{I} \\ 0 \end{Bmatrix}, \tag{11}$$

or written in a more compact form,

$$M\ddot{Y} + C\dot{Y} + K_0 Y + g(Y) = r(\tau), \tag{12}$$

where

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2\zeta_1 W & -2\mu\zeta_2 \\ -2\zeta_1 W & 2(1+\mu)\zeta_2 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} W^2 & -\mu \\ -W^2 & 1+\mu \end{bmatrix}, \quad g(X) = \begin{Bmatrix} \mu\eta Y_2^2 - \mu\varepsilon Y_2^3 \\ (1-\mu)\eta Y_2^2 + (1+\mu)\varepsilon Y_2^3 \end{Bmatrix},$$

$$Y = \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}, \quad r(\tau) = \begin{Bmatrix} e(\tau)\sqrt{I} \\ 0 \end{Bmatrix},$$

and where the over-dot and double over-dot denote, respectively, the first and second derivatives with respect to τ. In Eq. (11) the nonstationary random excitation is represented as a product of a deterministic amplitude modulating function $e(\tau)$ and a zero-mean narrow band random process which is related to the response from the filter. Note that the following definitions have been applied in Eqs. (11) and (12):

$$\omega_1^2 = K_1/M_1, \quad \omega_2^2 = K_2/M_2, \quad 2\zeta_1\omega_1 = C_1/M_1, \quad 2\zeta_2\omega_2 = C_2/M_2,$$

$$\eta = \eta'/M_2\omega_2^2, \quad \varepsilon = \varepsilon'/M_2\omega_2^2, \quad \mu = M_2/M_1, \quad W = \omega_1/\omega_2, \quad \tau = \omega_2 t, \tag{13}$$

such that Eqs. (11) and (12) are dimensionless. For easy reference, notations used here are identical to those in Refs. [3,7]. Therefore, the symbol τ in this section should not be confused with the τ used in the time coordinate transformation (TCT) [3].

The discretization of Eq. (12) in the τ domain leads to

$$M\ddot{Y}_s + C\dot{Y}_s + K_0 Y_s + g(Y_s) = r_s, \tag{14}$$

where the subscript s is a positive integer denoting the time step τ_s, such that $\Delta\tau = \tau_{s+1} - \tau_s$ and $\tau_0 = 0$.

Since this is a nonlinear system, the ensemble averages of its responses, in general, are nonzero and time dependent even if the excitations are of zero ensemble averages. Assume that the responses at every time step are Gaussian. Eq. (14) can be represented by the linearized equation

$$M\ddot{Y}_s + C\dot{Y}_s + K_{eq} Y_s = f_s, \tag{15}$$

where K_{eq} is the equivalent stiffness matrix which is time dependent. It is determined by the equation

$$(K_{eq})_{ij} = (K_0)_{ij} + \left\langle \frac{\partial g_i(Y(s))}{\partial Y_j(s)} \right\rangle, \quad i,j = 1,2. \tag{16}$$

Upon substituting the nonlinear term $g(Y)$ into Eq. (16) and carrying out the operation one has [3]

$$K_{eq} = K_{eq}(s) = K_0 + \begin{bmatrix} 0 & 2\mu\eta\langle Y_2(s)\rangle + 3\mu\varepsilon\langle Y_2^2(s)\rangle \\ 0 & 2(1-\mu)\eta\langle Y_2(s)\rangle + 3(1+\mu)\varepsilon\langle Y_2^2(s)\rangle \end{bmatrix}. \tag{17}$$

With Eq. (17), one can then apply the generalized ESCD method to Eq. (15) and carry out the mathematical operation to obtain the recursive expression which is identical to that described in Section 2 of the companion paper [1]. The modified ATS has to be employed with the generalized ESCD method to update the K_{eq} at every time step.

For the derivation of a recursive ensemble average expression, if Eq. (15) of Ref. [3] is directly applied it would result in a zero ensemble average of response even though the system is nonlinear.

This is because the ensemble average of Eq. (15) of Ref. [3] leads to

$$\bar{\mu}(s+1) = N_{2y}(s)\bar{\mu}(s) + N_{3y}\bar{\mu}(s-1). \tag{18}$$

If the system starts from rest, one obviously has the initial conditions

$$\bar{\mu}(0) = \bar{\mu}(1) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad R_y(0) = R_y(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_y(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{19}$$

which would lead to an incorrect zero ensemble average. To avoid this problem, the $g(Y)$ term of Eq. (12) can be moved to the right-hand side (RHS) such that

$$M\ddot{Y}(s) + C\dot{Y}(s) + K_0 Y(s) = r(s) - g(Y(s)). \tag{20}$$

Substituting the central difference approximation of velocity and acceleration terms,

$$\dot{Y}(s) = \frac{1}{2\Delta\tau}[Y(s+1) - Y(s-1)],$$

$$\ddot{Y}(s) = \frac{1}{(\Delta\tau)^2}[Y(s+1) - 2Y(s) + Y(s-1)], \tag{21}$$

into Eq. (20) one has

$$Y(s+1) = \bar{N}_{2y}Y(s) + N_{3y}Y(s-1) + (\Delta\tau)^2 N_{1y} r(s) - (\Delta\tau)^2 N_{1y} g(Y(s)), \tag{22}$$

where

$$\bar{N}_{2y} = N_{1y}[2M - (\Delta\tau)^2 K_0]. \tag{23}$$

Taking the ensemble average of Eq. (22) one obtains

$$\bar{\mu}(s+1) = \bar{N}_{2y}\bar{\mu}(s) + N_{3y}\bar{\mu}(s-1) - (\Delta\tau)^2 N_{1y}\langle g(Y(s))\rangle, \tag{24}$$

$$\langle g(Y(s))\rangle = \begin{Bmatrix} \mu\eta\langle Y_2(s)^2\rangle - \mu\varepsilon[3\langle Y_2(s)\rangle\langle Y_2(s)^2\rangle - 2\langle Y_2(s)\rangle^3] \\ (1-\mu)\eta\langle Y_2(s)^2\rangle + (1+\mu)\varepsilon[3\langle Y_2(s)\rangle\langle Y_2(s)^2\rangle - 2\langle Y_2(s)\rangle^3] \end{Bmatrix}.$$

Substituting Eq. (19) into Eq. (3) of the companion paper [1], and Eqs. (18) and (24) one can show that

$$R_y(2) = (\Delta\tau)^4 N_{1y} R_f(1) N_{1y}^{\mathrm{T}}, \quad \bar{\mu}(2) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{25a,b}$$

$$\bar{\mu}(3) = -(\Delta\tau)^2 \eta\langle X_2(s)^2\rangle N_{1y}\begin{Bmatrix} \mu \\ 1-\mu \end{Bmatrix}. \tag{25c}$$

As soon as the above nonzero ensemble averages are calculated, one can return to Eq. (18). For $s>3$, Eq. (18) can be applied to obtain the nonzero ensemble averages. Then the recursive mean squares can be computed with Eq. (3) of the companion paper [1].

Numerical examples are presented in the next two subsections. Parameters of the system are

$$W = \mu = 1.0, \quad \zeta_1 = \zeta_2 = 0.1. \tag{26}$$

The two natural frequencies of the corresponding linear system, that is when η and ε are both equal to zero, are $\omega_{s1} = 0.618$ rad/s and $\omega_{s2} = 1.618$ rad/s. The amplitude modulating function $e(\tau)$ is

$$e(\tau) = 4(e^{-0.05\tau} - e^{-0.1\tau}). \tag{27}$$

3.2. Narrow band responses with sdof filter

Numerical results presented in this subsection include two examples. The first example is the one with the center frequency of $\omega_f = 1.0$ and the second is with $\omega_f = 1.618$. Each example includes a 2dof system described in the last subsection and a single dof filter. The latter has a dimensionless natural frequency equal to one of the corresponding linear dimensionless natural frequencies of the nonlinear system. The other has a dimensionless natural frequency which is between the two corresponding linear dimensionless natural frequencies of the nonlinear system.

Damping ratios of the sdof filter for three cases were $\zeta_f = 0.01$, 0.1 and 1.0. The narrow band random excitation to the system was the response taken directly from the filter which was perturbed by a nonstationary Gaussian white noise. The spectral density of the Gaussian white noise was $S_0 = 0.00012$. The narrow band nonstationary random excitation to the system was applied at its base. Results by the MCS were also obtained for comparison to those by the generalized ESCD method. In the MCS 200 realizations, each having 25,600 points, were considered. It should be mentioned that in the following, the results obtained by the generalized ESCD method or simply called the ESCD method are those obtained with the ESCD method and modified ATS. Other system parameters were: $\eta = -1.0$ and $\varepsilon = 1.5$, and were included in the last subsection. Computed results are shown in Figs. 10–14. It was observed that the case where $\omega_f = 1.0$ and $\zeta_f = 0.01$ became computationally unstable in both the ESCD method and MCS. An attempt was also made to compute responses of the system excited by a narrow band process with $\omega_f = 0.618$. In this case, computational instability occurred when ζ_f equaled to 0.01 and 0.1. Note that the spectral density of the white noise was one order of magnitude smaller than what

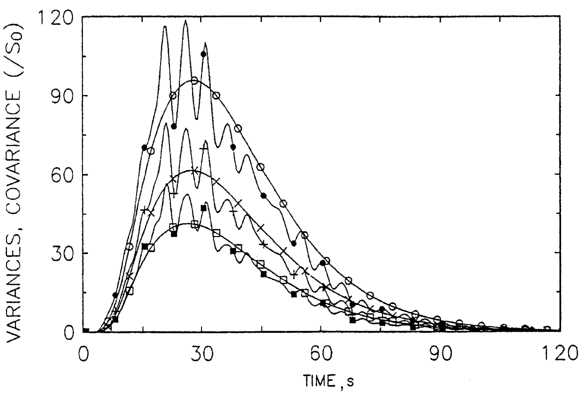

Fig. 10. Response variances and covariance of a nonlinear 2dof system with $\omega_f = 1.0$, $\zeta_f = 0.1$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1 Y_2 \rangle$ SCD (×), $\langle Y_1^2 \rangle$ MCS (●), $\langle Y_2^2 \rangle$ MCS (■), and $\langle Y_1 Y_2 \rangle$ MCS (+).

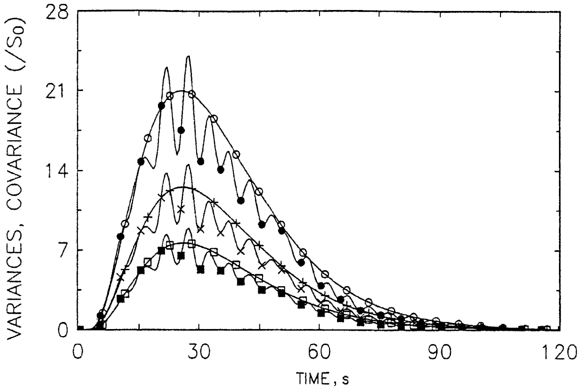

Fig. 11. Response variances and covariance of a nonlinear 2dof system with $\omega_f = 1.0$, $\zeta_f = 1.0$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1 Y_2 \rangle$ SCD (+), $\langle Y_1^2 \rangle$ MCS (●), $\langle Y_2^2 \rangle$ MCS (■), and $\langle Y_1 Y_2 \rangle$ MCS (×).

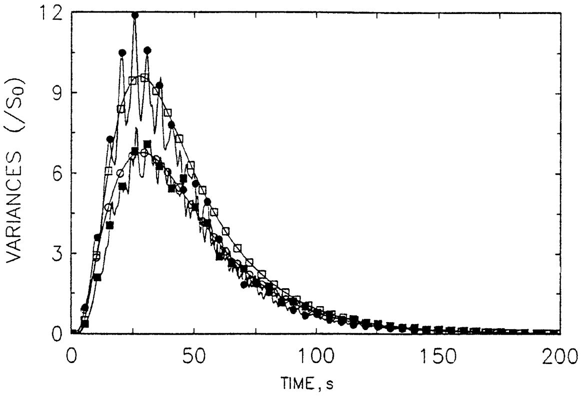

Fig. 12. Response variances of a nonlinear 2dof system with $\omega_f = 1.618$, $\zeta_f = 0.01$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1^2 \rangle$ MCS (●), and $\langle Y_2^2 \rangle$ MCS(■).

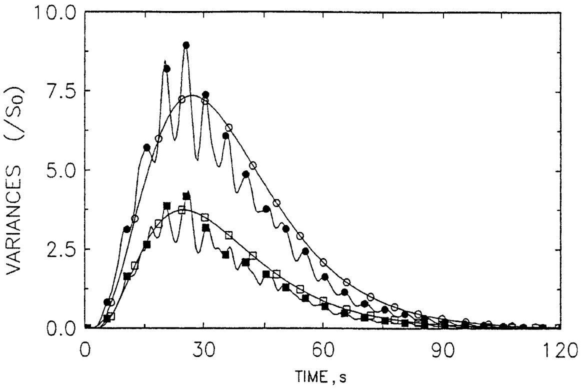

Fig. 13. Response variances of a nonlinear 2dof system with $\omega_f = 1.618$, $\zeta_f = 0.1$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1^2 \rangle$ MCS (●), and $\langle Y_2^2 \rangle$ MCS(■).

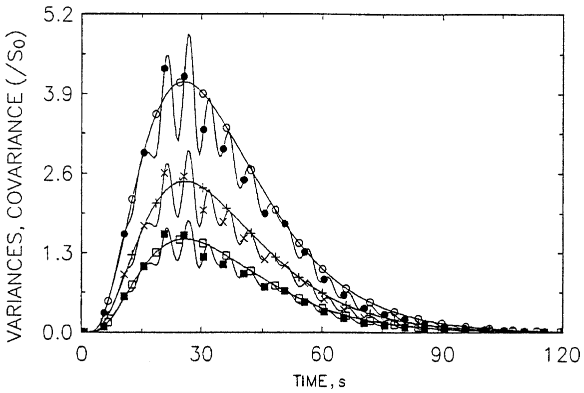

Fig. 14. Response variances and covariance of a nonlinear 2dof system with $\omega_f = 1.618$, $\zeta_f = 1.0$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1 Y_2 \rangle$ SCD (+), $\langle Y_1^2 \rangle$ MCS (●), $\langle Y_2^2 \rangle$ MCS (■), and $\langle Y_1 Y_2 \rangle$ MCS (×).

was used in Ref. [3]. This is because when $S_0 = 0.0012$ was used, the response of the system becomes unstable. In Figs. 12 and 13, for clarity, covariances of responses were not presented because they were very close to the variance $\langle Y_2^2 \rangle$.

From the figures, one observes that the results by the ESCD method have an excellent agreement with those using the MCS. Applying the same SGI machine as in the last section, the computational time employing the ESCD method was approximately 15 s, while that for the MCS was about 57 min. Consequently, one can conclude that the ESCD method is very efficient and accurate.

3.3. Narrow band responses with 2dof filter

This subsection deals with a 2dof system with a 2dof filter. Two cases are presented. The parameters of the 2dof system are identical to those in the last subsection. The parameters of the filters for the two cases are, respectively,

$$M_f = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \quad K_f = \begin{bmatrix} 4.0 & -1.0 \\ -1.0 & 2.0 \end{bmatrix}, \quad C_f = \lambda_k K_f,$$

where λ_k is a constant which is different in every example, and

$$M_f = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \quad K_f = \begin{bmatrix} 6.0 & -2.0 \\ -2.0 & 4.0 \end{bmatrix}, \quad C_f = \lambda_k K_f.$$

Other parameters of filter and system are listed in Table 2. The 2dof narrow band random excitations were the responses from the filter which was perturbed by a single modulated Gaussian white noise. The modulating function was that defined by Eq. (27) while the spectral density of the white noise was chosen as $S_0 = 0.0012$. Once again, the MCS results are employed for comparison to those by the generalized ESCD method. The objectives of the two cases were to examine the generalized ESCD method in terms of the responses of the system under narrow band nonstationary random excitations for the 2dof filter. In the original SCD method for wide band

Table 2
Parameters of system and filter

Filter		System	
Case no.	λ_k	η	ε
Case 1	0.02	−0.001	0.0015
	0.20	−0.001	0.0015
	2.00	−1.0	1.5
Case 2	0.02	−0.001	0.0015
	0.20	−0.001	0.0015
	2.00	−1.0	1.5

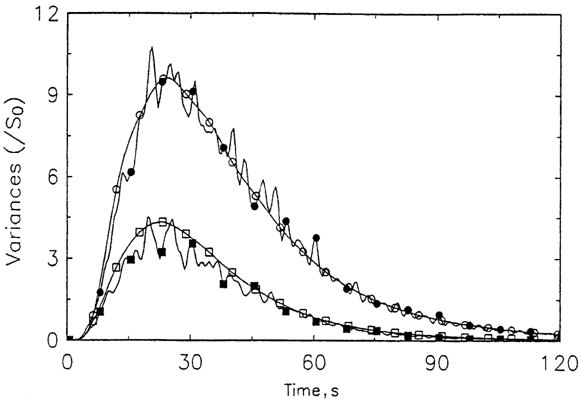

Fig. 15. Response variances of a nonlinear 2dof system for case 1 with $\lambda_k = 0.02$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1^2 \rangle$ MCS (•), and $\langle Y_2^2 \rangle$ MCS(■).

nonstationary random excitations, the terms for correlations of responses and excitations were zero [5,7]. It should be emphasized that in the generalized ESCD method, the correlations of responses and excitations could not be disregarded and therefore were retained in the studies.

Computed results by the generalized ESCD method and MCS are included in Figs. 15–20. Note that for clarity the covariances of responses in Figs. 15 and 18 are not included. From the figures, one can draw similar conclusions to those given in the previous section.

3.4. Effect of bandwidths of excitation processes

The examples in this subsection were included to demonstrate the effect of band widths of random excitations. The system, under a single narrow band nonstationary random excitation whose modulating function is that defined by Eq. (27), was considered. That is, the amplitude of the input to the system was controlled by the modulating function and the constant $(I)^{1/2}$. The

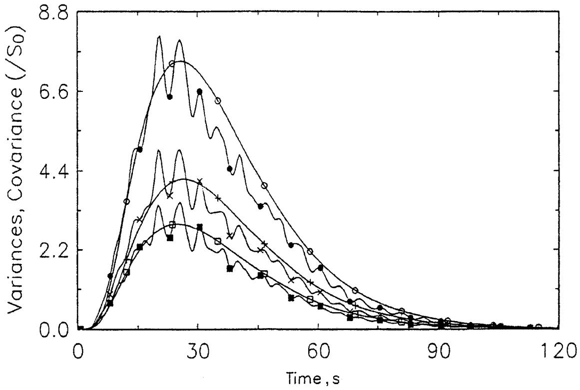

Fig. 16. Response variances and covariance of a nonlinear 2dof system for case 1 with $\lambda_k = 0.2$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1 Y_2 \rangle$ SCD (+), $\langle Y_1^2 \rangle$ MCS (•), $\langle Y_2^2 \rangle$ MCS (■), and $\langle Y_1 Y_2 \rangle$ MCS (×).

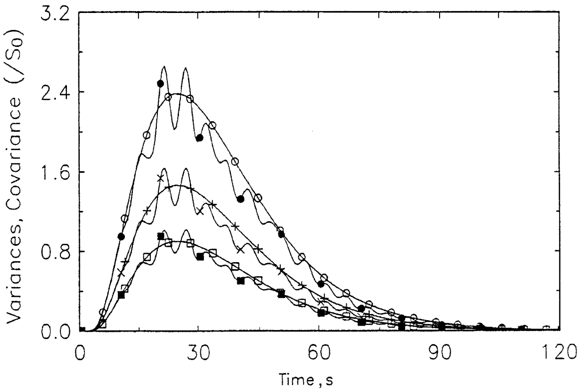

Fig. 17. Response variances and covariance of a nonlinear 2dof system for case 1 with $\lambda_k = 2.0$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1 Y_2 \rangle$ SCD (+), $\langle Y_1^2 \rangle$ MCS (•), $\langle Y_2^2 \rangle$ MCS (■), and $\langle Y_1 Y_2 \rangle$ MCS (×).

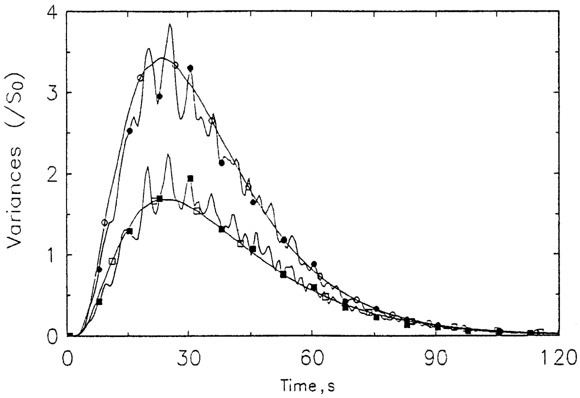

Fig. 18. Response variances of a nonlinear 2dof system for case 1 with $\lambda_k = 0.02$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1^2 \rangle$ MCS (•), and $\langle Y_2^2 \rangle$ MCS(■).

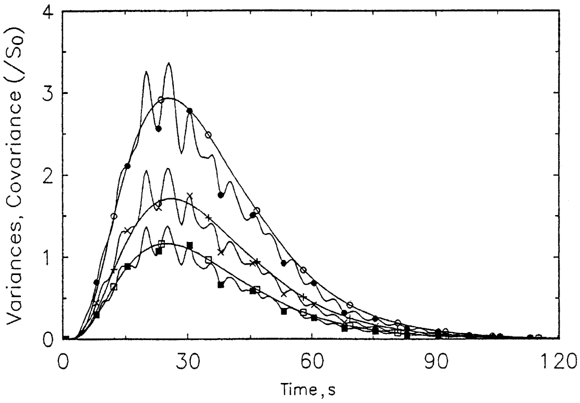

Fig. 19. Response variances and covariance of a nonlinear 2dof system for case 2 with $\lambda_k = 0.2$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1 Y_2 \rangle$ SCD (+), $\langle Y_1^2 \rangle$ MCS (•), $\langle Y_2^2 \rangle$ MCS (■), and $\langle Y_1 Y_2 \rangle$ MCS (×).

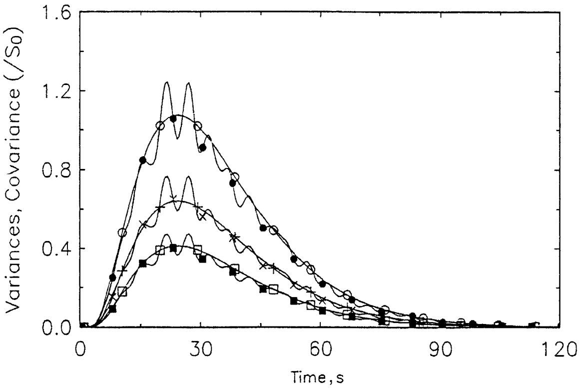

Fig. 20. Response variances and covariance of a nonlinear 2dof system for case 2 with $\lambda_k = 2.0$. $\langle Y_1^2 \rangle$ SCD (○), $\langle Y_2^2 \rangle$ SCD (□), $\langle Y_1 Y_2 \rangle$ SCD (+), $\langle Y_1^2 \rangle$ MCS (•), $\langle Y_2^2 \rangle$ MCS (■), and $\langle Y_1 Y_2 \rangle$ MCS (×).

constant $I = 0.00012 \times 2\pi$. The center frequency and bandwidth of the narrow band excitation process were controlled by the filter. The 2 sdof filters adopted in this subsection included one with dimensionless natural frequency of 1.0 and the other with 1.618. Damping ratios of each of the 2 sdof filters were 0.01, 0.1 and 1.0.

Computed responses of the system subjected to modulated zero mean Gaussian white noise are also presented so as to compare with those from the system under narrow band excitations. The system parameters were $\eta = -1.0, \varepsilon = 1.5$, and other pertinent data have already been provided in Section 2.4. Results for various damping ratios are presented in Figs. 21–23. From the figures, one observes that the amplitudes of system responses increase with decreasing bandwidth of the narrow band nonstationary random excitation. Compared with those of the modulated Gaussian white noise excitations, the responses of narrow band random excitations are much larger.

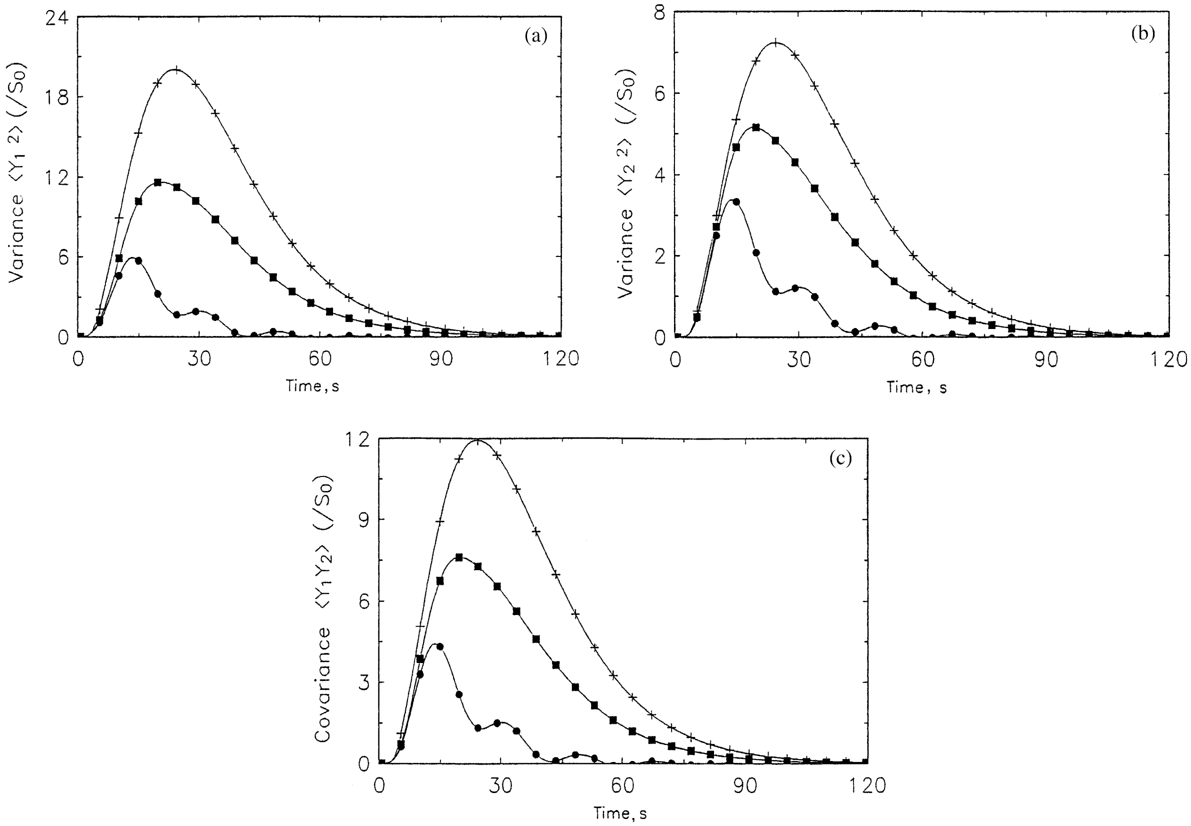

Fig. 21. Effect of bandwidth on response variances of a nonlinear 2dof system with $\omega_f = 1.0$, narrow band $\zeta_f = 0.01$ (+), $\zeta_f = 0.1$ (■), and $\zeta_f = 1.0$ (•): (a) variances of Y_1; (b) variances of Y_2; (c) covariances of Y_1 and Y_2.

4. Concluding remarks

The generalized extended stochastic central difference (ESCD) method was applied to the response analysis of a single degree-of-freedom (dof) nonlinear system and a 2dof nonsymmetric nonlinear system under narrow band nonstationary random excitations. It combines the ESCD method presented in the companion paper [1], the statistical linearization (SL) technique [2,5], modified adaptive time scheme (ATS) [3], and time coordinate transformation (TCT) [4] so that responses of stiff systems with large nonlinearities can be obtained without encountering computational instability. The latter frequently appears in techniques employing the conventional SL [2].

It has been demonstrated that numerical results computed by the generalized ESCD method are in excellent agreement with those by the Monte Carlo simulation (MCS). It is also very efficient compared with the MCS. Typically, the ratio of computational time employing the proposed approach to that of MCS for a 2dof nonlinear system is 1 to 200. The effect of bandwidths of excitation processes was also studied. It is observed that the amplitudes of system responses increase with decreasing bandwidth.

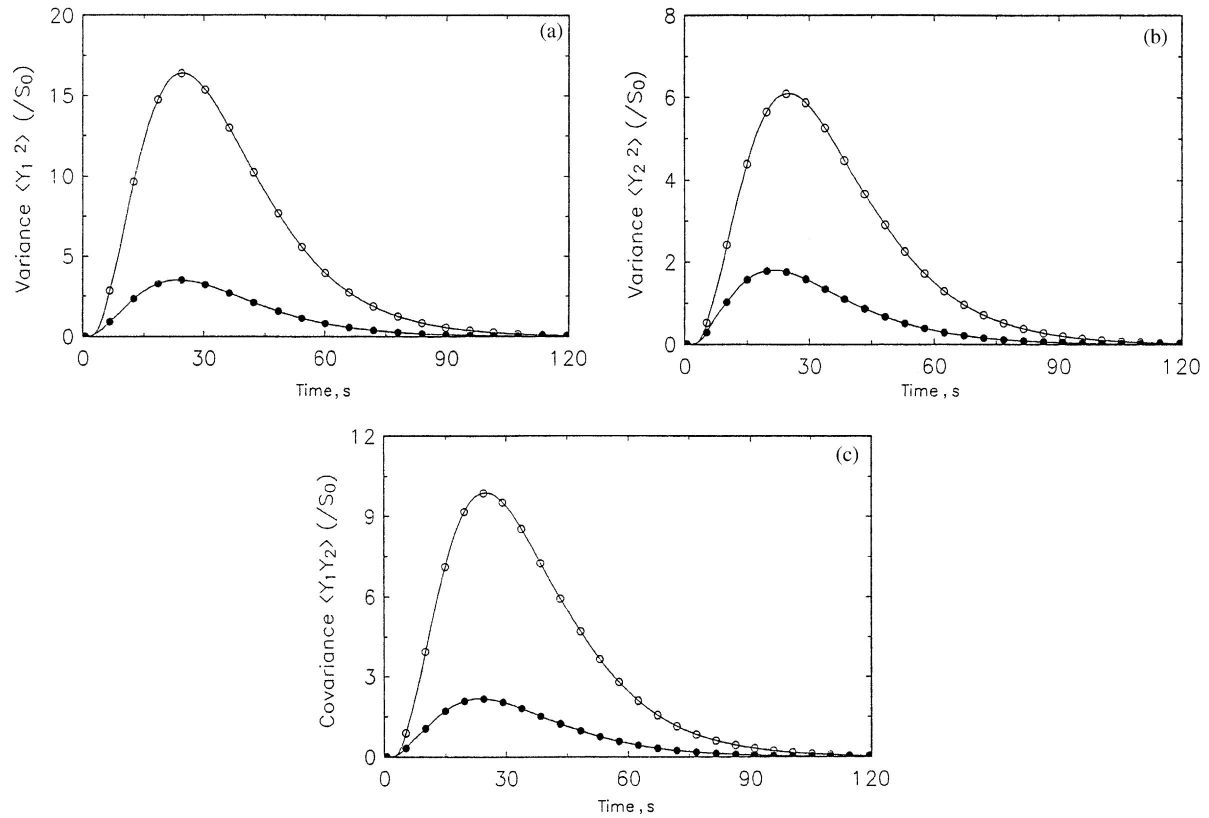

Fig. 22. Effect of bandwidth on response variances of a nonlinear 2dof system with $\omega_f = 1.618$, narrow band $\zeta_f = 0.01$ (○), and $\zeta_f = 0.1$ (●): (a) variances of Y_1; (b) variances of Y_2; (c) covariances of Y_1 and Y_2.

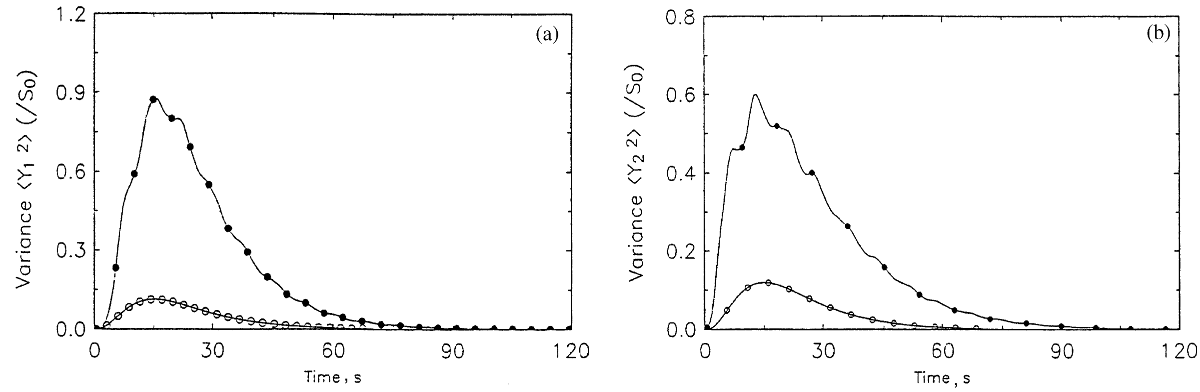

Fig. 23. Effect of bandwidth on response variances of a nonlinear 2dof system with $\omega_f = 1.618$, narrow band $\zeta_f = 1.0$ (●), and wide band (○): (a) variances of Y_1; (b) variances of Y_2.

An interesting observation is that responses of nonlinear systems to narrow band processes have multiple peaks as the natural frequencies of nonlinear systems are changing with the responses of the systems. Furthermore, for the systems studied here it was found that the higher the nonlinear coefficient ε was, the more peaks the variances of responses contained. The peaks of the variances of responses became smaller with higher ε. The results presented also lead one to conclude that for accurate response prediction the correct representation of the bandwidths of the random excitations is strictly necessary.

References

[1] Z. Chen, C.W.S. To, Responses of discretized systems under narrow band nonstationary random excitations. Part 1: linear problems, *Journal of Sound and Vibration* 287 (3) (2005) 433–458, this issue; doi:10.1016/j.jsv.2004.11.020.
[2] J.B. Roberts, P.D. Spanos, *Random Vibration and Statistical Linearization*, Wiley, New York, 1990.
[3] M.L. Liu, C.W.S. To, Adaptive time schemes for responses of non-linear multi-degree-of-freedom systems under random excitations, *Computers and Structures* 52 (3) (1994) 563–571.
[4] C.W.S. To, M.L. Liu, Random responses of discretized beams and plates by the stochastic central difference method with time co-ordinate transformation, *Computers and Structures* 53 (3) (1994) 727–738.
[5] C.W.S. To, Recursive expressions for random response of nonlinear systems, *Computers and Structures* 29 (3) (1988) 451–457.
[6] H.G. Davies, D. Nandlall, Phase plane for narrow band random excitation of a Duffing oscillator, *Journal of Sound and Vibration* 104 (2) (1986) 277–283.
[7] C.W.S. To, M.L. Liu, Recursive expressions for time dependent means and mean square responses of a multi-degree-of-freedom non-linear system, *Computers and Structures* 48 (6) (1993) 993–1000.
[8] K. Kimura, M. Sakata, Nonstationary response analysis of a nonsymmetric nonlinear multi-degree-of-freedom system to nonwhite random excitation, *Japanese Society of Mechanical Engineers, International Journal* 31 (4) (1988) 690–697.