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Short Communication

Design of a polynomially inhomogeneous bar with a tip mass for specified mode shape and natural frequency

I. Elishakoff*, A. Perez

Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0991, USA

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Abstract

In this study the closed form solution is obtained, apparently for the first time, for the free vibration inhomogeneous bar with a tip mass. It is remarkable that while a vibrations study the homogeneous bar with free tip leads to transcendental equation, here, in the case of the inhomogeneous bar, a closed form polynomial solution is obtained. If the mass ratio between the bar's mass and the concentrated mass is specified, numerical evaluation is needed for the axial rigidity coefficients' ratio; on the other hand, for specified mass ratios closed form solution is reported.

1. Introduction

Free longitudinal vibration of bars was studied in a number of papers. Various complicated effects were investigated by Gűrgöze and Ynceoglu [1], Li et al. [2] and Li [3,4]. In these papers exact solution has been derived for bars with uniform or non-uniform cross-section. Effect of the tip mass has been investigated in several textbooks, for the uniform bar. The reader may consult, for example, the text by Rao [5]. It makes sense to recapitulate some basic results from the uniform homogeneous bar. For the rod of uniform cross-section the governing differential equation reads

$$c^2\partial^2 u(x,t)/\partial x^2 = \partial^2 u(x,t)/\partial t^2, \tag{1}$$

*Corresponding author. Tel.: +1 561 297 2729; fax: +1 561 297 2825.
E-mail addresses: elishako@fau.edu (I. Elishakoff), aperezm@fau.edu (A. Perez).

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where $u(x,t)$ is the axial displacement, $c=(E/\rho)^{1/2}$ the speed of longitudinal waves, E the modulus of elasticity, A the cross-sectional area, ρ the mass density. The classical solution obtained by separation of variables is given by

$$u(x,t)=U(x)T(t)=[A\cos(\omega x/c)+B\sin(\omega x/c)](C\cos\omega t+D\sin\omega t). \tag{2}$$

For the bar that is fixed at $x=0$, the boundary condition is

$$u(0,t)=0. \tag{3}$$

For the case of the mass at the tip, the tensile force in the bar equals the inertial force of the mass M, so that

$$AE\,\partial u(L,t)/\partial x=-M\partial^2 u(L,t)/\partial t^2, \tag{4}$$

where L is the length of the rod. This leads to the transcendental equation

$$AE(\omega/c)\cos(\omega L/c)=M\omega^2\sin(\omega L/c). \tag{5}$$

We introduce the non-dimensional variable

$$\lambda=\omega L/c. \tag{6}$$

We get instead of Eq. (5)

$$\lambda\tan\lambda=1/\alpha, \tag{7}$$

where α is the mass ratio

$$\alpha=c^2M/AEL=M/\rho AL=M/m, \tag{8}$$

and where $m=\rho AL$ is the mass of the bar. Thus, the concentrated mass is expressed as fraction of the beam mass. Rao [5] lists the natural frequencies of the bar for the mass ratio α taking the values 0.01, 0.1, 1.0, 10.0 and 100.0.

Exact solution for the bars of varying cross-section has been reported by Graf [6]; namely, cross-sectional areas with linear, conical, exponential or catenoidal variation were studied. The solution was written in terms of Bessel functions. Abrate [7] obtained several elegant solutions for non-uniform rods. Eisenberger [8] devised as numerical scheme based on series solution, yielding arbitrarily small deviations from the exact solution. Candan and Elishakoff [9] furnished closed-form solutions for the inhomogeneous bars without a concentrated mass. They postulated the polynomial mode shape along with the polynomial flexural rigidity of the bar.

In this study the work by Candan and Elishakoff [9] is being generalized for the bar with tip as mass.

2. Basic equations

Consider free vibrations of an inhomogeneous bar with variable modulus of elasticity $E=E(x)$.

The governing differential equation reads

$$\frac{\partial}{\partial x}\left[D(x)\frac{\partial u}{\partial x}\right]=\rho(x)A(x)\frac{\partial^2 u}{\partial t^2}. \tag{9}$$

The axial rigidity $D(x) = E(x)A(x)$ is variable due to the axial variability of the elastic modulus. Cross-sectional areas A and material density ρ are considered as constants. For the free vibration we set

$$u(x) = U(x)\sin\omega t. \tag{10}$$

We considered a bar which is clamped at $x = 0$, so the boundary conditions given in Eq. (3) is applicable at $x = L$, where the mass M is attached, so that the boundary condition (4) holds.

We introduce the non-dimensional axial coordinate

$$\xi = x/L. \tag{11}$$

In view of Eqs. (10) and (11) the governing differential equation (9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[D(\xi)\frac{\mathrm{d}U}{\mathrm{d}\xi}\right] + \rho A L^2\omega^2 U = 0. \tag{12}$$

Following Ref. [10], we express the mode shape as a parabola:

$$U(\xi) = a_0 + a_1\xi + a_2\xi^2. \tag{13}$$

Satisfaction of boundary condition in Eq. (3) yields $a_0 = 0$. Satisfaction of boundary conditions (4) results in the mode shape

$$U(\xi) = -\frac{2(b_0 + b_1 + b_2) - ML\omega^2}{b_0 + b_1 + b_2 - ML\omega^2}\,\xi + \xi^2. \tag{14}$$

Note that for the bar without concentrated mass at the tip $M = 0$, we get $U(\xi) = -2\xi + \xi^2$ which is proportional to the mode shape obtained for this case in Ref. [10]. By setting and arbitrary coefficient a_2 to unity and bearing in mind that $A(\xi)E(\xi) = D(\xi)$, the axial rigidity is sought as a parabolic function

$$D(\xi) = b_0 + b_1\xi + b_2\xi^2, \tag{15}$$

where the coefficients b_0, b_1 and b_2 should be determined so that it constitutes a physically realizable quantity, i.e. is positive throughout the bar's axis for $\xi \in [0, 1]$.

Substitution of Eqs. (14) and (15) into the governing differential equation (12) leads to

$$c_0 + c_1\xi + c_2\xi^2 = 0, \tag{16}$$

where

$$c_0 = 2b_0 - (2b_0b_1 + 2b_1^2 + 2b_1b_2 + b_1ML\omega^2)/F_0 = 0, \tag{17}$$

$$\begin{aligned} c_1 = 4b_1 - (&4b_0b_2 + 4b_2^2 - 2b_2ML\omega^2 + 2\rho\omega^2L^2b_2 + 4b_1b_2 \\ &+ 2\rho\omega^2L^2b_0 + 2\rho\omega^2L^2b_1 - \rho\omega^4L^3M)/F_0, \end{aligned} \tag{18}$$

$$c_2 = 6b_2 + \rho\omega^2L^2, \tag{19}$$

where

$$F_0 = b_0 + b_1 + b_2 = ML\omega^2. \tag{20}$$

From Eq. (19) we conclude that

$$\omega^2 = -6b_2/\rho L^2. \tag{21}$$

In order for this natural frequency squared to correspond to the realistic problem, it is necessary for b_2 to be a negative quantity. With Eq. (20) in mind, Eq. (17) becomes

$$4b_1 + (8b_0b_2 + 8b_2^2 + 24\alpha b_2^2 + 8b_1b_2)/F_1 = 0, \tag{22}$$

where

$$F_1 = b_0 + b_1 + b_2 + 6\alpha b_2. \tag{23}$$

Eq. (22) is further simplified to

$$b_0b_1 + b_1^2 + 3b_1b_2 + 6\alpha b_1b_2 + 2b_0b_2 + 2b_2^2 + 6\alpha b_2^2 = 0. \tag{24}$$

We express b_0 from this equation as

$$b_0 = -[b_1^2 + 3b_1b_2(1 + 2\alpha) + 2b_2^2(1 + 3\alpha)]/(b_1 + 2b_2). \tag{25}$$

Eq. (17), namely $c_0 = 0$, becomes, after some algebra,

$$-b_1^3 + 6b_1b_2^2 + 12\alpha b_1b_2^2 + 4b_2^3 + 12\alpha b_2^3 = 0. \tag{26}$$

At this stage it is instructive to note that we have 4 unknowns, namely b_0, b_1, b_2 and ω^2, whereas we are in possession of three equations, $c_0 = 0, c_1 = 0, c_2 = 0$.

Thus, we have an infinite amount of solutions. If a unique solution is desired, one has to impose an additional constraint. Here we specify the value of the natural frequency: we demand that the fundamental natural frequency ω equals a pre-selected value ω_0. This leads to the expression for the coefficient b_2 in view of Eq. (21)

$$b_2 = -\rho L\omega^2/6. \tag{27}$$

The coefficient b_1 can be expressed in terms of b_2:

$$b_1 = gb_2. \tag{28}$$

Eq. (26) takes the form of a depressed cubic equation

$$-g^3 + 6g(1 + 2\alpha) + 4(1 + 3\alpha) = 0. \tag{29}$$

This equation can be written in the canonical form

$$g^3 + 3pg + 2q = 0, \tag{30}$$

where

$$p = -2(1 + 2\alpha), \quad q = -2(1 + 3\alpha). \tag{31}$$

According to Cardano's formula we form a discriminant

$$\Delta = q^2 + p^3, \tag{32}$$

which takes the following form, after substitution of Eq. (31)

$$\Delta = -4 - 24\alpha - 60\alpha^2 - 644\alpha^3. \tag{33}$$

The discriminant Δ turns to be negative for any value of the mass ratio α. Therefore, Eq. (30) has three real roots which are represented in the form

$$g_1 = u + v, \quad g_2 = \varepsilon_1 u + \varepsilon_2 v, \quad g_3 = \varepsilon_2 u + \varepsilon_1 v, \tag{34}$$

where

$$u = \sqrt[3]{-q + \sqrt{\Delta}}, \quad v = \sqrt[3]{-q - \sqrt{\Delta}}, \quad \varepsilon_1 = 0.5(-1 + \sqrt{3}\mathrm{i}), \quad \varepsilon_2 = -0.5(-1 + \sqrt{3}\mathrm{i}). \tag{35}$$

As it is seen, the three real roots are written in terms of complex numbers, because of the negativity of the discriminant. Therefore, it is preferable to use an alternative route of solution. We fix the value of α and evaluate the roots directly.

Once Eq. (29) is solved, for a specified mass ratio α, Eq. (25) gives the value of the coefficient b_0. For the value $\alpha = 1$, the Eq. (29) has three real roots:

$$g_1 = -3.69770, \quad g_2 = -0.93418, \quad g_3 = 4.63188. \tag{36}$$

For $\alpha = 2$, the ratios g_j are

$$g_1 = -4.93179, \quad g_2 = -0.96311, \quad g_3 = 5.89490, \tag{37}$$

while for $\alpha = 3$, the values of g_j become

$$g_1 = -5.93836, \quad g_2 = -0.97441, \quad g_3 = 6.91277. \tag{38}$$

Thus, the axial rigidity reads

$$D_j(\mathbf{x}) = \frac{b_2(-g_j^2 - 21g_j - 20 + g_j^2\mathbf{x} + 2g_j\mathbf{x} + \mathbf{x}^2 g_j + 2\mathbf{x}^2)}{g_j + 2} \quad (j = 1, 2, 3). \tag{39}$$

The above discussion shows that for each mass ratio α there exist three bars that represent the solutions of the problem. For $\alpha = 1$, the flexural rigidities are

$$D_1(\xi) = (-25.90487924 - 3.697700100\xi + \xi)^2 b_2, \tag{40}$$

$$D_2(\xi) = (-1.177402050 - 0.9341807585\xi + \xi^2) b_2, \tag{41}$$

$$D_2(\xi) = (-20.91771871 + 4.631880858\xi + \xi^2) b_2. \tag{42}$$

These are shown in Fig. 1. For $\alpha = 2$ the triplet of the axial rigidity is given by

$$D_1(\xi) = (-20.20780547 - 4.931789495\xi + b_2\xi^2) b_2, \tag{43}$$

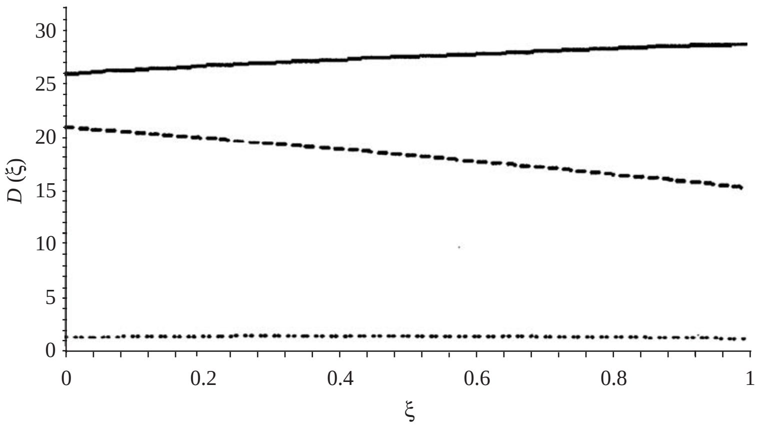

Fig. 1. Variation of flexural rigidity $D(\xi)$ vs. non-dimensional axial coordinate ξ for $\alpha = 1$ ($D_1(\xi)$, ——; $D_2(\xi)$, – – –; $D_3(\xi)$, ...).

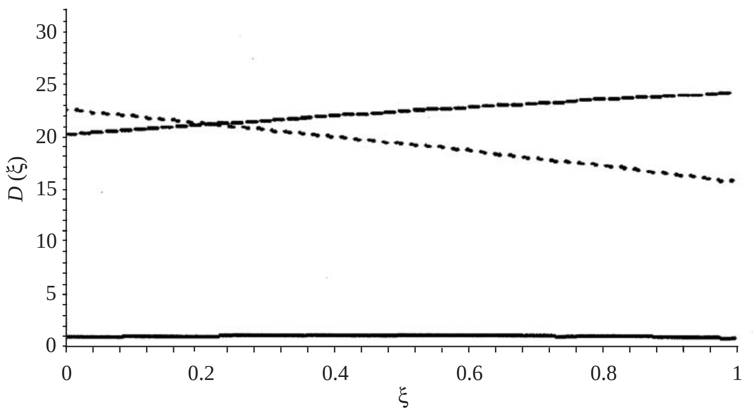

Fig. 2. Variation of flexural rigidity $D(\xi)$ vs. non-dimensional axial coordinate ξ for $\alpha = 2$ ($D_1(\xi)$, ——; $D_2(\xi)$, – – –; $D_3(\xi)$, ...).

$$D_2(\xi) = (-0.6772451108 - 0.9631122932\xi + \xi^2)b_2, \tag{44}$$

$$D_3(\xi) = (-22.61494942 + 5.894901788\xi + \xi^2)b_2. \tag{45}$$

They are portrayed in Fig. 2. For $\alpha = 3$, we get

$$D_1(\xi) = (-17.63206750 - 5.938361307\xi + \xi^2)b_2, \tag{46}$$

$$D_2(\xi) = (-0.4747363744 - 0.9744089218\xi + \xi^2)b_2, \tag{47}$$

$$D_3(\xi) = (-23.89319611 + 6.912770229\xi + \xi^2)b_2. \tag{48}$$

They are depicted in Fig. 3.

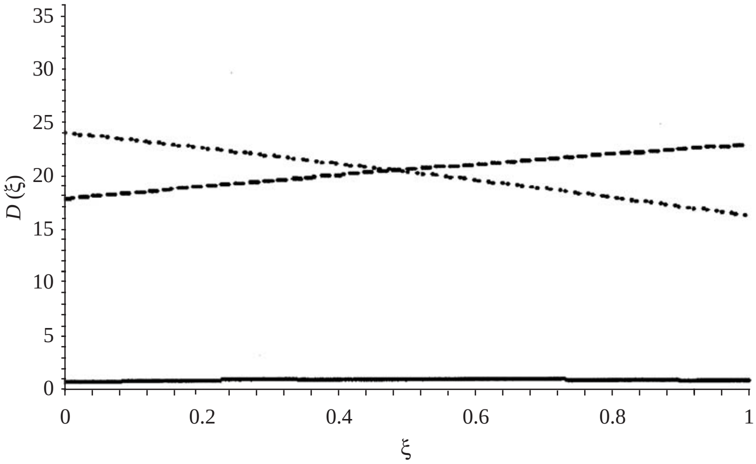

Fig. 3. Variation of flexural rigidity $D(\xi)$ vs. non-dimensional axial coordinate ξ for the mass ratio $\alpha = 3$ ($D_1(\xi)$, ——; $D_2(\xi)$, – – –; $D_3(\xi)$, ...).

3. Closed-form solutions

For specified values of the mass ratio α one can obtain closed-form solutions; indeed, from Eq. (29) we express α as a function of g:

$$\alpha = (g^3 - 6g - 4)/12(g + 1). \tag{49}$$

Therefore, if we pre-select $g = g^*$, we get the appropriate value of α from the latter equation yielding the closed-form solution. For example, for $g^* = -3$, we get $\alpha = 13/24$; for $g^* = -4$, $\alpha = 11/9$; for $g^* = -5$, $\alpha = 33/16$; for $g^* = -6$, $\alpha = 46/15$; for $g^* = 4$, $\alpha = 3/5$; for $g^* = 5$, $\alpha = 91/72$; for $g^* = 6$, $\alpha = 44/21$; for $g = 7$, $\alpha = 99/32$. The cases of $g^* = -3$, $g^* = -4$, $g^* = -5$ are shown in Fig. 4, as an example, $D(\xi)$ equals in these cases:

$$D(X) = 34 + 3X - X^2 \quad \text{for } g^* = -3, \tag{50}$$

$$D(X) = 24 + 4X - X^2 \quad \text{for } g^* = -4, \tag{51}$$

$$D(X) = 20 + 5X - X^2 \quad \text{for } g^* = -5. \tag{52}$$

The cases of $g^* = 3$, $g^* = 4$, $g^* = 5$, are shown in Fig. 5, as an example, $D(\xi)$ equals in these cases:

$$D(\xi) = 92/5 - 3\xi - \xi^2 \quad \text{for } g^* = 3, \tag{53}$$

$$D(\xi) = 20 - 4\xi - \xi^2 \quad \text{for } g^* = 4, \tag{54}$$

$$D(\xi) = 150/7 - 5\xi - \xi^2 \quad \text{for } g^* = 5. \tag{55}$$

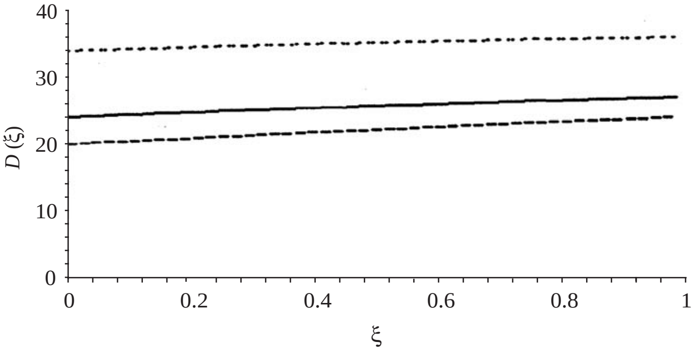

Fig. 4. Variation of flexural rigidity $D(\xi)$ vs. non-dimensional axial coordinate ξ for the mass ratio: ——, $g^* = -3$; – – –, $g^* = -4$; ……, $g^* = -5$.

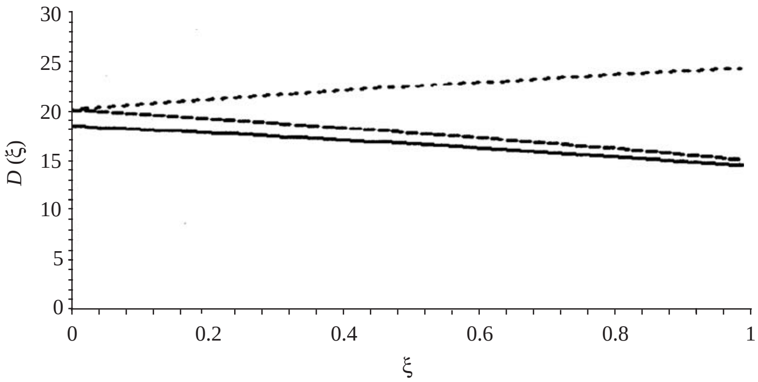

Fig. 5. Variation of flexural rigidity $D(\xi)$ vs. non-dimensional axial coordinate ξ for the mass ratio: ——, $g^* = 3$; – – –, $g^* = -4$; ……, $g^* = 5$.

4. Conclusion

To the best knowledge of the present authors no closed-form solution has been reported for inhomogeneous bars with tip mass, prior to this investigation. The solution appears to be stunningly simple. It appears that the material can be included even in the curriculum of an upper-level undergraduate or graduate vibration course.

A remarkable conclusion is reached: for any specified mass ratio and pre-set coefficient $b_2 < 0$, *three* beams are found that solve three posed problem. For the investigated values of α, the flexural rigidities turn out to be physically realizable, since they are positive in the range of interest $0 \leqslant \xi \leqslant 1$.

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