Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Sound and Vibration 287 (2005) 1027–1030
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Relationship between softening and stiffening effects in terms of Southwell coefficients

A. Bazoune*

Department of Mechanical Engineering, King Fahd University of Petroleum & Minerals, KFUPM Box 579, Dhahran 31261, Saudi Arabia

Received 5 April 2004; received in revised form 7 February 2005; accepted 28 February 2005
Available online 15 April 2005

Abstract

The Southwell coefficient (SC) represents an efficient way of presenting the relationship between rotating and non-rotating frequencies for rotating structures. For a given setting angle, a relationship between the in-plane frequencies and out-of-plane frequencies exists. In this paper, a proof of the relationship that exists between the in-plane and out-of-plane frequencies is given in terms of SC. With knowledge of the speed of rotation and the out-of-plane or in-plane frequencies along with this relationship in hand, the in-plane or out-of-plane frequency is easily investigated without recourse to further extensive calculations. Therefore, the Southwell relation provides a suitable tool for natural frequency estimates of rotating beams at a preliminary design stage.

1. Stiffening effect due to rotation

The stiffening effect of rotation was first acknowledged by Lamb and Southwell [1]. In their attempt to predict the fundamental frequency for rotating components, Lamb and Southwell [1] introduced the effect of stiffening caused by the spin of a disc through a general approximate relation of a lower bound form

$$\omega_{Ri}^2 \geqslant \omega_{Ni}^2 + \Omega^2, \tag{1}$$

*Tel.: +966 3 860 1330; fax: +966 3 860 2949.
E-mail address: abazoune@kfupm.edu.sa.

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.02.014

where Ω represents the speed of rotation and ω_{Ni} and ω_{Ri} are the natural frequencies of the ith mode of the non-rotating and rotating disc, respectively. In a subsequent paper, Southwell and Gough [2] extended the previous work to include many blade shapes such as the uniform cantilever and the knife-edged wedge. They used Rayleigh's quotient in order to estimate the fundamental natural frequency and established an upper bound relation of the form

$$\omega_{Ri}^2 \leqslant \omega_{Ni}^2 + S_i \Omega^2, \tag{2}$$

where S_i is a constant that corresponds to the ith mode. This is referred to as the Southwell's coefficient (SC) and depends on the form of the blade. Thus, the value of ω_{Ri} may be determined within upper and lower limits of error.

2. Softening effect due to the inclusion of a setting angle

The inclusion of a setting angle, ψ, in beam vibration problems originated in the work of Lo and Renbarger [3]. The setting angle in the helicopter area is analogous to the pitch angle of the blade, which must be continuously adjusted over the rotor disk in order to control the thrust and force distribution. Lo and Renbarger [3] derived a nonlinear differential equation for bending vibration of a rotating uniform Euler–Bernoulli beam clamped to a rigid hub, with the mid-plane of the beam cross-section making an angle ψ with the plane of rotation. For $\psi = 0^\circ$ the motion is described as out of the plane of rotation, and for $\psi = 90^\circ$ the motion is in the plane of rotation. Lo and Renbarger [3] showed that the frequencies at any arbitrary setting angle ψ are related to the frequencies at $\psi = 0^\circ$, by the relation

$$\omega_{Ri,\psi}^2 = \omega_{Ri,0}^2 - \Omega^2 \sin^2 \psi, \tag{3}$$

where $\omega_{Ri,0}^2$ and $\omega_{Ri,\psi}^2$ represent, respectively, the frequencies of the ith mode at zero and at an arbitrary setting angle. In the above, the term $\Omega^2 \sin^2 \psi$ causes a softening effect that opposes the stiffening one shown in Southwell's equation. For $\psi = 0^\circ$, this term vanishes and for $\psi = 90^\circ$ it is maximum and dominates, in certain cases, the stiffening effect. With knowledge of the speed of rotation and the out-of-plane or in-plane frequencies along with this relationship in hand, the in-plane or out-of-plane frequency is easily investigated without recourse to further extensive calculations.

3. In-plane and out-of-plane frequencies in terms of Southwell coefficients

In connection with the above Southwell relation, it can be easily shown that if the relationship given by Lo and Renbarger [3] is substituted into the Southwell relation, the resulting out-of-plane and in-plane Southwell coefficients differ by unity, regardless of the mode considered. This is true provided that the beam is slender and has constant stiffness and inertia properties. This characteristic was shown to be the case for rotating tapered Timoshenko beams as reported by Bazoune et al. [4].

Neglecting the Coriolis effect, the free vibrational motion of a spinning rotating Euler–Bernoulli beam that is making a setting angle ψ with the plane of rotation can be given by the following differential equation, as in Ref. [4]:

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{K} - \Omega^2 \mathbf{M} \sin^2 \psi)\mathbf{x} = 0, \tag{4}$$

where **M** and **K** are the mass and stiffness matrices, respectively. The out-of-plane (flapping) equation ($\psi = 0^\circ$) of the free vibrational motion can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0 \tag{5}$$

while the in-plane (lead-lag) ($\psi = 90^\circ$) equation of the free vibrational motion can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{K} - \Omega^2 \mathbf{M})\mathbf{x} = 0. \tag{6}$$

Assuming a solution of the form:

$$\mathbf{x} = \bar{\mathbf{x}} e^{j\omega t}, \tag{7}$$

where $\bar{\mathbf{x}}$ is the vector of displacement amplitudes and ω is the frequency of harmonic vibrations and $j = \sqrt{-1}$. Substitution of Eq. (7) into Eqs. (5) and (6) yields, respectively,

$$(\mathbf{K} - \omega_{Ri,0}^2 \mathbf{M})\,\bar{\mathbf{x}} = 0 \tag{8}$$

and

$$[\mathbf{K} - (\omega_{Ri,90}^2 + \Omega^2)\mathbf{M}]\,\bar{\mathbf{x}} = 0. \tag{9}$$

Since the matrices **K** and **M** are identical in Eqs. (8) and (9), the eigenvalues are also identical, i.e.,

$$\omega_{Ri,0}^2 = (\omega_{Ri,90}^2 + \Omega^2). \tag{10}$$

Notice that Eq. (10) can be obtained from Eq. (3) by substituting $\psi = 90^\circ$. The Southwell relationship can be written for both lead-lag and flapping motion as follows:

$$\omega_{Ri,90}^2 \leqslant \omega_{Ni,90}^2 + S_{i,90}\Omega^2 \tag{11}$$

and

$$\omega_{Ri,0}^2 \leqslant \omega_{Ni,0}^2 + S_{i,0}\Omega^2. \tag{12}$$

For a stationary beam $\omega_{Ni,0}^2 = \omega_{Ni,90}^2$, and by subtracting Eq. (11) from Eq. (12) one can get

$$\omega_{Ri,0}^2 - \omega_{Ri,90}^2 \leqslant (S_{i,0} - S_{i,90})\Omega^2. \tag{13}$$

But from Eq. (10) one has

$$\omega_{Ri,0}^2 - \omega_{Ri,90}^2 = \Omega^2. \tag{14}$$

Substitution of Eq. (14) into Eq. (13) yields

$$\Omega^2 \leqslant (S_{i,0} - S_{i,90})\Omega^2 \tag{15}$$

or

$$S_{i,0} - S_{i,90} \geqslant 1. \tag{16}$$

Thus the difference between the SC associated with the in-plane and that associated with the out-of-plane frequencies for slender beams is always greater than or equal to unity regardless of the mode considered.

Acknowledgment

The support provided by King Fahd University of Petroleum and Minerals is deeply appreciated.

References

[1] H. Lamb, R.V. Southwell, The vibrations of a spinning disc, *Proceedings of the Royal Society* 99 (1921) 272–280.
[2] R.V. Southwell, B.S. Gough, On the free transverse vibrations of airscrew blades, British A.R.C. Reports and Memoranda No. 766 (Ae. 26), 1921, pp. 358–368.
[3] H. Lo, J.L. Renbarger, Bending vibrations of a rotating beam, *First US National Congress of Applied Mechanics*, Chicago, IL, June 1951, pp. 75–79.
[4] A. Bazoune, Y.A. Khulief, N.G. Stephen, Further results on modal characteristics of rotating tapered Timoshenko beams, *Journal of Sound and Vibration* 219 (1999) 155–172.