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Short Communication

A generalized iteration procedure for calculating approximations to periodic solutions of "truly nonlinear oscillators"

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Abstract

An extended iteration method for calculating the periodic solutions of nonlinear oscillator equations is given. The procedure is illustrated by applying it to two "truly nonlinear oscillator" differential equations.

Consider a nonlinear oscillator modeled by the following differential equation:

$$\ddot{x} + g(x) = \varepsilon f(x, \dot{x}), \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

where ε is a positive parameter, not necessarily small, and the functions $g(x)$ and $f(x, \dot{x})$ have the properties:

$$g(-x) = -g(x), \tag{2a}$$

$$f(-x, -\dot{x}) = -f(x, \dot{x}). \tag{2b}$$

If $g(x)$ does not have for small x a dominant term proportional to x, then Eq. (1) is said to be a "truly nonlinear oscillator" (TNO). Two examples of such equations are

$$\ddot{x} + x^{1/3} = 0, \tag{3}$$

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$$\ddot{x} + x^3 = \varepsilon(1 - x^2)\dot{x}. \tag{4}$$

It is clear that none of the standard methods for constructing perturbation solutions to these equations (for the case where $0 < \varepsilon \ll 1$) apply since these procedures assume that when $\varepsilon = 0$ the resulting differential equation is that for the harmonic oscillator [1,2], i.e.,

$$\varepsilon = 0 : \ddot{x} + \omega^2 x = 0, \quad \omega = \text{constant}. \tag{5}$$

The following question now arises: Can a method be constructed for TNO equations that will allow the calculation of valid analytical approximations to their periodic solutions? A number of researchers have studied this problem and other related issues; a partial listing of such work includes the papers given in Refs. [1–10].

The purpose of this communication is to propose an iteration method that can be used to determine analytical approximations to the periodic solutions of TNO differential equations. This method, in principle, does not require the parameter ε to be small. The basis of the method is a result formulated by Mickens [3], which was then generalized by Lim and Wu [10]. It should be pointed out that while the so-called quasi-linearization method [11] shares some similar features with the proposed iteration scheme, the two procedures differ in their calculational philosophy and in the accuracy of the approximations to the periodic solutions for oscillatory systems; the paper of Krivec et al. can be consulted for references to the quasi-linearization method and how it has been applied.

To begin, let Eq. (1) be rewritten as

$$\ddot{x} + \Omega^2 x = G(x, \dot{x}), \tag{6}$$

where Ω^2 is not defined at this point and $G(x, \dot{x})$ is

$$G(x, \dot{x}) \equiv \Omega^2 x - g(x) + \varepsilon f(x, \dot{x}). \tag{7}$$

The proposed iteration scheme is

$$\begin{aligned} \ddot{x}_{k+1} + \Omega^2 x_{k+1} = {} & G(x_{k-1}, \dot{x}_{k-1}) + G_x(x_{k-1}, \dot{x}_{k-1})(x_k - x_{k-1}) \\ & + G_{\dot{x}}(x_{k-1}, \dot{x}_{k-1})(\dot{x}_k - \dot{x}_{k-1}), \end{aligned} \tag{8}$$

where

$$G_x(x, \dot{x}) = \frac{\partial G}{\partial x}, \quad G_{\dot{x}}(x, \dot{x}) = \frac{\partial G}{\partial \dot{x}}, \tag{9}$$

and k takes on the integer values $(0, 1, 2, \ldots)$. To start the iteration, $x_{-1}(t)$ and $x_0(t)$ need to be specified; they are taken to be

$$x_{-1}(t) = x_0(t) = A \cos(\Omega t). \tag{10}$$

Note, first, that for a given value of k, the solution must satisfy the initial conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0, \tag{11}$$

and, second, the differential equation to solve for $x_{k+1}(t)$ is second order, linear, inhomogeneous, where the inhomogeneous term is explicitly known in terms of previously calculated approximations.

The values for Ω^2 and for A, in the case of limit-cycle solutions, are determined by expanding G in a Fourier series and setting the coefficients of the $\cos(\Omega t)$ and $\sin(\Omega t)$ terms equal to zero. This requirement assures that the solution, $x_{k+1}(t)$, will not contain secular terms [12]. With these terms eliminated from the inhomogeneous differential equation, the particular solution, $x_{k+1}^{(P)}(t)$, can then be calculated. However, it should be stressed that the complete solution for $x_{k+1}(t)$ includes the homogeneous solution

$$x_{k+1}^{(H)}(t) = C_1 \cos(\Omega t) + C_2 \sin(\Omega t), \tag{12}$$

where C_1 and C_2 are arbitrary constants. Thus,

$$\begin{aligned} x_{k+1}(t) &= x_{k+1}^{(H)}(t) + x_{k+1}^{(P)}(t) \\ &= C_1 \cos(\Omega t) + C_2 \sin(\Omega t) + x_{k+1}^{(P)}(t). \end{aligned} \tag{13}$$

The constants C_1 and C_2 are determined from the initial conditions given in Eq. (11). This procedure can be extended to as large a value of k as needed. In practice, $k = 1$ or $k = 2$ will suffice to display all the essential features of the periodic solutions. However, the existence of computer algebra software allows the possibility for extending the iteration to large values of k.

It should be stated that in calculating $x_{k+1}(t)$ from the formula given by Eq. (8), both $x_{k-1}(t)$ and $x_k(t)$ are needed. These functions are known from the previous calculations. However, when they are substituted into the right-hand side of Eq. (8), the forms to be used should contain the amplitude, A, and angular frequency, Ω, unevaluated at the particular numerical values previously determined by the calculation. For example, the differential equation for $x_1(t)$ is dependent on knowing both $x_{-1}(t)$ and $x_0(t)$. At this stage of the calculation, they are taken (by assumption) to be $A \cos(\Omega t)$. The elimination of secular terms will either determine Ω as a function of A, for conservative systems, or A and Ω, separately, as will be the situation for a limit-cycle problem. The explicit solution for $x_1(t)$ can always be written in such a manner that it depends on A and Ω in unevaluated form. It is this structure that is to be used for the calculation of $x_2(t)$. The two worked examples will make explicit what this particular feature means. The reason for doing the calculation this way is because Ω is recalculated anew at each stage of the iteration by the requirement that no secular terms exist.

The details as to how to carry out this iteration scheme are illustrated in the following two examples. The first is the nonlinear conservative oscillator [1]

$$\ddot{x} + x^3 = 0. \tag{14}$$

For this case

$$g(x) = x^3, \quad f(x, \dot{x}) = 0, \tag{15}$$

and, consequently,

$$G(x, \dot{x}) = \Omega^2 x - x^3. \tag{16}$$

Since Eq. (14) has the first-integral

$$\frac{y^2}{2} + \frac{x^4}{4} = \text{constant}, \tag{17}$$

it can be concluded that all the solutions are periodic [12]. The corresponding iteration scheme is

$$\ddot{x}_{k+1} + \Omega^2 x_{k+1} = \Omega^2 x_{k-1} - (x_{k-1})^3 + (\Omega^2 - 3x_{k-1}^2)(x_k - x_{k-1}). \tag{18}$$

Using the results in Eq. (10), $x_1(t)$ satisfies the differential equation

$$\ddot{x}_1 + \Omega^2 x_1 = A\left[\Omega^2 - \frac{3A^2}{4}\right]\cos(\Omega t) - \left(\frac{A^3}{4}\right)\cos(3\Omega t). \tag{19}$$

The absence of a secular term requires that

$$A\left[\Omega^2 - \frac{3A^2}{4}\right] = 0, \tag{20}$$

or, for the nontrivial solution, i.e., $A \neq 0$,

$$\Omega \equiv \Omega(A) = \sqrt{\frac{3}{4}}A. \tag{21}$$

The particular solution for the resulting Eq. (19) is

$$x_1^{(P)}(t) = D\cos(3\Omega t), \tag{22}$$

where

$$D = \frac{A^3}{32\Omega^2} = \frac{A}{24}. \tag{23}$$

Thus, the complete solution for $x_1(t)$ is

$$x_1(t) = C_1\cos(\Omega t) + \left(\frac{A}{24}\right)\cos(3\Omega t). \tag{24}$$

Imposition of the initial conditions from Eq. (11) gives

$$x_1(0) = A = C + \frac{A}{24} \quad \text{or} \quad C = \left(\frac{23}{24}\right)A. \tag{25}$$

Therefore, $x_1(t)$ is

$$x_1(t) = \left(\frac{A}{24}\right)[23\cos(\Omega t) + \cos(3\Omega t)], \quad \Omega = \sqrt{\frac{3}{4}}A. \tag{26}$$

Continuing to $k = 1$ gives

$$\ddot{x}_2 + \Omega^2 x_2 = \Omega^2 x_0 - (x_0)^3 + (\Omega^2 - 3x_0^2)(x_1 - x_0). \tag{27}$$

Substituting into the right-hand side the results from Eqs. (10) and (26), and simplifying the resulting expression gives

$$\ddot{x}_2 + \Omega^2 x_2 = \left(\frac{23A}{24}\right)\left[\Omega^2 - \left(\frac{3\cdot 22}{4\cdot 23}\right)A^2\right]\cos(\Omega t) + \left(\frac{A}{24}\right)\left[\Omega^2 - \frac{27A^2}{4}\right]\cos(3\Omega t) - \left(\frac{A^3}{32}\right)\cos(5\Omega t). \tag{28}$$

Secular terms are eliminated by setting the coefficient of $\cos(\Omega t)$ equal to zero; doing this gives

$$\Omega(A) = \left(\frac{22}{23}\right)^{1/2} \sqrt{\frac{3}{4}} A. \tag{29}$$

It is straightforward to calculate the particular solution and, from it, the complete solution with the initial conditions, $x_2(0) = A$ and $\dot{x}_2(0) = 0$, satisfied; it is

$$x_2(t) = \left(\frac{A}{12672}\right) \{(12,094)\cos(\Omega t) + (555)\cos(3\Omega t) + (23)\cos(5\Omega t)\}, \tag{30}$$

with $\Omega(A)$ from Eq. (29).

Note that the nonlinear oscillator, given by Eq. (14), can be solved exactly as expressed in terms of a Jacobi elliptic function [12]. The exact period is four places [12].

$$T_{\text{exact}} = \frac{2\pi}{\Omega(A)} = \frac{7.4163}{A}. \tag{31}$$

Let $T_1(A)$ and $T_2(A)$ be the periods determined by the first and second iteration procedure given above; they are given by

$$AT_1(A) = 7.25519, \quad AT_2(A) = 7.41824. \tag{32}$$

Note that the fractional error for $T_2(A)$ is

$$\left|\frac{T_{\text{exact}} - T_2}{T_{\text{exact}}}\right| \times 100 \simeq 0.03\%. \tag{33}$$

This result is an indication of the accuracy of the proposed method as applied to this particular problem.

The second example is a TNO of the van der Pol type; it is given by the differential equation

$$\ddot{x} + x^3 = \varepsilon(1 - x^2)\dot{x}; \quad x(0) = A, \ \dot{x}(0) = 0, \tag{34}$$

where

$$g(x) = x^3, \quad f(x,\dot{x}) = \varepsilon(1 - x^2)\dot{x}, \quad G(x,\dot{x}) = \Omega^2 x - x^3 + \varepsilon(1 - x^2)\dot{x}. \tag{35}$$

For this case, the iteration scheme is

$$\begin{aligned} \ddot{x}_{k+1} + \Omega^2 x_{k+1} = {} & \Omega^2 x_{k-1} - x_{k-1}^3 + \varepsilon(1 - x_{k-1}^2)\dot{x}_{k-1} \\ & + \left[\Omega^2 - 3x_{k-1}^2 - 2\varepsilon x_{k-1}\dot{x}_{k-1}\right](x_k - x_{k-1}) \\ & + \varepsilon(1 - x_{k-1}^2)(\dot{x}_k - \dot{x}_{k-1}), \end{aligned} \tag{36}$$

with $x_{k+1}(0) = A$ and $\dot{x}_{k+1} = 0$. For $k = 0$, Eq. (36) becomes

$$\dot{x}_1 + \Omega^2 x_1 = \Omega^2 x_{-1} - x_{-1}^3 + \varepsilon(1 - x_{-1}^2)\dot{x}_{-1}. \tag{37}$$

Substituting Eq. (10) into the right-hand side gives, on simplification, the result

$$\ddot{x}_1 + \Omega^2 x_1 = A\left[\Omega^2 - \frac{3A^2}{4}\right] - \left(\frac{A^3}{4}\right)\cos(3\Omega t) - \varepsilon\Omega A\left[1 - \frac{A^2}{4}\right]\sin(\Omega t) + \left(\frac{\varepsilon\Omega A^3}{4}\right)\sin(3\Omega t). \tag{38}$$

No secular terms requires

$$A\left[\Omega^2 - \frac{3A^2}{4}\right] = 0, \quad \varepsilon\Omega A\left[1 - \frac{A^2}{4}\right] = 0, \tag{39}$$

and gives for A and Ω the values [2]

$$A = 2, \quad \Omega = \sqrt{3}. \tag{40}$$

The corresponding particular solution can be easily calculated and if added to the homogeneous solution gives

$$x_1(t) = C_1 \cos(\Omega t) + C_2 \sin(\Omega t) + \left(\frac{1}{12}\right)\cos(3\Omega t) - \left(\frac{\varepsilon}{4\sqrt{3}}\right)\sin(3\Omega t), \tag{41}$$

where C_1 and C_2 are constants determined by the initial conditions, $x_1(0) = A = 2$ and $\dot{x}_1(0) = 0$; they turn out to be

$$C_1 = \frac{23}{12}, \quad C_2 = \frac{\sqrt{3}\varepsilon}{4}. \tag{42}$$

Therefore, $x_1(t)$ is, on using $\Omega = \sqrt{3}$,

$$x_1(t) = \left(\frac{1}{12}\right)\left[(23)\cos\left(\sqrt{3}t\right) + \cos\left(3\sqrt{3}t\right)\right] + \left(\frac{\varepsilon}{4\sqrt{3}}\right)\left[3\sin\left(\sqrt{3}t\right) - \sin\left(3\sqrt{3}t\right)\right]. \tag{43}$$

Observe that $x_{-1}(t)$ and $x_0(t)$, with this information on the values of A and Ω, are

$$x_{-1}(t) = x_0(t) = 2\cos\left(\sqrt{3}t\right). \tag{44}$$

An effort to calculate $x_2(t)$ shows that while it is straightforward to perform, the calculations are algebraic intensive.

In summary, a modified iteration method for calculating analytical approximations to the periodic solutions of "truly nonlinear oscillators" (TNO) has been proposed. Its applicability has been demonstrated by means of two examples. The method builds on the previous work of Mickens [3] and Lim and Wu [10].

Future studies will involve the investigation of elastic force functions, $g(x)$, for which only a finite number of derivatives exist at $x = 0$. An example of such an equation is [6,9]

$$\ddot{x} + x^{1/3} = 0. \tag{45}$$

In general, this means that only a finite number of iterations can be performed. This fact may not be an essential difficulty since the evaluation of just $x_1(t)$ for a TNO can give a useful result if no other method is available.

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References

[1] L. Hill, R.E. Mickens, Investigation of the equation $\ddot{x} + x^3 = 0$, *Bulletin of the American Physical Society* 29 (1984) 63.

[2] R.E. Mickens, K. Oyedeji, Construction of approximate analytical solutions to a new class of non-linear oscillator equation, *Journal of Sound and Vibration* 102 (1985) 579–582.

[3] R.E. Mickens, Iteration procedure for determining approximate solutions to non-linear oscillator equations, *Journal of Sound and Vibration* 116 (1987) 185–187.

[4] M. Senator, C.N. Bapat, A perturbation technique that works even when the non-linearity is not small, *Journal of Sound and Vibration* 164 (1993) 1–27.

[5] R.E. Mickens, Generalization of the Senator–Bapat method to systems having limit cycles, *Journal of Sound and Vibration* 224 (1999) 167–171.

[6] R.E. Mickens, Analysis of nonlinear oscillations having nonpolynomial elastic terms, *Journal of Sound and Vibration* 255 (2002) 789–792.

[7] S.L. Das, A. Chatterjee, Multiple scales via Galerkin projections: approximate asymptotics for strongly nonlinear oscillations, *Nonlinear Dynamics* 32 (2003) 161–186.

[8] A. Chatterjee, Harmonic balance based averaging: approximate realizations of an asymptotic technique, *Nonlinear Dynamics* 32 (2003) 323–343.

[9] R.E. Mickens, A combined equivalent linearization and averaging perturbation method for non-linear oscillator equations, *Journal of Sound and Vibration* 264 (2003) 1195–1200.

[10] C.W. Lim, B.S. Wu, A modified Mickens procedure for certain non-linear oscillators, *Journal of Sound and Vibration* 257 (2002) 202–206.

[11] R. Krivec, et al., Numerical investigation of the quasilinearization method in quantum mechanics, *Computer Physics Communications* 138 (2001) 69–79.

[12] R.E. Mickens, *Nonlinear Oscillations*, Cambridge University Press, New York, 1981.