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Short Communication

Exact finite difference scheme for second-order, linear ODEs having constant coefficients

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Abstract

We construct an exact finite difference scheme for a second-order, linear equation that forms the basis for modeling and analyzing linear damped vibratory systems with forcing.

Second-order, linear ordinary differential equations arise as important mathematical models for a broad range of phenomena in vibration, acoustics, and seismology [1–4]. These equations, for the case of a single dependent variable, take the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a(t)\,\frac{\mathrm{d}x}{\mathrm{d}t} + b(t)x = f(t), \tag{1}$$

where the coefficients, $a(t)$ and $b(t)$, may be functions of the time t, and $f(t)$ is a forcing function. For this type of problem, the initial conditions are usually given, i.e.,

$$x(0) = A, \quad \frac{\mathrm{d}x(0)}{\mathrm{d}t} = B, \tag{2}$$

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where A and B have specified values. A more general form occurs in the consideration of singular boundary value problems. The equation now takes the form

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{3a}$$

$$y(0) = a_1, \qquad y(1) = a_2, \tag{3b}$$

where now x is a space coordinate, ε is a small parameter, i.e., $|\varepsilon| \ll 1$, the prime denotes taking the derivative with respect to x, i.e., $y'(x) = \mathrm{d}y(x)/\mathrm{d}x$; the x interval in [0, 1]; and (a_1, a_2) are constants.

For many applications, the data to be modeled and/or analyzed appear either in digital form and/or the system of interest has a non-elementary forcing function such that a discrete numerical integration method is needed to study the behavior of the system [6,7]. The main purpose of this short communication is to construct an exact finite difference scheme [8] for the homogeneous case where the coefficients are constant and then show how this result can be applied to construct discretizations of the full inhomogeneous case. The analysis begins by considering the following particular form of Eq. (3a):

$$cy''(x) + ay'(x) + by = 0, \tag{4}$$

where (a, b, c) are constants. Previous work by Mickens [7] builds on the results obtained by Ly [6]. Independently, Mickens [8] has also constructed an exact finite difference scheme for the damped harmonic oscillator written in the dimensionless form

$$\frac{\mathrm{d}^2 y(\bar{x})}{\mathrm{d}\bar{x}^2} + 2\varepsilon \frac{\mathrm{d}y(\bar{x})}{\mathrm{d}\bar{x}} + y(\bar{x}) = 0. \tag{5}$$

The answer provided by that calculation will be the starting point for constructing an exact finite difference scheme for second-order, linear ordinary differential equations having constant coefficients.

First, note that the transformation of variables

$$\bar{x} = \sqrt{\frac{b}{c}}x, \quad 2\varepsilon = \frac{1}{\sqrt{bc}}, \tag{6}$$

in Eq. (5), gives

$$c\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + a\frac{\mathrm{d}y(x)}{\mathrm{d}x} + by(x) = 0. \tag{7}$$

Second, the exact finite difference scheme for Eq. (5) is [8]

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{\phi^2} + 2\varepsilon\left(\frac{y_k - \psi y_{k-1}}{\phi}\right) + \frac{2(1-\psi)y_k + (\phi^2 + \psi^2 - 1)y_{k-1}}{\phi^2} = 0, \tag{8}$$

where

$$\bar{x} \to \bar{x}_k = \bar{h}k, \quad y(\bar{x}) \to y_k; \quad k = \text{integer}; \tag{9}$$

$\bar{h}$ is the step-size, i.e., $\bar{h} = \Delta\bar{x}$; and the functions ψ and ϕ are

$$\psi(\varepsilon, \bar{h}) = \frac{\varepsilon e^{-\varepsilon\bar{h}}}{\sqrt{1-\varepsilon^2}} + e^{-\varepsilon\bar{h}}\cos\left(\sqrt{1-\varepsilon^2}\cdot\bar{h}\right), \tag{10}$$

$$\phi(\varepsilon, \bar{h}) = \frac{e^{-\varepsilon\bar{h}}}{\sqrt{1-\varepsilon^2}} \sin\left(\sqrt{1-\varepsilon^2} \cdot \bar{h}\right). \tag{11}$$

In terms of (a, b, c), it follows that

$$\varepsilon^2 = \frac{a^2}{4bc}, \quad \varepsilon\bar{h} = \frac{ah}{2c}, \tag{12}$$

where, see Eq. (6),

$$x \to x_k = hk, \quad \bar{h} = \sqrt{\frac{b}{c}}h. \tag{13}$$

Now define $\psi_1(a, b, c, h)$ and $\phi_1(a, b, c, h)$ as

$$\psi_1(a, b, c, h) \equiv \psi(\varepsilon, \bar{h}) = \psi\left(\frac{a^2}{4bc}, \sqrt{\frac{b}{c}}h\right), \tag{14a}$$

$$\phi_1(a, b, c, h) \equiv \phi(\varepsilon, \bar{h}) = \psi\left(\frac{a^2}{4bc}, \sqrt{\frac{b}{c}}h\right). \tag{14b}$$

With these definitions, Eq. (8) can be rewritten, after some algebraic manipulations, to the form

$$c\left[\frac{y_{k+1} - 2y_k + y_{k-1}}{D_1}\right] + a\left[\frac{y_k - \psi_1 y_{k-1}}{D_2}\right] + b\left[\frac{2(1-\psi_1)y_k + (\phi_1^2 + \psi_1^2 - 1)y_{k-1}}{D_3}\right] = 0, \tag{15}$$

where

$$D_1 = D_1(a, b, c, h) = \left(\frac{c}{b}\right)[\phi_1(a, b, c, h)]^2, \tag{16a}$$

$$D_2 = D_2(a, b, c, h) = \sqrt{\frac{c}{b}}\phi_1(a, b, c, h), \tag{16b}$$

$$D_3 = D_3(a, b, c, h) = \phi_1(a, b, c, h). \tag{16c}$$

This is the exact finite scheme for the second-order, linear, constant coefficient differential equation

$$cy''(x) + ay'(x) + by(x) = 0. \tag{17}$$

An examination of the non-standard scheme [8], as represented in Eq. (15), gives the following features:

(i) A discrete second-order derivative, as given by standard methods [9], has the representation

$$y''(x) \to \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}. \tag{18}$$

Note, however, that the result in Eq. (15) has the step-size h replaced by the more complex expression $D_1(a, b, c, h)$ which not only has the property

$$D_1(a, b, c, h) = h^2 + O(h^4), \tag{19}$$

but, for finite h, depends also on the parameters (a,b,c).

(ii) The discrete first-derivative, for Eq. (17), is a backward-Euler type representation with the step-size in the denominator replaced by $D_2(a,b,c,h)$ with

$$D_2(a,b,c,h) = h + O(h^2). \tag{20}$$

(iii) The discrete representation for the y term is modeled by a linear combination of y_k and y_{k-1}. A direct calculation shows that

$$\operatorname*{Lim}_{h\to 0}\left[\frac{2(1-\psi_1)y_k + (\phi_1^2+\psi_1^2-1)y_{k-1}}{D_3}\right] = y. \tag{21}$$

(iv) With careful attention to taking the proper limits, it is easy to show that non-standard schemes for the differential equations

$$cy'' + ay' = 0, \tag{22a}$$

$$cy'' + by = 0, \tag{22b}$$

$$ay' + by = 0 \tag{22c}$$

are obtained by considering, respectively, the limits as $b \to 0$, $a \to 0$, and $c \to 0$.

For the case where the coefficients depend on x, i.e.,

$$c(x)y''(x) + a(x)y'(x) + b(x)y(x) = 0, \tag{23}$$

a non-standard finite difference scheme will incorporate the features of Eq. (15) along with selecting a method for the discretization of $a(x)$, $b(x)$, and $c(x)$. One possibility is to make the following replacements in Eq. (15):

$$c(x) \to \frac{c_{k+1} + 2c_k + c_{k-1}}{4}, \tag{24a}$$

$$a(x) \to \frac{a_k + \psi_1 a_{k-1}}{1+\psi_1}, \tag{24b}$$

$$b(x) \to \frac{2(1-\psi_1)b_k + (\phi_1^2+\psi_1^2-1)b_{k-1}}{2(1-\psi_1) + (\phi_1^2+\psi_1^2-1)}, \tag{24c}$$

where $c_k \equiv c(x_k)$, $b_k \equiv b(x_k)$, $a_k = a(x_k)$, and replace (D_1, D_2, D_3) by the expressions, see Eqs. (16),

$$D_1 \to D_1(a_k, b_k, c_k, h), \tag{25a}$$

$$D_2 \to D_2(a_k, b_k, c_k, h), \tag{25b}$$

$$D_3 \to D_3(a_k, b_k, c_k, h). \tag{25c}$$

Similarly, the inhomogeneous differential equation

$$c(x)y''(x) + a(x)y'(x) + b(x)y(x) = f(x). \tag{26}$$

A partial test of the above proposed numerical integration schemes was carried out by Patidar [5] for several special test cases of singular perturbation problems, i.e., situations for which $c(x) = \varepsilon$, where the parameter ε can be very small. The properly selected non-standard schemes gave extremely accurate solutions and, in general, provided better numerical solutions in comparison with several standard discretizations.

In summary, an exact finite difference model was constructed for second-order, linear, homogeneous differential equations where the coefficients are constant. Using this form, a discretization for the case where the coefficients depended on x was proposed, along with a possible non-standard extension for the inhomogeneous equation. These schemes may provide improved numerical solutions to certain problems arising in several areas of acoustics, vibrations and seismology.

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