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Short Communication

Image solution for clamped finite beams

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Abstract

An image solution for the forced response of a clamped finite beam is developed. The finite beam is replaced by an infinite beam under spatially periodic excitation and periodic support reactions. The response is then found by a Fourier transform approach, making use of the periodicity. An explicit expression is found, using the Poisson sum formula. The aim is to describe the potential of using the image method for clamped structural acoustic and vibration problems.

1. Introduction

The method of images, or mirrors, is a well-known idea in theoretical physics [1, pp. 812–820] and has commonly been employed in disciplines such as acoustics, electro-magnetics, and optics. When considering the problem of obtaining Green's function for a bounded domain, the reflection is described by one or more image sources, and the position and sign of the image sources is chosen so that the boundary conditions will be fulfilled. The Green's function of the bounded problem can then be found as the superposition of the free Green's function for the two sources. A consequence of the method of images is that the domain of the problem is expanded from a part-space to the entire space. The problem is then suitable for the spatial Fourier transform.

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Even if it is not common, the method of images can also be used in structural vibration problems. Examples are Gunda et al. [2,3], using free Green's functions for the infinite plate. However, only solutions to the simply supported and guided boundary conditions has been reported, as shown in Fig. 1a and b. Gunda et al. [3] treated the clamped and free boundary conditions by means of adding a near-field function to the image solution. The present paper describes an image method for the clamped boundary condition. This is performed by means of transforming the clamped boundary conditions to simple supports (see Fig. 1c) and then using a transform method.

The aim of the paper is to describe the potential of using the image method for clamped structural acoustic problems. Especially finite periodic problems are likely to be successfully treated with this approach because of the periodic nature of the expanded problem and the solution technique used. Thus, the approach can be used in combination with the models for sound insulation in wooden joist structures developed by the present author [4–7]. The approach is, however, best described in connection with a simple problem.

This paper provides an example of the method of images; the method is applied to a structural acoustic problem with a known solution. The problem is a finite beam, clamped at the boundaries and driven by a point force. The displacement field is solved for. In order to make use of periodicity, the region along the x-axis is expanded. An infinite number of image sources is added to the original exciting force. The image expansion is even, and the clamped boundaries convert to simple supports, as seen in Fig. 1c. The supports are introduced in the governing equation as infinite sums of reaction forces. The system is now infinite, implying Fourier transforms are to be used on the spatial coordinate. The transformed displacement is solved for and formally inverse transformed, making use of periodic theory [8,9]. The support forces are related to the beam displacement so that the yet unknown reaction forces are solved for. The resulting infinite sums are given explicitly using Poisson sum formula and contour integration.

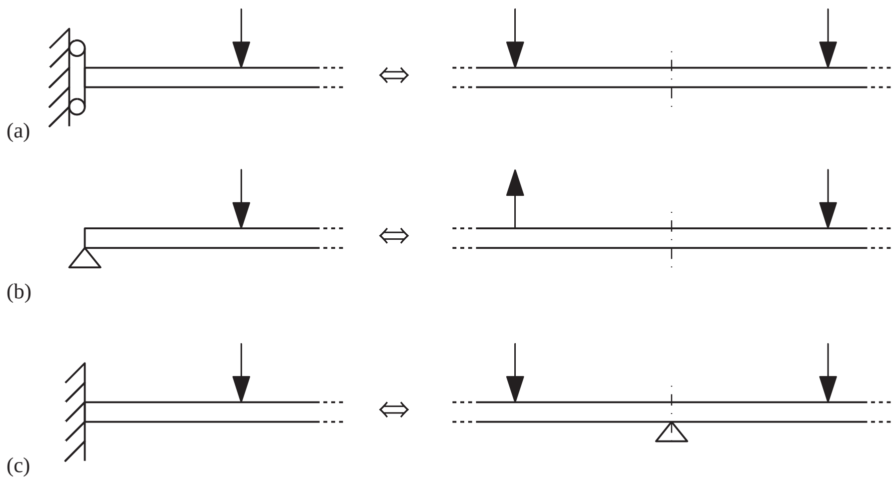

Fig. 1. Examples of boundary conditions and their equivalent image representation. Case (a) guided and (b) simply supported, has been used previously [2], but case (c) clamped, is new.

2. Theory

2.1. Problem description

As the original problem under study, consider a finite beam, clamped at the boundaries (see Fig. 2). The beam is described by the Euler–Bernoulli beam equation. The time convention used is described by the factor $e^{i\omega t}$, which is henceforth suppressed. The domain under consideration is $\Omega_1 : x \in [0, l]$, $\partial\Omega_1 = \partial\Omega_{1a} \cup \partial\Omega_{1b}$. The clamped boundaries condition implies that

$$C: w|_{\partial\Omega_1} = 0, \quad \left.\frac{\partial w}{\partial x}\right|_{\partial\Omega_1} = 0. \tag{1}$$

The beam is driven by a point force, $Q\delta(x - x_0)$, where $x_0 \in \Omega_1$.

In order to make use of periodicity, expand the region along the x-axis, see Fig. 3. An infinite number of mirror sources is added to the original force. The expanded domain is $\Omega: x \in \mathbb{R}$. The mirror expansion is even, $w(-x) = w(x)$, and the clamped boundaries convert to simple supports, which implies that

$$SS: w|_{\partial\Omega_1} = 0, \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{\partial\Omega_1 -} = \left.\frac{\partial^2 w}{\partial x^2}\right|_{\partial\Omega_1 +}. \tag{2}$$

The second boundary condition is fulfilled automatically by means of the mirror sources. The displacement of the system will satisfy the Euler–Bernoulli beam equation inside Ω,

$$D\frac{\partial^4 w}{\partial x^4} - A\rho\omega^2 w = p_e + \sum_{n=-\infty}^{\infty} F_n \delta(x - nl) \tag{3}$$

and the boundary conditions,

$$\begin{aligned} &w(x + 2nl) = w(x), \quad w(x + (2n-1)l) = w(l - x), \\ &w(nl) = 0, \quad n \in \mathbb{Z} \end{aligned} \tag{4}$$

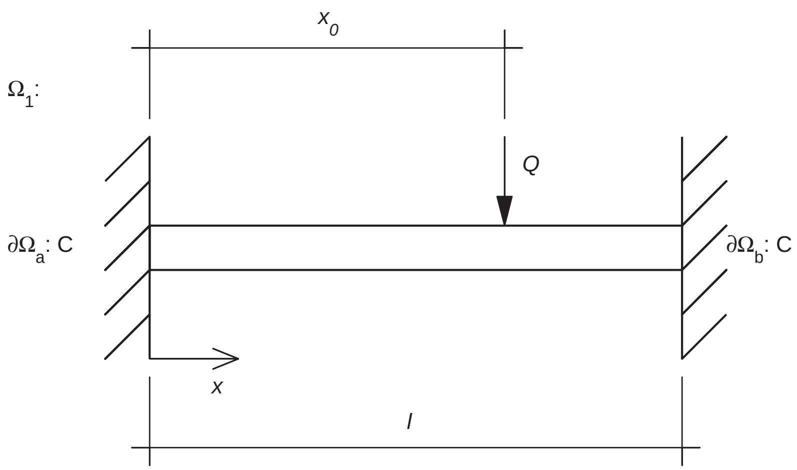

Fig. 2. The original system, a clamped beam with a point force Q located at x_0.

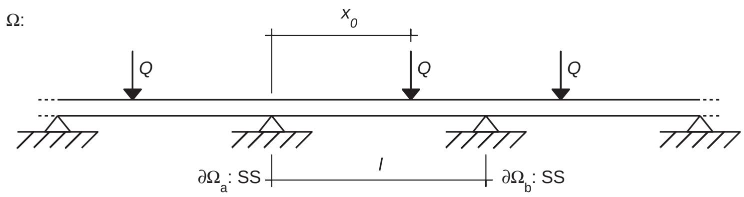

Fig. 3. The expanded system, infinite in extent and periodic. The boundary conditions are simple supports.

of which the first is the periodic condition and the second is the absence of displacement at the supports. $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is the integer numbers. The periodic length of the expanded system is $2l$. In Eq. (3) p_e is the driving point force including the mirror sources and F_n is the reaction force from the nth support. $D = EI$ is the bending stiffness, E the Young's modulus, I the moment of inertia of the area, A the cross-section area, ρ the density, ω the angular frequency, and $w(x)$ the displacement at position x. The reaction from the surrounding acoustic pressure is omitted.

The system is now infinite, and we can use Fourier transforms on the spatial coordinate. The transformation of the displacement and its inversion is defined as

$$\tilde{w}(\alpha) = \int_{-\infty}^{\infty} w(x) \mathrm{e}^{\mathrm{i}\alpha x} \, \mathrm{d}x, \quad w(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{w}(\alpha) \mathrm{e}^{-\mathrm{i}\alpha x} \, \mathrm{d}\alpha. \tag{5}$$

The periodic condition in Eq. (4) can also be expressed as

$$w(x + 2nl) = w(x)\mathrm{e}^{-\mathrm{i}2n\pi} = w(x), \tag{6}$$

while the support reactions are related to the reactions in the original supports,

$$F_n = \begin{cases} F_0, & n \text{ even}, \\ F_1, & n \text{ odd}, \end{cases} \tag{7}$$

where F_0 and F_1 are yet unknown. These forces are to be determined in the next section.

2.2. The point force and support reactions

The point force in the original system Ω_1 is given as $p_e(x) = Q\delta(x - x_0)$. In the expanded system Ω, the exciting force and the corresponding mirror forces become

$$p_e(x) = Q\left(\sum_{n=-\infty}^{\infty} \delta(x - x_0 - 2nl) + \sum_{n=-\infty}^{\infty} \delta(x + x_0 - 2nl) \right). \tag{8}$$

The Fourier transform of Eq. (8) is

$$\tilde{p}_e(\alpha) = Q(\mathrm{e}^{\mathrm{i}\alpha x_0} + \mathrm{e}^{-\mathrm{i}\alpha x_0}) \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\alpha 2nl}. \tag{9}$$

A spatial Fourier transformation of the support reaction yields $\mathscr{F}(F_n \delta(x - nl)) = F_n \mathrm{e}^{\mathrm{i}\alpha nl}$, where $\mathscr{F}$ is the Fourier operator in Eq. (5). If the periodic condition (6) and (7) is considered, the reaction forces are related back to the origin.

The Poisson sum rule [1, p. 467] can be used to give the following relationship:

$$\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}mkl} = 2\pi \sum_{m=-\infty}^{\infty} \delta(2m\pi - kl). \tag{10}$$

(See also the more general result in Eq. (18).) Eq. (9) can therefore be rewritten as

$$\tilde{p}_e(\alpha) = 2\pi Q(\mathrm{e}^{\mathrm{i}\alpha x_0} + \mathrm{e}^{-\mathrm{i}\alpha x_0}) \sum_{n=-\infty}^{\infty} \delta(2n\pi - 2\alpha l) \tag{11}$$

and the sum of reaction forces can be expressed as

$$\begin{aligned}\sum_{n=-\infty}^{\infty} F_n \mathrm{e}^{\mathrm{i}\alpha n l} &= F_0 \sum_{2n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\alpha l} + F_1 \sum_{2n-1=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\alpha l} \\ &= 2\pi(F_0 + F_1 \mathrm{e}^{-\mathrm{i}\alpha l}) \sum_{n=-\infty}^{\infty} \delta(2\pi n - 2\alpha l) \end{aligned} \tag{12}$$

using Eq. (7) in the first equality, and variable changes and Eq. (10) in the last equality.

2.3. Inverse transformation and solution

Summing up the result so far, the transformed version of the governing equation (3) is found by applying Eq. (5) and using Eqs. (11) and (12) for the forces. Solving for the transformed displacement yields

$$\tilde{w}(\alpha) = \frac{2\pi}{D} \frac{Q(\mathrm{e}^{\mathrm{i}\alpha x_0} + \mathrm{e}^{-\mathrm{i}\alpha x_0}) - F_0 - F_1 \mathrm{e}^{-\mathrm{i}\alpha l}}{\alpha^4 - k_B^4} \sum_{n=-\infty}^{\infty} \delta(2n\pi - 2\alpha l), \tag{13}$$

where the bending wavenumber of the beam, k_B, is defined as $k_B^4 \equiv A\rho\omega^2/D$.

The inverse Fourier transform of Eq. (13) can be found, using Eq. (5). The integral so found is easily calculated by changing the order of integration and summation, giving the beam displacement as infinite sums

$$\begin{aligned} w(x) = {} & \frac{2Q}{D2l} \sum_{n=-\infty}^{\infty} \frac{\cos(\pi n x_0/l)\mathrm{e}^{-\mathrm{i}\pi n x/l}}{(\pi n/l)^4 - k_B^4} \\ & - \frac{F_0}{D2l} \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\pi n x/l}}{(\pi n/l)^4 - k_B^4} - \frac{F_1}{D2l} \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\pi n(x+l)/l}}{(\pi n/l)^4 - k_B^4}. \end{aligned} \tag{14}$$

In the final step of the solution the support forces are related to the plate displacement so that the unknown reaction forces F_0 and F_1 is solved for. Thus, using $w(0) = w(l) = 0$ in Eq. (14) leads to an equation system. Using the identities $\mathrm{e}^{-\mathrm{i}2n\pi} = 1$ and $\mathrm{e}^{-\mathrm{i}n\pi} = (-1)^n$, the equation system is

$$\begin{bmatrix} S(2\pi, 0) & S(\pi, 0) \\ S(\pi, 0) & S(2\pi, 0) \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = 2Q \begin{bmatrix} S(2\pi, \pi x_0/l) \\ S(\pi, \pi x_0/l) \end{bmatrix}, \tag{15}$$

where the following definition has been introduced:

$$S(p,q) \equiv \left(\frac{l}{\pi}\right)^4 \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}pn}\cos(nq)}{n^4-(lk_B/\pi)^4}. \tag{16}$$

Note that also the sums in Eq. (14) can be expressed in terms of $S(p,q)$. Thus, the final result reads

$$w(x) = \frac{Q}{Dl} S(\pi x/l, \pi x_0/l) - \frac{F_0}{D2l} S(\pi x/l, 0) - \frac{F_1}{D2l} S(\pi(x+l)/l, 0) \tag{17}$$

and

$$\begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = \frac{2Q}{S(2\pi,0)^2 - S(\pi,0)^2} \begin{bmatrix} S(2\pi,\pi x_0/l)S(2\pi,0) - S(\pi,\pi x_0/l)S(\pi,0) \\ -S(2\pi,\pi x_0/l)S(\pi,0) + S(\pi,\pi x_0/l)S(2\pi,0) \end{bmatrix}.$$

2.4. The infinite sums

Matrix inversion then yields the reaction forces. Sum (16) can be truncated as the order of the polynom of the denominator is 4. However, explicit formulas can be found, as described in this section. The approach is the same as in Refs. [9,10].

A more general form of the Poisson sum formula [1], as compared to Eq. (10), states that

$$\sqrt{a} \sum_{n=-\infty}^{\infty} g(na) = \sqrt{\frac{b}{2\pi}} \sum_{n=-\infty}^{\infty} \tilde{g}(nb), \tag{18}$$

where $ab = 2\pi$ and $\tilde{g}(\xi)$ is the Fourier transform of $g(k)$, as defined in Eq. (5). In view of Eq. (16), define the function $g(k)$ to be

$$g(k) \equiv \frac{\mathrm{e}^{-\mathrm{i}pk}\cos(kq)}{k^4 - c^4},$$

where $c = lk_B/\pi$ and $k \in \mathbb{C}$. Making use of contour calculus, see e.g. Refs. [9,10], taking into account the effect of damping, the transformed function $\tilde{g}(\xi)$ can be found. Using Eq. (18) and the expression for the geometric series

$$\sum_{n=0}^{\infty} \mathrm{e}^{-\alpha n} = \frac{1}{1-\mathrm{e}^{-\alpha}}, \quad |\mathrm{e}^{-\alpha}| < 1$$

then, after some algebraic manipulations, result in an explicit expression for the infinite sum function (16),

$$\begin{aligned} S(p,q)\frac{4k_B^3}{l} = & \frac{-2\sin k_B l}{1-\cos 2k_B l}(\cos k_B l(2m_1 + q/\pi + 1 - p/\pi) + \cos k_B l(2m_2 - q/\pi + 1 - p/\pi)) \\ & + \frac{2\sinh k_B l}{1-\cosh 2k_B l}(\cosh k_B l(2m_1 + q/\pi + 1 - p/\pi) + \cosh k_B l(2m_2 - q/\pi + 1 - p/\pi)), \end{aligned} \tag{19}$$

using the fact that only $0<x<l$ and $0<x_0<l$ is of interest, and where the integers m_1 and m_2 are chosen to the floor of $p/(2\pi) \pm q/(2\pi)$,

$$m_1 = \left\lfloor \frac{p}{2\pi} - \frac{q}{2\pi} \right\rfloor, \quad m_2 = \left\lfloor \frac{p}{2\pi} + \frac{q}{2\pi} \right\rfloor.$$

(The floor function $\lfloor x \rfloor$ is the largest integer $\leqslant x$, see Ref. [11].) For real values of p, q and k, also $S(p,q)$ is real. In particular, the following expressions can be found:

$$S(2\pi, \pi x_0/l) = \frac{l}{2k_B^3}\left(\frac{-\sin k_B l}{1-\cos 2k_B l}\left(\cos k_B(x_0-l) + \cos k_B(l-x_0)\right) + \frac{\sinh k_B l}{1-\cosh 2k_B l}\left(\cosh k_B(x_0-l) + \cosh k_B(l-x_0)\right)\right)$$

$$S(\pi, \pi x_0/l) = \frac{l}{k_B^3}\left(-\frac{\sin k_B l \cos x_0 k_B}{1-\cos 2k_B l} + \frac{\sinh k_B l \cosh x_0 k_B}{1-\cosh 2k_B l}\right)$$

$$S(\pi x/l, \pi x_0/l) = \frac{l}{2k_B^3}\left(\frac{-\sin k_B l}{1-\cos 2k_B l}\left(\cos(k_B l - k_B|x_0-x|) + \cos(k_B l - k_B(x_0+x))\right) + \frac{\sinh k_B l}{1-\cosh 2k_B l}\left(\cosh(k_B l - k_B|x_0-x|) + \cosh(k_B l - k_B(x_0+x))\right)\right)$$

$$S(\pi x/l, 0) = \frac{l}{k_B^3}\left(-\frac{\sin k_B l \cos k_B(l-x)}{1-\cos 2k_B l} + \frac{\sinh k_B l \cosh k_B(l-x)}{1-\cosh 2k_B l}\right)$$

$$S(\pi(x+l)/l, 0) = \frac{l}{k_B^3}\left(-\frac{\sin k_B l \cos k_B x}{1-\cos 2k_B l} + \frac{\sinh k_B l \cosh k_B x}{1-\cosh 2k_B l}\right), \tag{20}$$

which can be used directly in the solution, Eq. (17).

2.5. Correction check

The correctness of the present solution can be checked by comparing it with other solutions. A straightforward approach is to assume there to be two travelling and two decaying waves with unknown complex amplitudes, together with the free Green's function for a beam, as found in e.g. Ref. [12], so that the total field is

$$w(x) = a\mathrm{e}^{-\mathrm{i}k_B x} + b\mathrm{e}^{\mathrm{i}k_B(x-l)} + c\mathrm{e}^{-k_B x} + d\mathrm{e}^{k_B(x-l)} + \frac{Q}{\mathrm{i}4Bk_B^3}\left(\mathrm{e}^{-\mathrm{i}k_B|x-x_0|} - \mathrm{i}\mathrm{e}^{-k_B|x-x_0|}\right), \tag{21}$$

where the constants a–d are yet unknown. Using the boundary conditions in the original problem, Eq. (1), $w(0) = w(l) = 0$ and $w'(0) = w'(l) = 0$, yields the system of equation

$$\begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}k_B l} & 1 & \mathrm{e}^{-k_B l} \\ \mathrm{e}^{-\mathrm{i}k_B l} & 1 & \mathrm{e}^{-k_B l} & 1 \\ -\mathrm{i} & \mathrm{i}\mathrm{e}^{-\mathrm{i}k_B l} & -1 & \mathrm{e}^{-k_B l} \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}k_B l} & \mathrm{i} & -\mathrm{e}^{-k_B l} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{Q}{4Bk_B^3} \begin{bmatrix} \mathrm{i}\mathrm{e}^{-\mathrm{i}k_B x_0} + \mathrm{e}^{-k_B x_0} \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}k_B(l-x_0)} + \mathrm{e}^{-k_B(l-x_0)} \\ -\mathrm{e}^{-\mathrm{i}k_B x_0} + \mathrm{e}^{-k_B x_0} \\ \mathrm{e}^{-\mathrm{i}k_B(l-x_0)} - \mathrm{e}^{-k_B(l-x_0)} \end{bmatrix}, \tag{22}$$

which, if solved, results in exactly the same result as is found by Eq. (17).

3. Concluding remarks

The paper shows that an image approach can be used for clamped structural acoustic problems, and that the exact solution can be obtained in explicit form. Even though the solution to the present problem is more cumbersome than when using conventional methods, the approach can be preferable if transform methods are to be used. An example is periodically stiffened plate structures, where the advantages of an infinite structure can be applied to a finite structure, using the image approach.

Acknowledgements

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