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Generalized hypergeometric function solutions for transverse vibration of a class of non-uniform annular plates

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Abstract

Free vibration analysis of thin annular plate with thickness varying monotonically in arbitrary power form is presented. Transformation of variable is introduced to translate the governing equation for the free vibration of thin annular plate into a fourth-order generalized hypergeometric equation. The analytical solutions in terms of generalized hypergeometric function taking either logarithmic or non-logarithmic forms are proposed, which encompass existing published solutions as special cases. To illustrate the use of the closed form solutions presented, free vibration analyses of a thin annular ultra-high-molecular weight polyethylene and a steel plate with linear and nonlinear thickness variation are performed. The results are compared with those from FE analysis based on Kirchhoff thin plate theory and 3D elasticity theory indicating good agreement.

1. Introduction

The transverse vibration of plates of various shapes has been studied by many researchers over a long period of time owing to its wide applications in engineering design. The simplicity and

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widespread use of circular plates are borne by the many publications on their behavior under different boundary conditions. For circular plate with uniform thickness, Airey [1] and Carrington [2] gave exact solutions in terms of Bessel functions. Other related references may be found in the well-known work of Leissa [3] and his subsequent articles [4–9].

While considerable work has been done on vibration of circular plates with uniform thickness, there is no lack of publications on the vibration of thin circular and annular plates with variable thickness either. Since the response of a plate with non-uniform thickness can be formulated as a set of differential equations with variable coefficients, many approximate solutions have been proposed. Raleigh–Ritz method has been applied to obtain approximate frequencies and mode shapes of circular annular plate with various forms of thickness variations [10–15]. Perturbation method [16] has been employed in analyzing the axi-symmetric free vibration of a circular plate with arbitrary but slow variation in thickness. The generalized differential quadrature rule (GDQR) was utilized by Wu and Liu [17] for the free vibration of solid circular plates with variable thickness and elastic constants. In their work, the thickness of the circular plates can vary radially in specific continuous form such as exponential and linear form. However, relatively few *analytical* solutions are available for plates with variable thickness. Analytical solutions in terms of Bessel functions for axi-symmetric vibrations of circular plate with linear varying thickness and Poisson ratio $\mu = 1/3$ were given by Conway et al. [18]. Exact closed form solutions, in terms of the power of the radius, were obtained by Lenox et al. [19] for the transverse vibrations of a thin annular plate having a parabolic thickness variation. Wang [20] gave a power series solution method for the axi-symmetric vibration of a thin annular plate whose thickness is constant in the circumferential direction but varies arbitrary in the radial direction.

In this paper, the free vibration analysis of thin annular plate with thickness varying monotonically in the radial direction in arbitrary power form is presented. Transformation of variable is introduced such that the governing equation for the free vibration of varying thickness in power form can be transformed into a fourth-order generalized hypergeometric equation. The corresponding analytical solution in terms of generalized hypergeometric function is proposed, which encompass existing published solutions as special cases. As an illustration, the free vibration solutions of thin annular plate with three types of thickness variations based on the presented solutions are discussed, namely, variation with power of (a) 1 (i.e. linearly increasing thickness), (b) 1/2 (nonlinear increasing thickness), and (c) $-1/2$ (nonlinear decreasing thickness). The results are compared with those from three-dimensional (3D) finite element method (FEM).

2. Transformation of governing equation

Consider an annular plate generated by rotating the line $z = \pm\frac{1}{2}h_0(r/a)^m$ about the z-axis, $0 < b \leqslant r \leqslant a$, where b and a are the inner and outer radius of the plate, respectively, m is a positive real number and h_0 is the maximum thickness which occurs at the outer radius of the annular plate. When $m < 0$, the rotating line is modified as $z = \pm\frac{1}{2}h_0(r/b)^m$, where the method of analysis is the same as that when $m > 0$ by replacing 'a' with 'b'; hence, only the case $m > 0$ is presented herein. The governing equation using the cylindrical coordinate system for the free vibration of such thin annular plate can be expressed as [19]

$$r^4\frac{\partial^4 w}{\partial r^4}+(6m+2)r^3\frac{\partial^3 w}{\partial r^3}+r^2\left[(9m^2+3m\mu+3m-1)\frac{\partial^2 w}{\partial r^2}+2\frac{\partial^4 w}{\partial\theta^2\partial r^2}\right]$$
$$+r\left[(9m^2-3m\mu-3m+1)\frac{\partial w}{\partial r}+(6m-2)\frac{\partial^3 w}{\partial\theta^2\partial r}\right]$$
$$+(9m^2\mu-9m-3m\mu+4)\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial^4 w}{\partial\theta^4}+\frac{12\rho(1-\mu^2)a^4}{Eh_0^2}\left(\frac{r}{a}\right)^{4-2m}\frac{\partial^2 w}{\partial t^2}=0, \tag{1}$$

where E is Young's modulus, μ the Poisson ratio and w the transverse displacement of the plate.

Assume the displacement takes on the separable form:

$$w(r,\theta,t)=z(r)e^{ip\theta}e^{i\omega t}. \tag{2}$$

Substituting Eq. (2) into Eq. (1) leads to a homogeneous linear ordinary differential equation with variable coefficients

$$r^4\frac{d^4 z}{dr^4}+(6m+2)r^3\frac{d^3 z}{dr^3}+r^2(9m^2+3m\mu+3m-1-2p^2)\frac{d^2 z}{dr^2}$$
$$+r[(9m^2-3m\mu-3m+1)-(6m-2)p^2]\frac{dz}{dr}$$
$$+\left[p^4-(9m^2\mu-9m-3m\mu+4)p^2-\frac{12\rho(1-\mu^2)a^4\omega^2}{Eh_0^2}\left(\frac{r}{a}\right)^{4-2m}\right]z(r)=0. \tag{3}$$

Solutions for specific simplified forms of Eq. (3) have been presented in published literature. When $m=0$ (i.e. uniform thickness), Eq. (3) takes on the usual Bessel function solutions. When $m=1$ (i.e. linearly varying thickness), $p=0$ (axi-symmetric vibration), and $\mu=1/3$, Eq. (3) can be simplified to a fourth-order Bessel equation [18]. When $m=2$ (i.e. parabolic thickness variation), Eq. (3) can be simplified to a fourth-order Euler equation [19]. There appears to be no other published closed form solutions for annular plate with thickness varying in power form with arbitrary constants. In this paper, a variable transformation is defined such that Eq. (3) can be transformed into a fourth-order generalized hypergeometric equation, which covers all cases, except for $m=2$, given by

$$x=\frac{1}{(4-2m)^4}\left(\frac{\omega}{\omega_0}\right)^2\left(\frac{r}{a}\right)^{4-2m}, \tag{4}$$

where

$$\omega_0=\frac{h_0}{a^2}\sqrt{\frac{E}{12\rho(1-\mu^2)}}.$$

Through this transformation, Eq. (3) can be written as

$$\left\{1-\frac{1}{x}\prod_{i=1}^{4}(\vartheta+\gamma_i-1)\right\}z(x)=0, \tag{5}$$

where

$$\vartheta = x\frac{\partial}{\partial x}$$

and

$$\gamma_i = 1 - \frac{a_i}{2m-4}, \quad i = 1\ldots 4,$$

$$a_1, a_2 = -1 + \frac{3}{2}m \mp \frac{1}{2}\sqrt{\Delta_1 + 2\sqrt{\Delta_2}},$$

$$a_3, a_4 = -1 + \frac{3}{2}m \mp \frac{1}{2}\sqrt{\Delta_1 - 2\sqrt{\Delta_2}},$$

$$\Delta_1 = 9m^2 - 6(1+\mu)m + 4(1+p^2),$$

$$\Delta_2 = (9(1-\mu)^2 + 36\mu p^2)m^2 - 24(1+\mu)p^2 m + 16p^2. \tag{6}$$

3. Closed form solutions

Eq. (5) is a generalized hypergeometric equation. According Frobenius theory, if no two values of γ_i are equal or differ by an integer value, the solutions of Eq. (5) are non-logarithmic and may be written in the form [21,22]

$$\begin{aligned}
z_1(x) &= x^{1-\gamma_1}{}_0F_3([\,],[1+\gamma_2-\gamma_1, 1+\gamma_3-\gamma_1, 1+\gamma_4-\gamma_1], x),\\
z_2(x) &= x^{1-\gamma_2}{}_0F_3([\,],[1+\gamma_1-\gamma_2, 1+\gamma_3-\gamma_2, 1+\gamma_4-\gamma_2], x),\\
z_3(x) &= x^{1-\gamma_3}{}_0F_3([\,],[1+\gamma_1-\gamma_3, 1+\gamma_2-\gamma_3, 1+\gamma_4-\gamma_3], x),\\
z_4(x) &= x^{1-\gamma_4}{}_0F_3([\,],[1+\gamma_1-\gamma_4, 1+\gamma_2-\gamma_4, 1+\gamma_3-\gamma_4], x),
\end{aligned} \tag{7}$$

where ${}_0F_3([\,],[1+\gamma_2-\gamma_1, 1+\gamma_3-\gamma_1, 1+\gamma_4-\gamma_1], x)$ is the generalized hypergeometric function. The series form of the function ${}_pF_q$ is given by

$${}_pF_q([a_1, a_2, \ldots, a_p],[b_1, b_2, \ldots, b_q], x) = 1 + \sum_{k=1}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_k x^k}{\prod_{j=1}^{q}(b_j)_k k!}, \tag{8}$$

where

$$(a_i)_k = \frac{\Gamma(a_i+k)}{\Gamma(a_i)} = a_i(a_i+1)\cdots(a_i+k-1).$$

The complete solution of Eq. (5) can be expressed as

$$z(x) = \sum_{i=1}^{4} c_i z_i(x), \tag{9}$$

where c_i are non-zero constants. Since the infinite series of Eq. (8) converges for all finite x if $p \leqslant q$ [23], the solutions given by Eq. (7) are convergent.

If only λ numbers ($\lambda = 2, 3$ or 4 in the case plate vibration) of γ_i are equal or differ by an integer value, there is no loss of generality in taking λ numbers of γ_i's as $\gamma_1, \gamma_2, \ldots, \gamma_\lambda$, arranged with their real parts in ascending order. Under these conditions, according to the theory of Frobenius [22], the solutions $z_j(x)$ $(j = 1, \lambda + 1, \ldots, 4)$ of Eq. (5) are given by Eq. (7) with the remaining $z_j(x)$ $(j = 2, \ldots, \lambda)$ in logarithmic form. The detailed derivations of the logarithmic solutions are presented in Appendix A. The results for three cases, which span all possible combinations of γ_i, are given as follows:

(I) *When two* γ_i's *are equal or differ by an integer value*: Under this case, $\lambda = 2$. Then $z_1(x), z_3(x)$ and $z_4(x)$ are non-logarithmic solution expressed by Eq. (7). The logarithmic solution, $z_2(x)$, is given by

$$z_2(x) = z_1(x) \ln x + x^{1-\gamma_1} \sum_{s=0}^{\infty} \Psi_{0s}^{10} x^s \prod_{i=1}^{4} \frac{\Gamma(1 - \gamma_1 + \gamma_i)}{\Gamma(1 - \gamma_1 + \gamma_i + s)} + \prod_{i=2}^{4} (\gamma_i - \gamma_1) \frac{1}{x^{\gamma_1}} {}_5F_0\left([1, 1, 1 + \gamma_1 - \gamma_2, 1 + \gamma_1 - \gamma_3, 1 + \gamma_1 - \gamma_4], [], \frac{1}{x} \right), \tag{10}$$

where Ψ_{nk}^{ij} is listed in Appendix A.

(II) *When three* γ_i's *are equal or differ by an integer value*: Under this case, $\lambda = 3$. Then $z_1(x)$ and $z_4(x)$ are non-logarithmic solutions given by Eq. (7). There are two logarithmic solutions, namely $z_2(x)$ given by Eq. (10), and $z_3(x)$ which is given by

$$\begin{aligned} z_3(x) = {} & 2\bar{z}_2(x) \ln x - z_1(x) \ln^2 x + x^{1-\gamma_1} \sum_{s=0}^{\infty} [(\Psi_{0s}^{20})^2 + \Psi_{1s}^{20} + 2\pi^2] x^s \frac{\prod_{i=1}^{4} \Gamma(1 - \gamma_1 + \gamma_i)}{\prod_{i=1}^{4} \Gamma(1 - \gamma_1 + \gamma_i + s)} \\ & + 2x^{1-\gamma_2} \sum_{s=1}^{\gamma_2 - \gamma_1} (-1)^{1-s} \Gamma(s) \Psi_{0s}^{21} x^{-s} \prod_{i=2}^{4} \frac{\Gamma(1 - \gamma_1 + \gamma_i)}{\Gamma(1 - \gamma_1 + \gamma_i - s)} \\ & + 2(-1)^{\gamma_1 + \gamma_2} \Gamma(1 - \gamma_1 + \gamma_2) \frac{\prod_{i=2}^{4} \Gamma(1 - \gamma_1 + \gamma_i)}{\prod_{i=3}^{4} \Gamma(\gamma_i - \gamma_2)} \\ & \times x^{-\gamma_2} {}_5F_0\left([1, 1, 1 + \gamma_2 - \gamma_1, 1 + \gamma_2 - \gamma_3, 1 + \gamma_2 - \gamma_4], [], \frac{1}{x} \right), \end{aligned} \tag{11}$$

where $\bar{z}_2(x)$ is listed in Appendix A.

(III) *When four* γ_i's *are equal or differ by an integer value*: Under this case, $\lambda = 4$. Then $z_1(x)$ is the only non-logarithmic solution given by Eq. (7). There are three logarithmic solutions, namely $z_2(x)$ given by Eq. (10), $z_3(x)$ by Eq. (11) and $z_4(x)$ which is given by

$$\begin{aligned} z_4(x) = {} & z_1(x) \ln^3 x - 3\bar{\bar{z}}_2(x) \ln^2 x + 3\bar{z}_3(x) \ln x + \frac{6(-1)^{\gamma_1 + \gamma_2}}{\Gamma(\gamma_4 - \gamma_3)} \prod_{i=2}^{4} \Gamma(1 - \gamma_1 + \gamma_i) \\ & \times \prod_{i=1}^{2} \Gamma(1 - \gamma_i + \gamma_3) x^{-\gamma_3} {}_5F_0\left([1, 1, 1 + \gamma_3 - \gamma_1, 1 + \gamma_3 - \gamma_2, 1 + \gamma_3 - \gamma_4], [], \frac{1}{x} \right) \end{aligned}$$

$$+ 6(-1)^{\gamma_1+\gamma_2}x^{1-\gamma_2}\prod_{i=2}^{4}\Gamma(1-\gamma_1+\gamma_i)\sum_{s=1}^{\gamma_3-\gamma_2}\frac{\Gamma(\gamma_2-\gamma_1+s)\Gamma(s)\Psi_{0s}^{32}x^{-s}}{\prod_{i=3}^{4}\Gamma(1+\gamma_i-\gamma_2-s)}$$
$$+ 3x^{1-\gamma_1}\sum_{s=1}^{\gamma_2-\gamma_1}\Gamma(s)(-1)^{1-s}x^{-s}[(\Psi_{0s}^{31})^2+\Psi_{1s}^{31}+2\pi^2]\prod_{i=2}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i-s)}$$
$$+ x^{1-\gamma_1}\sum_{s=0}^{\infty}x^s[(\Psi_{0s}^{30})^3+\Psi_{2s}^{30}+3\Psi_{0s}^{30}(\Psi_{1s}^{30}+3\pi^2)]\prod_{i=1}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i+s)}, \quad (12)$$

where $\bar{\bar{z}}_2(x)$ and $\bar{z}_3(x)$ are listed in Appendix A.

4. Some special cases

The generalized hypergeometric function is of a very general form, and encompasses many other special functions. Thus the proposed solutions can be reduced to other types of special functions for certain combinations of the parameters m, p, and μ. To compare the present solutions with existing published results, some special cases are considered.

First consider a uniform plate, that is $m = 0$, for which

$$\gamma_1 = \frac{1}{2}-\frac{p}{4}, \quad \gamma_2 = \frac{1}{2}+\frac{p}{4}, \quad \gamma_3 = 1-\frac{p}{4}, \quad \gamma_4 = 1+\frac{p}{4}. \quad (13)$$

It can be shown that

$$\left.\begin{aligned}\gamma_1 &= \gamma_2 - n\\ \gamma_3 &= \gamma_4 - n\end{aligned}\right\}\ \text{if } p \text{ is odd,}$$
$$\left.\begin{aligned}\gamma_1 &= \gamma_4 - n\\ \gamma_3 &= \gamma_2 - n\end{aligned}\right\}\ \text{if } p \text{ is even,} \quad (14)$$

where n is a non-negative integer.

Thus whenever p is odd or even, $z_1(x)$ and $z_3(x)$ are always of non-logarithmic form given by Eq. (7) while $z_2(x)$ and $z_4(x)$ are always of logarithmic form given by Eq. (10). For purpose of simplification, the relationship between hypergeometric functions $z_1(x)$, $z_3(x)$ and Bessel function are shown in the following.

Substituting Eq. (13) into Eq. (7) gives

$$z_1(x) = x^{(1/2+(p/4))}{}_0F_3\left([\,],\left[1+\frac{p}{4},\frac{3}{2},\frac{3}{2}+\frac{p}{4}\right],x\right)$$
$$= \frac{(p+1)!}{2^{(p+2)}}\sum_{k=0}^{\infty}\frac{(2x^{1/4})^{[2(2k+1)+p]}}{(2k+1+p)!(2k+1)!}, \quad (15)$$

$$z_3(x) = x^{p/4}{}_0F_3\left([\,],\left[\frac{1}{2}+\frac{p}{2},\frac{1}{2},1+\frac{p}{2}\right],x\right)$$
$$= \frac{p!}{2^p}\sum_{k=0}^{\infty}\frac{(2x^{1/4})^{[2(2k)+p]}}{(2k+p)!(2k)!}. \quad (16)$$

The combination of Eqs. (15) and (16) may be re-written as a combination of the series

$$\sum_{k=0}^{\infty} \frac{(2x^{1/4})^{(2k+p)}}{(k+p)!(k)!} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(-1)^k (2x^{1/4})^{(2k+p)}}{(k+p)!(k)!} \tag{17}$$

or in Bessel function form as

$$I_p(4x^{1/4}) \quad \text{and} \quad J_p(4x^{1/4}) \tag{18}$$

where J is Bessel functions and I is modified Bessel functions of the first kind.

These Bessel function solutions of Eq. (5) considering variable transformation (4) are the conventional solutions for uniform thickness plate [24]. The relationship between hypergeometric functions $z_2(x)$, $z_4(x)$ and Bessel functions can be similarly shown.

Another special case is for plate with linearly varying thickness. When $\mu = 1/3, m = 1$ (linearly varying thickness) and $p = 0$ (axi-symmetric vibration), the solution may be written in terms of Bessel functions [18]. For this case, γ_i can be obtained according to Eq. (6) as $\gamma_1 = 1/2, \gamma_2 = 3/2$ $\gamma_3 = 1, \gamma_4 = 2$. Since $\gamma_2 - \gamma_1 = 1, \gamma_4 - \gamma_3 = 1$, $z_1(x)$ and $z_3(x)$ are of non-logarithmic form given by Eq. (7). Substituting $\gamma_1 = 1/2, \gamma_2 = 3/2$, $\gamma_3 = 1, \gamma_4 = 2$ into Eq. (7) gives

$$z_1(x) = x^{1/2}{}_0F_3\left([\,], \left[\frac{3}{2}, 2, \frac{5}{2}\right], x\right) = \frac{3}{2}\sum_{k=0}^{\infty} \frac{(4x^{1/2})^{(2k+1)}}{(2k+1)!(2k+3)!}, \tag{19}$$

$$z_3(x) = {}_0F_3\left([\,], \left[\frac{1}{2}, \frac{3}{2}, 2\right], x\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(4x^{1/2})^{(2k)}}{(2k)!(2k+2)!}, \tag{20}$$

$z_2(x)$ and $z_4(x)$ are of logarithmic form given by Eq. (10). Substituting $\gamma_1 = 1/2, \gamma_2 = 3/2, \gamma_3 = 1, \gamma_4 = 2$ into Eq. (10) gives

$$z_2(x) = z_1(x)\ln x + \sqrt{x}\sum_{s=0}^{\infty}\prod_{i=1}^{4} \frac{\Gamma\left(\frac{1}{2}+\gamma_i\right)\Psi_{0s}^{10}x^s}{\Gamma\left(\frac{1}{2}+\gamma_i+s\right)} + \prod_{i=2}^{4}\left(\gamma_i - \frac{1}{2}\right)\frac{1}{\sqrt{x}}\,{}_5F_0\left(\left[1, 1, 0, \frac{1}{2}, -\frac{1}{2}\right], [\,]\frac{1}{x}\right). \tag{21}$$

To obtain $z_4(x)$, re-arrange γ_i in the order $\gamma_1 = 1, \gamma_2 = 2, \gamma_3 = 1/2, \gamma_4 = 3/2$ and substitute into Eq. (10), giving

$$z_4(x) = z_3(x)\ln x + \sum_{s=0}^{\infty}\prod_{i=1}^{4} \frac{\Gamma(\gamma_i)\Psi_{0s}^{10}x^s}{\Gamma(\gamma_i+s)} + \prod_{i=2}^{4}(\gamma_i - 1)\frac{1}{x}\,{}_5F_0\left(\left[1, 1, 0, \frac{3}{2}, \frac{1}{2}\right], [\,], \frac{1}{x}\right). \tag{22}$$

The combination of Eqs. (19) and (20) may be re-written as a combination of the series

$$\sum_{k=0}^{\infty} \frac{(4x^{1/2})^k}{k!(k+2)!} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(-1)^k (4x^{1/2})^k}{k!(k+2)!} \tag{23}$$

or in Bessel function form as

$$\frac{J_2(4x^{1/4})}{\sqrt{x}} \quad \text{and} \quad \frac{I_2(4x^{1/4})}{\sqrt{x}}. \tag{24}$$

In the same manner, the combination of the other two series in Eqs. (21) and (22) is a linear combination of the solution

$$\frac{K_2(4x^{1/4})}{\sqrt{x}} \quad \text{and} \quad \frac{Y_2(4x^{1/4})}{\sqrt{x}}, \tag{25}$$

where Y is Bessel functions and K is modified Bessel functions, both of the second kind. These Bessel function solutions of Eq. (5) considering variable transformation (4) are the forms presented by Conway et al. [18].

5. Numerical examples

To check the correctness of the proposed solutions presented in the paper, the axi-symmetric free vibration of a ultra-high molecular weight polyethylene (UHMWPE [25,26]) plate is studied under two types of boundary conditions: C–C, F–C, where the first and second letter denotes the edge condition at the inner and outer edge, respectively, and C denotes clamped and F denotes free. The material properties and geometry of the UHMWPE plate are shown in Table 1. The reason to choose such material and geometry is that for $m = 6/5, \mu = -19/45$ (negative Poisson ratio) and $p = 0$ (axi-symmetric vibration), γ_i according to Eq. (6) are $\gamma_1 = 0, \gamma_2 = 1, \gamma_3 = 2, \gamma_4 = 3$, which is the most complex case in the proposed solutions, that is, the free vibration solutions can be written in terms of one non-logarithmic form $z_1(x)$ given by Eq. (7) and three logarithmic forms $z_2(x)$, $z_3(x)$ and $z_4(x)$ given by Eqs. (10)–(12), respectively. A finite element model is also prepared using ABAQUS 6.3 to assess the validity of the results provided by the analytical approach. The annular plate of varying thickness with $m = 6/5$ shown in Fig. 1 is represented by a mesh of 13,659 triangular shell elements STRI3 which is based on Kirchhoff thin plate theory (CPT). Lanczos iterative technique was adopted to compute the fundamental natural frequency of the plate. The comparison between the analytical and numerical results is shown in Table 2. The good agreement of less than 1% maximum difference indicates that the correctness of proposed solutions in this paper, especially for the case of materials with negative Poisson's ratio.

To investigate the application of the proposed solutions in conventional materials, consider an annular steel plate where Young's modulus, mass density and geometric parameters are listed in Table 1. Fig. 1 plots the geometry of each annular plate and its corresponding FEM mesh (using 3D solid element with 20 nodes, C3D20R, and Lanczos iterative technique) with $m = 1$ (linear

Table 1
Material and geometrical properties of annular plate

	Steel	UHMWPE [25]
Young's module ($\mathrm{N\,m^{-2}}$)	210×10^9	3×10^9
Mass density ($\mathrm{kg\,m^{-3}}$)	7800	800
Outer radius a (m)	1.0	1.0
Inner radius b (m)	0.1	0.5
Poisson ratio	0.3	$-\frac{19}{45}$

increasing thickness, 1242 elements), $m = 1/2$ (nonlinear increasing thickness, positive power, 3195 elements), and $m = -1/2$ (nonlinear decreasing thickness, negative power, 2880 elements). The frequencies for free vibration of the above plate with 0–2 diametrical nodes and 0–2 nodal circles are investigated using the solution of Eq. (7) and compare well with those from 3D FEM obtained using ABAQUS 6.3, as summarized in Tables 3,4,5. For example, for $m = 1$ (Table 3), $m = 1/2$ (Table 4), and $m = -1/2$ (Table 5), the respective maximum errors of 2.4%, 6.8% and 3.4% occur at $p = 2$ and $n = 2$ under clamped–clamped boundary condition, respectively. Such agreement shows that the proposed solutions based on CPT are closed to that from FE analysis based on 3D elasticity theory.

The variation of the ratio of frequencies of varying cross section plates with the maximum thickness $h_0 = 1/15$ to those of a plate with uniform thickness $h_0 = 1/15$ under clamped-clamped boundary conditions is plotted in Fig. 2. The variation of the first two frequencies with the taper (represented by the power of thickness function) of the plate is illustrated. When the power m is in

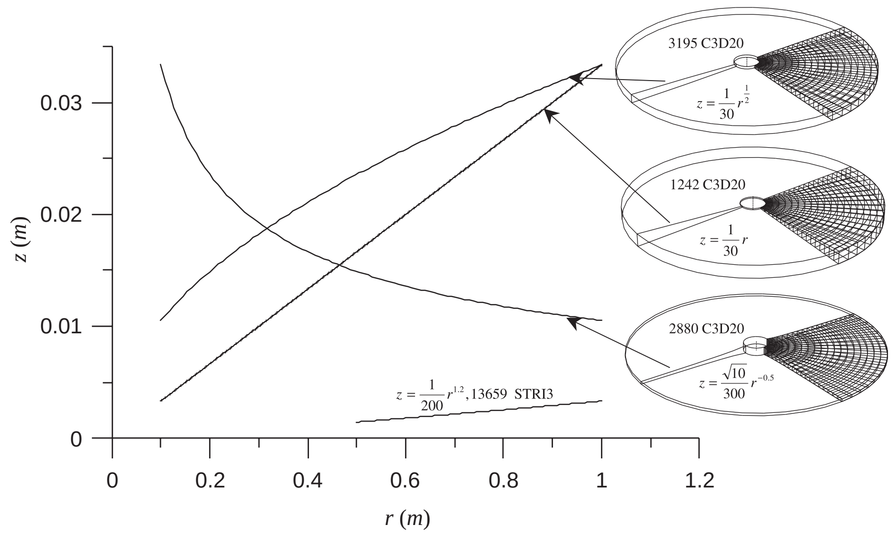

Fig. 1. Geometry of annular plate with $m = 1, 1/2, -1/2, 6/5$.

Table 2
Comparison of frequencies (Hz) of annular plate under C–C, F–C boundary conditions between CPT FEM and proposed results for UHMWPE plate

n	p	C–C[a]			F–C		
		FEM	Proposed	Error (%)	FEM	Proposed	Error (%)
0	0	61.991	62.037	0.07	20.102	20.123	0.10
	1	166.42	168.04	0.96	76.381	76.609	0.30
	2	324.10	326.97	0.88	182.70	183.74	0.57

p = number of nodal diameters; n = number of nodal circles; C = clamped, F = free.
[a]The first letter denotes the condition at the inner edge.

Table 3
Comparison of frequencies (Hz) of annular plate under C–C, F–C boundary conditions between 3D FEM and proposed results for $m = 1$ (linear increasing)

n	p	C–C[a]			F–C		
		FEM	Proposed	Error (%)	FEM	Proposed	Error (%)
0	0	223.460	223.772	0.14	149.250	149.603	0.24
	1	258.170	258.972	0.31	218.330	219.379	0.48
	2	363.450	366.295	0.78	352.450	355.647	0.91
1	0	580.010	582.352	0.40	382.440	385.354	0.76
	1	618.090	622.285	0.68	468.700	473.471	1.02
	2	737.360	747.363	1.36	665.410	675.195	1.47
2	0	1097.600	1114.610	1.55	750.720	762.903	1.62
	1	1136.600	1156.676	1.77	833.160	848.420	1.83
	2	1257.900	1287.404	2.35	1055.90	1079.504	2.24

p = number of nodal diameters; n = number of nodal circles; C = clamped, F = free.
[a]The first letter denotes the condition at the inner edge.

Table 4
Comparison of frequencies (Hz) of annular plate under C–C, F–C boundary conditions between 3D FEM and proposed results for $m = 1/2$ (nonlinear increasing)

n	p	C–C[a]			F–C		
		FEM	Proposed	Error (%)	FEM	Proposed	Error (%)
0	0	302.120	306.027	1.29	153.210	153.859	0.42
	1	336.990	342.054	1.50	273.710	276.750	1.11
	2	459.550	468.545	1.96	444.780	452.562	1.75
1	0	821.290	849.045	3.38	491.620	499.910	1.69
	1	869.350	900.462	3.58	670.370	688.640	2.73
	2	1027.200	1069.060	4.08	956.330	989.164	3.43
2	0	1573.400	1669.238	6.09	1044.400	1081.779	3.58
	1	1626.600	1728.966	6.29	1229.000	1286.276	4.66
	2	1797.400	1919.878	6.81	1610.100	1699.110	5.53

p = number of nodal diameters; n = number of nodal circles; C = clamped, F = free.
[a]The first letter denotes the condition at the inner edge.

the range of -1 to 0, the thickness tapers from a value of h at the centre to a smaller value at the circumference. The natural frequency, say ω_N, will be lower than that of a uniform plate of thickness h, say ω_U. As m increases from -1 to 0, the taper reduces until the plate thickness

Table 5
Comparison of frequencies (Hz) of annular plate under C–C, C–F boundary conditions between 3D FEM and proposed results for $m = -1/2$ (nonlinear decreasing)

n	p	C–C[a]			C–F		
		FEM	Proposed	Error (%)	FEM	Proposed	Error (%)
0	0	223.240	224.421	0.53	47.783	47.813	0.06
	1	227.610	229.730	0.93	235.320	237.822	1.06
	2	261.450	264.359	1.11	277.680	281.165	1.26
1	0	584.750	594.186	1.61	227.030	228.568	0.68
	1	597.870	609.392	1.93	605.200	617.117	1.97
	2	658.090	671.925	2.10	669.630	684.120	2.16
2	0	1112.500	1145.732	2.99	590.600	600.318	1.65
	1	1132.200	1168.831	3.24	1139.100	1175.481	3.19
	2	1212.400	1253.376	3.38	1222.500	1263.459	3.35

p = number of nodal diameters; n = number of nodal circles; C = clamped, F = free.
[a]The first letter denotes the condition at the inner edge.

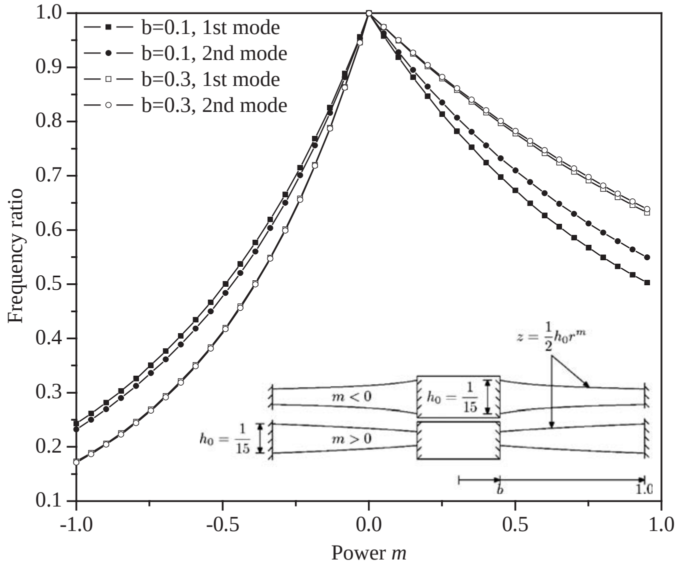

Fig. 2. Frequency ratio (varying thickness to uniform plate) for different m.

reaches h when $m = 0$, implying that the natural frequency increases from ω_N to ω_U. This is reflected in Fig. 2. Similarly, when the power m is in the range from 0 to 1, the thickness tapers from h at the outer edge to a smaller value at the centre. The natural frequency, say ω_P, will be lower than ω_U. As m decreases from 1 to 0, the taper reduces until the plate thickness reaches h

when $m = 0$, implying that the natural frequency increases from ω_P to ω_U. This is again consistent with the results of Fig. 2. Since $\omega_P > \omega_N$ as the plate is stiffer where more materials are concentrated towards the circumference, it is consistent that the negative power of thickness function have much effect on the frequencies of the plate than positive power. In addition, when the inner radius b increases, the variation of the frequencies of the plate with negative power varying thickness became larger while that of the plate with positive power varying thickness decreases. This is because the mass of the plate with negative power varying thickness decreases much more than that of the plate with positive power varying thickness with the increased inner radius b. Another issue to note is the different convergence conditions of hypergeometric functions under negative and positive powers. From Eqs. (7) and (A.46), one can see the rate of convergence of hypergeometric function is dependent on these difference $g_1 = \gamma_2 - \gamma_1$, $g_2 = \gamma_3 - \gamma_1$ and $g_3 = \gamma_4 - \gamma_1$.

Fig. 3 plots the summation of g_i $(i = 1, \ldots, 3)$ and their bi- and tri-product. For $p = 0$, it is easy to see the slowest convergence rate occurs at m near zero and such conclusion may not be hold for $p = 1$, and trial and error is necessary to ensure convergence. In this paper, all hypergeometric functions are calculated using 20 items because the 21st item is less than 10^{-30} even for the slowest convergence case.

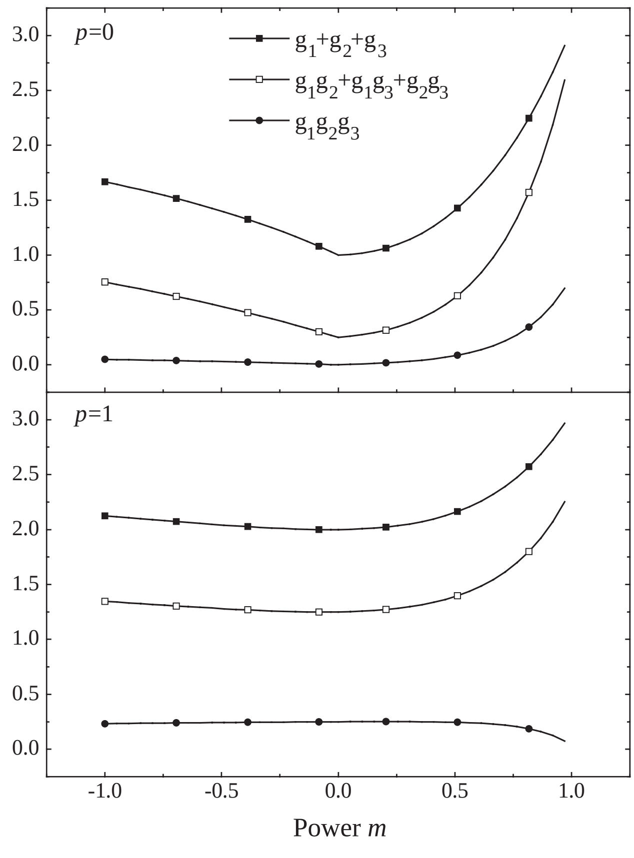

Fig. 3. Convergence conditions for different m and p (where $g_1 = \gamma_2 - \gamma_1, g_2 = \gamma_3 - \gamma_1, g_3 = \gamma_4 - \gamma_1$).

6. Conclusions

The general analytical solutions in terms of generalized hypergeometric function for the free vibration of thin annular plate with thickness varying monotonically in arbitrary power form are presented, which includes published solutions as special cases. The solutions are verified by comparing with those from Kirchoff-based and 3D FEM for plates with linear increasing, nonlinear increasing and nonlinear decreasing thicknesses in the radial direction. The results are consistent, indicating that the negative power of thickness function have much effect on the frequencies of the plate than positive power. In addition, when the inner radius b increases, the variation of the frequencies of the plate with negative power varying thickness became larger while that of the plate with positive power varying thickness decreases. Although the solution technique presented in this paper is based on Kirchhoff plate model, the same approach can be used to solve the free vibration problem of thick plate with varying thickness based on Mindlin plate model.

Appendix A. Logarithmic solutions of generalized hypergeometric equation when $p = 0$ and $q = 3$

In Eq. (5), the generalized hypergeometric equation of $p = 0$ and $q = 3$, is an ordinary differential equation with a regular singular point at the origin, and assume to have a solution of the form

$$z(x) = \sum_{k=0}^{\infty} c_k x^{\rho+k}, \quad c_0 \neq 0. \tag{A.1}$$

Substituting Eq. (A.1) into Eq. (5) yields

$$\begin{aligned}\sum_{k=0}^{\infty} c_k \left[x^{\rho+k} - \prod_{i=1}^{4} (\rho + k + \gamma_i - 1) x^{\rho+k-1} \right] \\ = \sum_{k=1}^{\infty} \left[c_{k-1} - c_k \prod_{i=1}^{4} (\rho + k + \gamma_i - 1) \right] x^{\rho+k-1} - c_0 \prod_{i=1}^{4} (\rho + \gamma_i - 1) x^{\rho-1}.\end{aligned} \tag{A.2}$$

Then, the indicial equation (or characteristic equation) is

$$c_0 \prod_{i=1}^{4} (\rho - 1 + \gamma_i) = 0. \tag{A.3}$$

Since $c_0 \neq 0$, Eq. (A.3) yields four values of ρ, namely,

$$\rho_i = 1 - \gamma_i, \quad i = 1 \ldots 4. \tag{A.4}$$

The coefficients c_k satisfy the recurrence formula

$$c_k = \frac{c_{k-1}}{\prod_{i=1}^{4} (\rho + k - 1 + \gamma_i)}, \tag{A.5}$$

which leads to

$$c_k = c_0 \prod_{i=1}^{4} \frac{\Gamma(\rho - 1 + \gamma_i)}{\Gamma(\rho - 1 + \gamma_i + k)}. \tag{A.6}$$

Let $c_0 = 1$ and substitute Eq. (A.6) into Eq. (A.1) gives

$$z(x) = \sum_{k=0}^{\infty} x^{\rho+k} \prod_{i=1}^{4} \frac{\Gamma(\rho+\gamma_i)}{\Gamma(\rho+\gamma_i+k)}. \tag{A.7}$$

If no two values of γ_i are equal or differ by an integer, from Eq. (A.7), the various solutions may be obtained by setting ρ equal to the roots of Eq. (A.4). This leads to Eq. (7).

If only λ numbers ($\lambda = 2, 3$ or 4 in the case plate vibration) of γ_i are equal or differ by an integer (as discussed in Section 3), Eq. (5) has $\lambda - 1$ logarithmic solutions. For the case of vibration of plate, the largest value of λ is 4. When λ is 2, the logarithmic solutions have been given by Smith [22], MacRobert [27] and Wang [28]. The logarithmic solutions are derived here for λ equal to 3 or 4. For completeness, the solutions for $\lambda = 2$ are also presented. Thus, $z_2(x), z_3(x), z_4(x)$ can be written according to the theory of Frobenius [22] as

$$z_2(x) = v'(\rho)_{\rho=1-\gamma_2} = \sum_{k=0}^{\infty} f_k(\Gamma) x^{\rho+k} (c_0 \ln x + g_k^1), \tag{A.8}$$

$$z_3(x) = v''(\rho)_{\rho=1-\gamma_3} = \sum_{k=0}^{\infty} f_k(\Gamma) x^{\rho+k} (c_0 \ln^2 x + 2g_k^1 \ln x + g_k^2), \tag{A.9}$$

$$z_4(x) = v'''(\rho)_{\rho=1-\gamma_4} = \sum_{k=0}^{\infty} f_k(\Gamma) x^{\rho+k} (c_0 \ln^3 x + 3g_k^1 \ln^2 x + 3g_k^2 \ln x + g_k^3), \tag{A.10}$$

where

$$v(\rho) = c_0(\rho) \sum_{k=0}^{\infty} f_k(\Gamma) x^{\rho+k},$$

$$c_0(\rho) = c_0'(\rho + \gamma_r - 1)^{\lambda-1},$$

$$f_k(\Gamma) = \prod_{i=1}^{4} \frac{\Gamma(\rho+\gamma_i)}{\Gamma(\rho+\gamma_i+k)},$$

$$g_k^1 = \frac{\partial c_0}{\partial \rho} + \Phi_{0k}^{ij} c_0,$$

$$g_k^2 = \frac{\partial^2 c_0}{\partial^2 \rho} + 2\Phi_{0k}^{ij} \frac{\partial c_0}{\partial \rho} + [(\Phi_{0k}^{ij})^2 + \Phi_{0k}^{ij}] c_0,$$

$$g_k^3 = \frac{\partial^3 c_0}{\partial^3 \rho} + 3\Phi_{0k}^{ij} \frac{\partial^2 c_0}{\partial^2 \rho} + 3[(\Phi_{0k}^{ij})^2 + \Phi_{1k}^{ij}] \frac{\partial c_0}{\partial \rho} + [(\Phi_{0k}^{ij})^3 + 3\Phi_{0k}^{ij}\Phi_{1k}^{ij} + \Phi_{2k}^{ij}] c_0,$$

$$\Phi_{nk}^{ij} = \Psi_{nk}^{ij} + \sum_{t=1}^{j} \pi \frac{\partial^n}{\partial^n \rho} \cot \pi(\rho + \gamma_t + k) - \sum_{t=1}^{i} \pi \frac{\partial^n}{\partial^n \rho} \cot \pi(\rho + \gamma_t),$$

$$\Psi_{nk}^{ij} = \sum_{t=1+i}^{4} \varphi_n(\rho+\gamma_t) + \sum_{t=1}^{i} (-1)^n \varphi_n(1-\rho-\gamma_t) - \sum_{t=1+j}^{4} \varphi_n(\rho+\gamma_t+k)$$
$$- \sum_{t=1}^{j} (-1)^n \varphi_n(1-\rho-\gamma_t-k), \quad n=0,1,2; \; i,j=0,1,2,3, \tag{A.11}$$

$\varphi_n(z)$ is a polygamma function.

In the above development, the following formula has been used:

$$\varphi_n(1-z) + (-1)^{n+1}\varphi_n(z) = (-1)^n \pi \frac{\mathrm{d}^n}{\mathrm{d}z^n} \cot \pi z, \quad n=0,1,2\ldots . \tag{A.12}$$

The logarithmic solutions of Eqs. (A.8)–(A.10) are in a general form which cannot be used directly. The specific forms are be derived in the following:

A.1. $z_2(x)$

Under this case, $c_0 = c_0'(\rho+\gamma_2-1)$, where c_0' is an arbitrary constant independent of ρ, and $z_2(x)$ consists of two parts depending on the range of k. When $k \geqslant \gamma_2-\gamma_1$, using the relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \tag{A.13}$$

where z is an arbitrary complex number, gives

$$f_k(\Gamma)_{\rho=1-\gamma_2} = \frac{\prod_{i=2}^{4}\Gamma(\rho+\gamma_i)}{\prod_{i=1}^{4}\Gamma(\rho+\gamma_i+k)} \frac{1}{\rho+\gamma_2-1} \lim_{\rho\to 1-\gamma_2} \Gamma(\rho+\gamma_1)(\rho+\gamma_2-1)$$
$$= \frac{\prod_{i=2}^{4}\Gamma(1-\gamma_2+\gamma_i)}{\prod_{i=1}^{4}\Gamma(1-\gamma_2+\gamma_i+k)} \frac{(-1)^{1-\gamma_2+\gamma_1}}{\Gamma(\gamma_2-\gamma_1)} \frac{1}{\rho+\gamma_2-1}, \tag{A.14}$$

$$\left(\frac{1}{\rho+\gamma_2-1} c_0\right)_{\rho=1-\gamma_2} = c_0'. \tag{A.15}$$

The coefficient of ln x in Eq. (A.8) can be calculated as follows:

$$\left(\sum_{k=\gamma_2-\gamma_1}^{\infty} f_k(\Gamma) x^{\rho+k} c_0\right)_{\rho=1-\gamma_2} = c_0' \frac{(-1)^{1-\gamma_2+\gamma_1}\prod_{i=2}^{4}\Gamma(1-\gamma_2+\gamma_i)}{\Gamma(\gamma_2-\gamma_1)\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_i)} z_1(x), \tag{A.16}$$

c_0' is chosen to make the coefficient of $z_1(x)$ equal to 1. That is,

$$c_0' = \frac{\Gamma(\gamma_2-\gamma_1)\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_i)}{(-1)^{1-\gamma_2+\gamma_1}\prod_{i=2}^{4}\Gamma(1-\gamma_2+\gamma_i)}. \tag{A.17}$$

Since

$$\left(\frac{1}{\rho+\gamma_2-1}g_k^1\right)_{\rho=1-\gamma_2} = \frac{c_0'}{\rho+\gamma_2-1}+c_0'\Phi_{0k}^{10} = \frac{c_0'}{\rho+\gamma_2-1}+c_0'(\Psi_{0k}^{10}-\pi\ \cot\ \pi(\rho+1)) = c_0'\Psi_{0k}^{10} \tag{A.18}$$

the non-logarithmic terms in Eq. (A.8) can be obtained:

$$\left(\sum_{k=\gamma_2-\gamma_1}^{\infty} f_k(\Gamma)x^{\rho+k}g_k^1\right)_{\rho=1-\gamma_2} = \sum_{s=0}^{\infty} f_s(\Gamma)x^{\rho+\gamma_2-\gamma_1+s}c_0'\Psi_{0s}^{10} = x^{1-\gamma_1}\sum_{s=0}^{\infty}\Psi_{0s}^{10}x^s\prod_{i=1}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i+s)}. \tag{A.19}$$

When $0\leqslant k\leqslant \gamma_2-\gamma_1-1$,

$$f_k(\Gamma)_{\rho=1-\gamma_2} = \prod_{i=2}^{4}\frac{\Gamma(\rho+\gamma_i)}{\Gamma(\rho+\gamma_i+k)}\lim_{\rho\to 1-\gamma_2}\frac{\Gamma(\rho+\gamma_1)}{\Gamma(\rho+\gamma_1+k)} = (-1)^k\prod_{i=2}^{4}\frac{\Gamma(1-\gamma_2+\gamma_i)}{\Gamma(1-\gamma_2+\gamma_i+k)}\frac{\Gamma(\gamma_2-\gamma_1-k)}{\Gamma(\gamma_2-\gamma_1)}, \tag{A.20}$$

$$\begin{aligned}(g_k^1)_{\rho=1-\gamma_2} &= c_0'+c_0'(\rho+\gamma_2-1)\Phi_{0k}^{11}\\ &= c_0'+c_0'(\rho+\gamma_2-1)\left(\Psi_{0k}^{11}+\lim_{\rho\to 1-\gamma_2}(\pi\ \cot\ \pi(\rho+1+k)-\pi\cot\pi(\rho+1))\right)\\ &= c_0'+c_0'(\rho+\gamma_2-1)\Psi_{0k}^{11}\\ &= c_0'\end{aligned} \tag{A.21}$$

then

$$\sum_{k=0}^{\gamma_2-\gamma_1-1} f_k(\Gamma)x^{\rho+k}g_k^1 = \sum_{s=1}^{\gamma_2-\gamma_1} f_s(\Gamma)x^{\rho+\gamma_2-\gamma_1-s}g_s^1 = \prod_{i=2}^{4}(\gamma_i-\gamma_1) \times\frac{1}{x^{\gamma_1}}\,{}_5F_0\left([1,1,1+\gamma_1-\gamma_2,1+\gamma_1-\gamma_3,1+\gamma_1-\gamma_4],[\,],\frac{1}{x}\right), \tag{A.22}$$

$$(c_0)_{\rho=1-\gamma_2} = c_0'(\rho+\gamma_2-1)=0. \tag{A.23}$$

Hence

$$z_2(x) = z_1(x)\ln x + x^{1-\gamma_1}\sum_{s=0}^{\infty}\Psi_{0s}^{10}x^s\prod_{i=1}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i+s)} + \prod_{i=2}^{4}(\gamma_i-\gamma_1)\frac{1}{x^{\gamma_1}}\,{}_5F_0\left([1,1,1+\gamma_1-\gamma_2,1+\gamma_1-\gamma_3,1+\gamma_1-\gamma_4],[\,],\frac{1}{x}\right). \tag{A.24}$$

A.2. $z_3(x)$

Under this case $c_0 = c_0'(\rho+\gamma_3-1)^2$ where c_0' is an arbitrary constant independent of ρ. For $k \geqslant \gamma_3-\gamma_1$, the coefficient of $\ln^2 x$ in Eq. (A.9) can be calculated using

$$\left(\sum_{k=\gamma_3-\gamma_1}^{\infty} f_k(\Gamma)x^{\rho+k}c_0\right)_{\rho=1-\gamma_3} = \left(\sum_{s=0}^{\infty} f_s(\Gamma)x^{\rho+\gamma_3-\gamma_1+s}c_0\right)_{\rho=1-\gamma_3} = \frac{c_0'(-1)^{\gamma_2+\gamma_1}\Gamma(1-\gamma_3+\gamma_4)}{\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_i)\prod_{i=1}^{2}\Gamma(\gamma_3-\gamma_i)} z_1(x), \quad \text{(A.25)}$$

c_0' is chosen by making the coefficient of $z_1(x)$ equal to 1:

$$c_0' = \frac{\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_i)\prod_{i=1}^{2}\Gamma(\gamma_3-\gamma_i)}{(-1)^{\gamma_2+\gamma_1}\Gamma(1-\gamma_3+\gamma_4)} \quad \text{(A.26)}$$

then

$$\left(\sum_{k=\gamma_3-\gamma_1}^{\infty} f_k(\Gamma)x^{\rho+k}g_k^1\right)_{\rho=1-\gamma_3} = \left(\sum_{s=0}^{\infty} f_s(\Gamma)x^{\rho+\gamma_3-\gamma_1+s}g_s^1\right)_{\rho=1-\gamma_3} = x^{1-\gamma_1}\sum_{s=0}^{\infty}\Psi_{0s}^{20}x^s\frac{\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_i)}{\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_i+s)}, \quad \text{(A.27)}$$

$$\left(\sum_{k=\gamma_3-\gamma_1}^{\infty} f_k(\Gamma)x^{\rho+k}g_k^2\right)_{\rho=1-\gamma_3} = \left(\sum_{s=0}^{\infty} f_s(\Gamma)x^{\rho+\gamma_3-\gamma_1+s}g_s^2\right)_{\rho=1-\gamma_3} = x^{1-\gamma_1}\sum_{s=0}^{\infty}[(\Psi_{0s}^{20})^2+\Psi_{1s}^{20}+2\pi^2]x^s\frac{\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_i)}{\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_i+s)}. \quad \text{(A.28)}$$

When $\gamma_3-\gamma_2 \leqslant k \leqslant \gamma_3-\gamma_1-1$,

$$\left(\sum_{k=\gamma_3-\gamma_2}^{\gamma_3-\gamma_1-1} f_k(\Gamma)x^{\rho+k}g_k^1\right)_{\rho=1-\gamma_3} = \left(\sum_{s=1}^{\gamma_2-\gamma_1} f_s(\Gamma)x^{\rho+\gamma_3-\gamma_1-s}g_s^1\right)_{\rho=1-\gamma_3} = \prod_{i=2}^{4}(\gamma_i-\gamma_1)\frac{1}{x^{\gamma_1}}\,{}_5F_0\left([1,1,1+\gamma_1-\gamma_2,1+\gamma_1-\gamma_3,1+\gamma_1-\gamma_4],[]\frac{1}{x}\right), \quad \text{(A.29)}$$

$$\left(\sum_{k=\gamma_3-\gamma_2}^{\gamma_3-\gamma_1-1} f_k(\Gamma)x^{\rho+k}g_k^2\right)_{\rho=1-\gamma_3} = \left(\sum_{s=1}^{\gamma_2-\gamma_1} f_s(\Gamma)x^{\rho+\gamma_3-\gamma_1-s}g_s^2\right)_{\rho=1-\gamma_3} = 2x^{1-\gamma_2}\sum_{s=1}^{\gamma_2-\gamma_1}\Psi_{0s}^{21}x^{-s}\frac{(-1)^{1-s}\Gamma(s)\prod_{i=2}^{4}\Gamma(1-\gamma_1+\gamma_i)}{\prod_{i=2}^{4}\Gamma(1-\gamma_1+\gamma_i-s)}. \quad \text{(A.30)}$$

When $0 \leqslant k \leqslant \gamma_3 - \gamma_2 - 1$,

$$\left(\sum_{k=0}^{\gamma_3-\gamma_2-1} f_k(\Gamma) x^{\rho+k} g_k^2\right)_{\rho=1-\gamma_3} = \left(\sum_{s=1}^{\gamma_3-\gamma_2} f_s(\Gamma) x^{\rho+\gamma_3-\gamma_{2-s}} g_k^2\right)_{\rho=1-\gamma_3}$$
$$= 2(-1)^{\gamma_1+\gamma_2}\Gamma(1-\gamma_1+\gamma_2)\frac{\prod_{i=2}^{4}\Gamma(1-\gamma_1+\gamma_i)}{\prod_{i=3}^{4}\Gamma(\gamma_1-\gamma_2)}$$
$$\times \frac{1}{x^{\gamma_2}}\, {}_5F_0\left([1,1,1+\gamma_2-\gamma_1,1+\gamma_2-\gamma_3,1+\gamma_2-\gamma_4],[\,],\frac{1}{x}\right). \tag{A.31}$$

Then $z_3(x)$ can be expressed as

$$z_3(x) = \sum_{k=0}^{\infty} f_k(\Gamma) x^{\rho_3+k}(c_0 \ln^2 x + 2g_k^1 \ln x + g_k^2)$$
$$= 2\bar{z}_2(x)\ln x - z_1(x)\ln^2 x + x^{1-\gamma_1}\sum_{s=0}^{\infty}[(\Psi_{0s}^{20})^2 + \Psi_{1s}^{20} + 2\pi^2]x^s \frac{\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_i)}{\prod_{i=1}^{4}\Gamma(1-\gamma_1+\gamma_1+s)}$$
$$+ 2x^{1-\gamma_2}\sum_{s=1}^{\gamma_2-\gamma_1}(-1)^{1-s}\Gamma(s)x^{-s}\Psi_{0s}^{21}\prod_{i=2}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i-s)}$$
$$+ 2(-1)^{\gamma_1+\gamma_2}\Gamma(1-\gamma_1+\gamma_2)\frac{\prod_{i=2}^{4}\Gamma(1-\gamma_1+\gamma_i)}{\prod_{i=3}^{4}\Gamma(\gamma_i-\gamma_2)}$$
$$\times \frac{1}{x^{\gamma_2}}\, {}_5F_0\left([1,1,1+\gamma_2-\gamma_1,1+\gamma_2-\gamma_3,1+\gamma_2-\gamma_4],[\,],\frac{1}{x}\right), \tag{A.32}$$

where $\bar{z}_2(x)$ can be obtained by substituting Ψ_{**}^{2*} for Ψ_{**}^{1*} in $z_2(x)$ given by Eq. (A.24).

A.3. $z_4(x)$

Under this case $c_0 = c_0'(\rho + \gamma_4 - 1)^3$ where c_0' is an arbitrary constant independent of ρ. For $k \geqslant \gamma_4 - \gamma_1$, the coefficient of $\ln^3 x$ of Eq. (A.10) are calculated as

$$\left(\sum_{k=\gamma_4-1}^{\infty} f_k(\Gamma) x^{\rho+k} c_0\right)_{\rho=1-\gamma_4} = \frac{c_0'(-1)^{1+\gamma_1+\gamma_2+\gamma_3-\gamma_4}}{\prod_{i=1}^{3}\Gamma(\gamma_4-\gamma_i)\prod_{i=2}^{4}\Gamma(1-\gamma_1+\gamma_i)} z_1(x), \tag{A.33}$$

c_0' is chosen to make the coefficient of $z_1(x)$ equal to 1:

$$c_0' = (-1)^{1+\gamma_1+\gamma_2+\gamma_3-\gamma_4}\prod_{i=1}^{3}\Gamma(\gamma_4-\gamma_i)\prod_{i=2}^{4}\Gamma(1-\gamma_1+\gamma_i), \tag{A.34}$$

then

$$\left(\sum_{k=\gamma_4-\gamma_1}^{\infty} f_k(\Gamma)x^{\rho+k}g_k^3\right)_{\rho=1-\gamma_4} = \left(\sum_{s=0}^{\infty} f_s(\Gamma)x^{\rho+\gamma_4-\gamma_1+s}g_s^3\right)_{\rho=1-\gamma_4}$$

$$= x^{1-\gamma_1}\sum_{s=0}^{\infty}[(\Psi_{0s}^{30})^3 + \Psi_{2s}^{30} + 3\Psi_{0s}^{30}(\Psi_{1s}^{30} + 3\pi^2)]x^s \times \prod_{i=1}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i+s)}, \quad \text{(A.35)}$$

$$\left(\sum_{k=\gamma_4-\gamma_1}^{\infty} f_k(\Gamma)x^{\rho+k}g_k^2\right)_{\rho=1-\gamma_4} = \left(\sum_{s=0}^{\infty} f_s(\Gamma)x^{\rho+\gamma_4-\gamma_1+s}g_s^2\right)_{\rho=1-\gamma_4}$$

$$= x^{1-\gamma_1}\sum_{s=0}^{\infty}[(\Psi_{0s}^{30})^2 + \Psi_{1s}^{30} + 3\pi^2]x^s\prod_{i=1}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i+s)}, \quad \text{(A.36)}$$

$$\left(\sum_{k=\gamma_4-\gamma_1}^{\infty} f_k(\Gamma)x^{\rho+k}g_k^1\right)_{\rho=1-\gamma_4} = \left(\sum_{s=0}^{\infty} f_s(\Gamma)x^{\rho+\gamma_4-\gamma_1+s}g_s^1\right)_{\rho=1-\gamma_4}$$

$$= x^{1-\gamma_1}\sum_{s=0}^{\infty}\Psi_{0s}^{30}x^s\prod_{i=1}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i+s)}. \quad \text{(A.37)}$$

When $\gamma_4-\gamma_2 \leqslant k \leqslant \gamma_4-\gamma_1-1$,

$$\left(\sum_{k=\gamma_4-\gamma_2}^{\gamma_4-\gamma_1-1} f_k(\Gamma)x^{\rho+k}g_k^1\right)_{\rho=1-\gamma_4} = \left(\sum_{s=1}^{\gamma_2-\gamma_1} f_s(\Gamma)x^{\rho+\gamma_4-\gamma_1-s}g_s^1\right)_{\rho=1-\gamma_4}$$

$$= \prod_{i=2}^{4}(\gamma_i-\gamma_1)\frac{1}{x^{\gamma_1}} \times {}_5F_0\left([1,1,1+\gamma_1-\gamma_2,1+\gamma_1-\gamma_3,1+\gamma_1-\gamma_4],[]\frac{1}{x}\right), \quad \text{(A.38)}$$

$$\left(\sum_{k=\gamma_4-\gamma_2}^{\gamma_4-\gamma_1-1} f_k(\Gamma)x^{\rho+k}g_k^2\right)_{\rho=1-\gamma_4} = \left(\sum_{s=1}^{\gamma_2-\gamma_1} f_s(\Gamma)x^{\rho+\gamma_4-\gamma_1-s}g_s^2\right)_{\rho=1-\gamma_4}$$

$$= 2x^{1-\gamma_1}\sum_{s=1}^{\gamma_2-\gamma_1}\Gamma(s)(-1)^{1-s}x^{-s}\Psi_{0s}^{31}\prod_{i=2}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i-s)}, \quad \text{(A.39)}$$

$$\left(\sum_{k=\gamma_4-\gamma_2}^{\gamma_4-\gamma_1-1} f_k(\Gamma)x^{\rho+k}g_k^3\right)_{\rho=1-\gamma_4} = \left(\sum_{s=1}^{\gamma_2-\gamma_1} f_s(\Gamma)x^{\rho+\gamma_4-\gamma_1-s}g_s^3\right)_{\rho=1-\gamma_4}$$
$$= 3x^{1-\gamma_1}\sum_{s=1}^{\gamma_2-\gamma_1}\Gamma(s)(-1)^{1-s}x^{-s}[(\Psi_{0s}^{31})^2+\Psi_{1s}^{31}+2\pi^2]$$
$$\times\prod_{i=2}^{4}\frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i-s)}. \quad \text{(A.40)}$$

When $\gamma_4-\gamma_3 \leqslant k \leqslant \gamma_4-\gamma_2-1$,

$$\left(\sum_{k=\gamma_4-\gamma_3}^{\gamma_4-\gamma_2-1} f_k(\Gamma)x^{\rho+k}g_k^2\right)_{\rho=1-\gamma_4} = \left(\sum_{s=1}^{\gamma_3-\gamma_2} f_s(\Gamma)x^{\rho+\gamma_4-\gamma_2-s}g_s^2\right)_{\rho=1-\gamma_4}$$
$$= 2(-1)^{\gamma_1+\gamma_2}\Gamma(1-\gamma_1+\gamma_2)\frac{\prod_{i=2}^4\Gamma(1-\gamma_1+\gamma_i)}{\prod_{i=3}^4\Gamma(\gamma_i-\gamma_2)}$$
$$\times\frac{1}{x^{\gamma_2}}\,{}_5F_0\left([1,1,1+\gamma_2-\gamma_1,1+\gamma_2-\gamma_3,1+\gamma_2-\gamma_4],[\,],\frac{1}{x}\right), \quad \text{(A.41)}$$

$$\left(\sum_{k=\gamma_4-\gamma_3}^{\gamma_4-\gamma_2-1} f_k(\Gamma)x^{\rho+k}g_k^3\right)_{\rho=1-\gamma_4} = \left(\sum_{s=1}^{\gamma_3-\gamma_2} f_s(\Gamma)x^{\rho+\gamma_4-\gamma_2-s}g_s^3\right)_{\rho=1-\gamma_4}$$
$$= 6(-1)^{\gamma_1+\gamma_2}x^{1-\gamma_2}\prod_{i=2}^{4}\Gamma(1-\gamma_1+\gamma_i)$$
$$\times\sum_{s=1}^{\gamma_3-\gamma_2}\frac{\Gamma(\gamma_2-\gamma_1+s)\Gamma(s)x^{-s}\Psi_{0s}^{32}}{\prod_{i=3}^4\Gamma(1+\gamma_i-\gamma_2-s)}. \quad \text{(A.42)}$$

When $0 \leqslant k \leqslant \gamma_4-\gamma_3-1$,

$$\left(\sum_{k=0}^{\gamma_4-\gamma_3-1} f_k(\Gamma)x^{\rho+k}g_k^3\right)_{\rho=1-\gamma_4} = \left(\sum_{s=1}^{\gamma_4-\gamma_3} f_k(\Gamma)x^{\rho+\gamma_4-\gamma_3-s}g_s^3\right)_{\rho=1-\gamma_4}$$
$$= 6(-1)^{\gamma_1+\gamma_2}\frac{\prod_{i=2}^4\Gamma(1-\gamma_1+\gamma_i)\prod_{i=1}^2\Gamma(1-\gamma_i+\gamma_3)}{\Gamma(\gamma_4-\gamma_3)}$$
$$\times\frac{1}{x^{\gamma_3}}\,{}_5F_0\left([1,1,1+\gamma_3-\gamma_1,1+\gamma_3-\gamma_2,1+\gamma_3-\gamma_4],[\,],\frac{1}{x}\right) \quad \text{(A.43)}$$

then

$$
\begin{aligned}
z_4(x) = & \sum_{k=0}^{\infty} f_k(\Gamma) x^{\rho_4+k}(c_0 \ln^3 x + 3g_k^1 \ln^2 x + 3g_k^2 \ln x + g_k^3) \\
= & z_1(x)\ln^3 x - 3\bar{\bar{z}}_2(x)\ln^2 x + 3\bar{z}_3(x) \ln x \\
& + 6(-1)^{\gamma_1+\gamma_2} \frac{\prod_{i=2}^4 \Gamma(1-\gamma_1+\gamma_i)\prod_{i=1}^2 \Gamma(1-\gamma_i+\gamma_3)}{\Gamma(\gamma_4-\gamma_3)} \\
& \times \frac{1}{x^{\gamma_3}} {}_5F_0\left([1,1,1+\gamma_3-\gamma_1, 1+\gamma_3-\gamma_2, 1+\gamma_3-\gamma_4],[], \frac{1}{x}\right) \\
& + 6(-1)^{\gamma_1+\gamma_2} x^{1-\gamma_2} \prod_{i=2}^4 \Gamma(1-\gamma_1+\gamma_i) \sum_{s=1}^{\gamma_3-\gamma_2} \frac{\Gamma(\gamma_2-\gamma_1+s)\Gamma(s)x^{-s}\Psi_{0s}^{32}}{\prod_{i=3}^4 \Gamma(1+\gamma_i-\gamma_2-s)} \\
& + 3x^{1-\gamma_1} \sum_{s=1}^{\gamma_2-\gamma_1} \Gamma(s)(-1)^{1-s} x^{-s}[(\Psi_{0s}^{31})^2 + \Psi_{1s}^{31} + 2\pi^2] \prod_{i=2}^4 \frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i-s)} \\
& + x^{1-\gamma_1} \sum_{s=0}^{\infty} x^s[(\Psi_{0s}^{30})^3 + \Psi_{2s}^{30} + 3\Psi_{0s}^{30}(\Psi_{1s}^{30}+3\pi^2)] \prod_{i=1}^4 \frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i+s)},
\end{aligned}
\tag{A.44}
$$

where $\bar{\bar{z}}_2(x)$ can be obtained by substituting Ψ_{**}^{3*} for Ψ_{**}^{1*} in $z_2(x)$ given by Eq. (A.24) and $\bar{z}_3(x)$ can be obtained by substituting Ψ_{**}^{3*} and $3\pi^2$ for Ψ_{**}^{2*} and $2\pi^2$ respectively in $z_3(x)$ given by Eq. (A.32).

A.4. Convergence conditions

For checking the convergence condition of logarithmic solutions $z_2(x)$, $z_3(x)$ and $z_4(x)$ given by Eqs. (A.24), (A.32) and (A.44), respectively, the infinite series included in $z_2(x)$, $z_3(x)$ and $z_4(x)$ expressed by Eqs. (A.19), (A.27), (A.28), (A.35), (A.36) and (A.37) will be checked. These infinite series can be expressed in the general form:

$$
x^{1-\gamma_1} \sum_{s=0}^{\infty} x^s \left[\sum \Psi_{*s}^{**}\right] \prod_{i=1}^4 \frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i+s)}, \tag{A.45}
$$

where $\sum \Psi_{*s}^{**}$ is the summation of polygamma functions. Let u_s denotes the s term in Eq. (A.45) and define

$$
d = \lim_{s\to\infty} \frac{u_{s+1}}{u_s} = \lim_{s\to\infty} \frac{\sum \Psi_{*s+1}^{**}}{\sum \Psi_{*s}^{**}} \frac{x}{\prod_{i=1}^4 (1-\gamma_1+\gamma_i+s)}. \tag{A.46}
$$

Since $\lim_{s\to\infty}(\sum \Psi_{*s+1}^{**}/\sum \Psi_{*s}^{**}) = 1$, hence if x is finite, $d<1$. Then the infinite series in Eq. (A.45) converge for all finite x (i.e. all cases except for $m \neq 2$).

To illustrate, Ψ_{0s}^{10} in relation to $\sum \Psi_{*s}^{**}$ in Eq. (A.19) will be used as an example to prove that $\lim_{s\to\infty}(\sum \Psi_{*s+1}^{**}/\sum \Psi_{*s}^{**}) = 1$. The proofs for other $\sum \Psi_{*s}^{**}$ terms appearing in Eqs. (A.27), (A.28), (A.35), (A.36), and (A.37) follow the same procedure. Ψ_{0s}^{10} can be expressed according to Eq. (A.11) as

$$\Psi_{0s}^{10} = \sum_{t=2}^{4} \varphi_0(1-\gamma_2+\gamma_t) + \varphi_0(\gamma_2-\gamma_1) - \sum_{t=1}^{4} \varphi_0(1-\gamma_1+\gamma_t+s). \tag{A.47}$$

Since

$$\varphi_n(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}, \tag{A.48}$$

$$\Psi_{0s}^{10} = \sum_{k=0}^{\infty} \left[\sum_{t=1}^{4} \frac{1}{1-\gamma_1+\gamma_t+s+k} - \sum_{t=2}^{4} \frac{1}{1-\gamma_2+\gamma_t+k} - \frac{1}{\gamma_2-\gamma_1+k} \right], \tag{A.49}$$

then

$$\lim_{s\to\infty} \frac{\Psi_{0s+1}^{10}}{\Psi_{0s}^{10}} = 1. \tag{A.50}$$

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