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Dynamic analysis of an inclined beam due to moving loads

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Abstract

It has been found that if each of the moving loads on the beam is considered as a *moving mass element*, then one can easily formulate the problem with all the pertinent factors relating to the multiple moving loads considered. To this end, the property matrices of the *moving mass element* are derived by taking account of the effects of inertial force, Coriolis force and centrifugal force induced by the moving mass. Combination of the element property matrices for each of the moving loads and the associated overall property matrices for the inclined beam itself determines the overall effective property matrices of the entire vibrating system. Since the property matrices of each moving mass element are dependent on the instantaneous position of the moving load on the inclined beam, they are time-dependent and so are the overall effective mass, damping and stiffness matrices of the entire vibrating system. To validate the presented theory, the dynamic responses of a horizontal pinned–pinned beam subjected to a moving load are determined and compared with those of the existing literature and good agreement is achieved. Finally, the following factors having something to do with the title problem are studied: the moving-load speed, the Coriolis force, centrifugal force, the frictional force, the inclined angle of the beam and the total number of moving loads. Numerical results reveal that all the above-mentioned parameters have significant influence on both the vertical ($\bar{y}$) and the horizontal ($\bar{x}$) dynamic responses of the inclined beam except the frictional force.

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1. Introduction

The literature concerning the moving-load-induced-vibration problem is numerous. For example, Xu et al. [1] have used the finite difference method and the perturbation technique to study the longitudinal and transverse motions of a finite elastic beam subjected to a moving mass. Michaltsos et al. [2] have studied the influence of mass and velocity of the moving load on the dynamic response of a simply supported beam. Esmailzadeh and Ghorashi [3,4] have investigated the dynamic behavior of a beam traversed by uniformly and partially distributed moving masses. Mofid and Shadnam [5], Cifuentes [6], Frýba [7], Foda and Abduljabbar [8], Akin and Mofid [9], Ichikawa et al. [10], Wu [11], Stanisik and Hardin [12] and Beneditti [13] have performed the dynamic analysis of beams due to moving masses by means of various analytical or numerical methods.

From the review of the existing literature, it is found that all the above-mentioned researches are limited to the cases that the external load is moving on the *horizontal* beams. If the last horizontal beams are replaced by the *inclined* ones, then the approaches presented in the foregoing researches cannot be directly applied to solve the problem. For this reason, this paper presents the concept of *moving mass element*, so that one can easily determine the dynamic characteristics of the inclined beams subjected to moving loads, with the effects of inertia force, Coriolis force and centrifugal force considered.

Firstly, under the assumption of all the moving loads being always in close contact with the inclined beam, each of the moving loads is considered as a *moving mass element* and the related element property matrices are derived based on the local (xy) coordinates of the beam element on which the moving load applies. Next, the last matrices are transformed into those with respect to the global ($\bar{x}\bar{y}$) coordinates of the entire vibrating system. It has been found that, using the presented concept, one can easily take account of the effects of inertia force, Coriolis force and centrifugal force induced by all the moving loads by directly adding the last property matrices of each moving mass element to the overall ones of the entire inclined beam itself. Because the property matrices of each *moving mass element* are dependent on the instantaneous position of the moving load on the inclined beam, they are time-dependent and so are the overall effective mass, damping and stiffness matrices of the entire vibrating system.

Before the title problem is studied, the dynamic responses of a horizontal pinned–pinned beam subjected to a moving load are determined and compared with those of the existing literature and good agreement is achieved. In addition to present the theory regarding the *moving mass element*, the factors having something to do with the title problem are also investigated. Among which, the moving-load speed, the Coriolis force, the centrifugal force, the inclined angle of the beam and the total number of moving loads are found to have significant influence on both the vertical ($\bar{y}$) and the horizontal ($\bar{x}$) dynamic responses of the inclined beam. However, the effect of frictional force is negligible.

2. Property matrices of a moving mass element

Fig. 1(a) shows a concentrated mass m_c moving on an inclined beam element. If, at any instant of time, the moving mass m_c is located at point i of the beam element, then the interaction forces

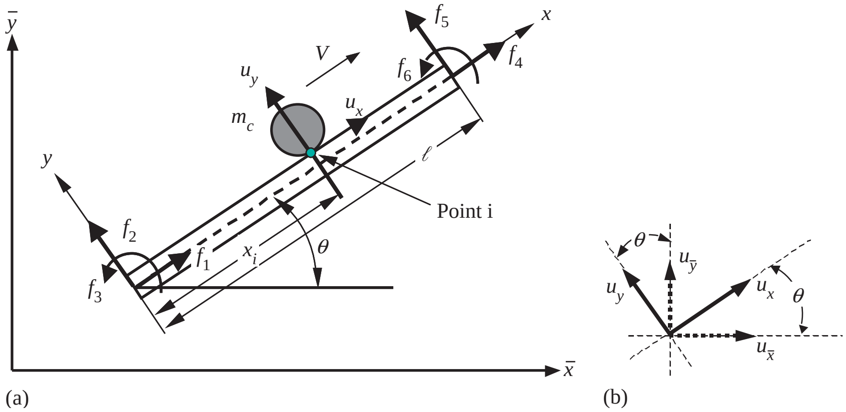


Fig. 1. (a) The equivalent nodal forces (f_1–f_6) of an inclined beam element due to a moving concentrated mass m_c; (b) For any point on the inclined beam, its displacement components in the x and y directions (u_x and u_y) and those in the $\bar{x}$ and $\bar{y}$ directions ($u_{\bar{x}}$ and $u_{\bar{y}}$) are related by $u_{\bar{x}} = u_x \cos\theta - u_y \sin\theta$ and $u_{\bar{y}} = u_x \sin\theta + u_y \cos\theta$.

in the x and y directions, induced by the moving mass m_c, are respectively given by [6]

$$F_x = m_c \ddot{u}_x, \tag{1a}$$

$$F_y = m_c(\ddot{u}_y + 2V\dot{u}'_y + V^2 u''_y), \tag{1b}$$

where the overhead dot ($\cdot$) and prime ($'$) represent the differentiations with respect to (w.r.t.) time t and coordinate x, respectively, V represents the velocity of the moving mass m_c in the local x direction, u_x and u_y represent the displacement components of contact point i in the local x and y directions of the beam element, respectively, while $m_c\ddot{u}_y$, $2m_c V\dot{u}'_y$ and $m_c V^2 u''_y$ represent the inertia force, Coriolis force and centrifugal force (due to the fact that the mass is moving along the deformed shape of the beam), respectively. Note that Eq. (1) is obtained under the assumption that the moving mass and the beam element are always in close contact.

For an arbitrary point on the inclined beam, its displacement components in the local x and y directions, u_x and u_y, and those in the global $\bar{x}$ and $\bar{y}$ directions, $u_{\bar{x}}$ and $u_{\bar{y}}$, have the following relationships (cf. Fig. 1(b)):

$$u_{\bar{x}} = u_x \cos\theta - u_y \sin\theta, \tag{1c}$$

$$u_{\bar{y}} = u_x \sin\theta + u_y \cos\theta. \tag{1d}$$

It is noted that u_x and u_y are in the axial and transverse directions of the inclined beam, while $u_{\bar{x}}$ and $u_{\bar{y}}$ are in the horizontal and vertical directions, respectively. Eqs. (1c) and (1d) are two important expressions, because based on which most of the phenomena appearing in the numerical results may be easily explained.

The equivalent nodal forces of the beam element induced by the two forces given by Eqs. (1a) and (1b) are given by (cf. Fig. 1(a)) [14]

$$f_k = \phi_k F_x \quad (k = 1, 4), \tag{2a}$$

$$f_k = \phi_k F_y \quad (k = 2, 3, 5, 6), \tag{2b}$$

where ϕ_k ($k = 1$ to 6) are the shape functions defined by [14]

$$\phi_1 = 1 - \varsigma,$$

$$\phi_2 = 1 - 3\varsigma^2 + 2\varsigma^3,$$

$$\phi_3 = (\varsigma - 2\varsigma^2 + \varsigma^3)\ell,$$

$$\phi_4 = \varsigma,$$

$$\phi_5 = 3\varsigma^2 - 2\varsigma^3,$$

$$\phi_6 = (-\varsigma^2 + \varsigma^3)\ell \tag{3a}$$

with

$$\varsigma = \frac{x_i}{\ell}. \tag{3b}$$

In the last expressions, ℓ represents the total length of the beam element on which the moving mass m_c applies, while x_i represents the local x coordinate of the moving mass m_c with respect to the left end of the beam element (see Fig. 1(a)).

Based on the definition of shape functions [15], the displacement components of the contact point i in the x and y directions, u_x and u_y, can be obtained from

$$u_x = \phi_1 u_1 + \phi_4 u_4, \tag{4a}$$

$$u_y = \phi_2 u_2 + \phi_3 u_3 + \phi_5 u_5 + \phi_6 u_6, \tag{4b}$$

where u_i ($i = 1, \ldots, 6$) are the nodal displacements of the beam element on which the moving mass m_c applies.

The time derivatives of Eq. (4) give

$$\ddot{u}_x = \phi_1 \ddot{u}_1 + \phi_4 \ddot{u}_4, \tag{5a}$$

$$\ddot{u}_y = \phi_2 \ddot{u}_2 + \phi_3 \ddot{u}_3 + \phi_5 \ddot{u}_5 + \phi_6 \ddot{u}_6 \tag{5b}$$

Substituting Eq. (5) into Eqs. (1a), (1b) and (2), and writing the resulting expressions in matrix form yield

$$\{f\} = [m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\}, \tag{6}$$

where

$$\{f\} = [f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6]^{\mathrm{T}}, \tag{7a}$$

$$\{\ddot{u}\} = [\ddot{u}_1 \quad \ddot{u}_2 \quad \ddot{u}_3 \quad \ddot{u}_4 \quad \ddot{u}_5 \quad \ddot{u}_6]^{\mathrm{T}}, \tag{7b}$$

$$\{\dot{u}\} = [\dot{u}_1 \quad \dot{u}_2 \quad \dot{u}_3 \quad \dot{u}_4 \quad \dot{u}_5 \quad \dot{u}_6]^{\mathrm{T}}, \tag{7c}$$

$$\{u\} = [u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6]^{\mathrm{T}}, \tag{7d}$$

$$[m] = m_c \begin{bmatrix} \phi_1^2 & 0 & 0 & \phi_1\phi_4 & 0 & 0 \\ 0 & \phi_2^2 & \phi_2\phi_3 & 0 & \phi_2\phi_5 & \phi_2\phi_6 \\ 0 & \phi_3\phi_2 & \phi_3^2 & 0 & \phi_3\phi_5 & \phi_3\phi_6 \\ \phi_4\phi_1 & 0 & 0 & \phi_4^2 & 0 & 0 \\ 0 & \phi_5\phi_2 & \phi_5\phi_3 & 0 & \phi_5^2 & \phi_5\phi_6 \\ 0 & \phi_6\phi_2 & \phi_6\phi_3 & 0 & \phi_6\phi_5 & \phi_6^2 \end{bmatrix}, \tag{8a}$$

$$[c] = 2m_c V \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_2\phi_2' & \phi_2\phi_3' & 0 & \phi_2\phi_5' & \phi_2\phi_6' \\ 0 & \phi_3\phi_2' & \phi_3\phi_3' & 0 & \phi_3\phi_5' & \phi_3\phi_6' \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_5\phi_2' & \phi_5\phi_3' & 0 & \phi_5\phi_5' & \phi_5\phi_6' \\ 0 & \phi_6\phi_2' & \phi_6\phi_3' & 0 & \phi_6\phi_5' & \phi_6\phi_6' \end{bmatrix}, \tag{8b}$$

$$[k] = m_c V^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_2\phi_2'' & \phi_2\phi_3'' & 0 & \phi_2\phi_5'' & \phi_2\phi_6'' \\ 0 & \phi_3\phi_2'' & \phi_3\phi_3'' & 0 & \phi_3\phi_5'' & \phi_3\phi_6'' \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_5\phi_2'' & \phi_5\phi_3'' & 0 & \phi_5\phi_5'' & \phi_5\phi_6'' \\ 0 & \phi_6\phi_2'' & \phi_6\phi_3'' & 0 & \phi_6\phi_5'' & \phi_6\phi_6'' \end{bmatrix}. \tag{8c}$$

For convenience of application of the finite element method (FEM), two coordinate systems are introduced in Fig. 1(a). Where xy represents the local coordinate system of the beam element and $\bar{x}\bar{y}$ represents the global one of the entire vibrating system. If u_i $(i = 1, \ldots, 6)$ denote the nodal displacements w.r.t. the local xy coordinate system and $\bar{u}_i$ $(i = 1, \ldots, 6)$ denote the corresponding ones w.r.t. the global $\bar{x}\bar{y}$ coordinate system, then according to Ref. [16], one has

$$\{u\} = [T]\{\bar{u}\}, \tag{9}$$

where

$$\{u\} = [u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6]^{\mathrm{T}}, \tag{10a}$$

$$\{\bar{u}\} = [\bar{u}_1 \quad \bar{u}_2 \quad \bar{u}_3 \quad \bar{u}_4 \quad \bar{u}_5 \quad \bar{u}_6]^{\mathrm{T}}, \tag{10b}$$

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

In the last expression, θ is the inclined angle of the beam element (see Fig. 1(a)), while $[T]$ is the transformation matrix between the local xy coordinate system and the global $\bar{x}\bar{y}$ one.

Similarly, the nodal forces in local xy coordinate system, f_i $(i = 1, \ldots, 6)$, can be transformed into those in global $\bar{x}\bar{y}$ coordinate system, $\bar{f}_i$ $(i = 1, \ldots, 6)$, by using

$$\{f\} = [T]\{\bar{f}\}, \tag{12}$$

where $\{f\}$ and $[T]$ are given by Eqs. (7a) and (11), respectively, while $\{\bar{f}\}$ takes the form

$$\{\bar{f}\} = [\bar{f}_1 \quad \bar{f}_2 \quad \bar{f}_3 \quad \bar{f}_4 \quad \bar{f}_5 \quad \bar{f}_6]^{\mathrm{T}}. \tag{13}$$

Differentiating Eq. (9) w.r.t. time yields

$$\{\dot{u}\} = [T]\{\dot{\bar{u}}\}, \tag{14a}$$

$$\{\ddot{u}\} = [T]\{\ddot{\bar{u}}\}, \tag{14b}$$

where

$$\{\dot{\bar{u}}\} = [\dot{\bar{u}}_1 \quad \dot{\bar{u}}_2 \quad \dot{\bar{u}}_3 \quad \dot{\bar{u}}_4 \quad \dot{\bar{u}}_5 \quad \dot{\bar{u}}_6]^{\mathrm{T}}, \tag{15a}$$

$$\{\ddot{\bar{u}}\} = [\ddot{\bar{u}}_1 \quad \ddot{\bar{u}}_2 \quad \ddot{\bar{u}}_3 \quad \ddot{\bar{u}}_4 \quad \ddot{\bar{u}}_5 \quad \ddot{\bar{u}}_6]^{\mathrm{T}}. \tag{15b}$$

Introducing Eqs. (9), (12) and (14) into Eq. (6) leads to

$$\{\bar{f}\} = [\bar{m}]\{\ddot{\bar{u}}\} + [\bar{c}]\{\dot{\bar{u}}\} + [\bar{k}]\{\bar{u}\}, \tag{16}$$

where

$$[\bar{m}] = [T]^{\mathrm{T}}[m][T], \tag{17a}$$

$$[\bar{c}] = [T]^{\mathrm{T}}[c][T], \tag{17b}$$

$$[\bar{k}] = [T]^{\mathrm{T}}[k][T]. \tag{17c}$$

In Eq. (17), the right superscript T of $[T]$ denotes the transpose of the matrix $[T]$, while $[\bar{m}]$, $[\bar{c}]$ and $[\bar{k}]$ are the mass, damping and stiffness matrices of the *moving mass element* w.r.t. the global $\bar{x}\bar{y}$ coordinate system. Because the numerical values of the last property matrices for a moving mass element are dependent on the shape functions of the beam element on which the associated moving mass m_c applies (see Eq. (8)), they are to vary with the position of moving mass m_c on the beam.

3. Equations of motion of the entire vibrating system

For a multiple-degree-of-freedom damped structural system, its equation of motion is given by

$$[\bar{M}(t)]\{\ddot{\bar{q}}(t)\} + [\bar{C}(t)]\{\dot{\bar{q}}(t)\} + [\bar{K}(t)]\{\bar{q}(t)\} = \{\bar{F}(t)\}, \tag{18}$$

where $[\bar{M}(t)]$, $[\bar{C}(t)]$ and $[\bar{K}(t)]$ are the *instantaneous* overall mass, damping and stiffness matrices, respectively; $\{\ddot{\bar{q}}(t)\}$, $\{\dot{\bar{q}}(t)\}$ and $\{\bar{q}(t)\}$ are the acceleration, velocity and displacement vectors, respectively; while $\{\bar{F}(t)\}$ is the *instantaneous* external force vector. It is worthy of mention that the symbols $[\bar{M}(t)]$, $[\bar{C}(t)]$ and $[\bar{K}(t)]$ in Eq. (18) are called the *instantaneous* matrices, because they are time-dependent and composed of the *constant* overall mass and stiffness matrices of the entire inclined beam itself and the *time-dependent* element property matrices of the moving mass element. Since the position of the moving concentrated mass m_c changes from time to time, $\{\bar{F}(t)\}$ is also a *time-dependent* external force vector.

3.1. Overall property matrices

To take the effects of inertia force and centrifugal force of the moving load into account, one must add the contribution of the mass and stiffness matrices of the moving mass element, $[\bar{m}]$ and $[\bar{k}]$, to the overall corresponding ones of the entire inclined beam itself, $[M_b]$ and $[K_b]$. In other words, the *instantaneous* overall mass matrix $[\bar{M}_{(t)}]$ and stiffness matrix $[\bar{K}_{(t)}]$ of the entire vibrating system are established by

$$[\bar{M}_{(t)}]_{n\times n} = [M_b]_{n\times n} + [\bar{m}]_{6\times 6}, \tag{19a}$$

$$[\bar{K}_{(t)}]_{n\times n} = [K_b]_{n\times n} + [\bar{k}]_{6\times 6}, \tag{19b}$$

where

$$\bar{M}_{ij} = M_{b,ij} \quad (i,j = 1,\ldots,n), \tag{20a}$$

$$\bar{K}_{ij} = K_{b,ij} \quad (i,j = 1,\ldots,n) \tag{20b}$$

except

$$\bar{M}_{s_i s_j} = M_{b,s_i s_j} + \bar{m}_{ij} \quad (i,j = 1,\ldots,6), \tag{21a}$$

$$\bar{K}_{s_i s_j} = K_{b,s_i s_j} + \bar{k}_{ij} \quad (i,j = 1,\ldots,6). \tag{21b}$$

In the last equations, n represents the total degrees of freedom of the entire vibrating system, $[M_b]$ and $[K_b]$ represent the overall mass and stiffness matrices of the inclined beam obtained by assembling all its element mass and stiffness matrices [16], respectively, while the subscripts s_i and s_j $(i,j = 1,\ldots,6)$ represent the numberings for the 6 dof of the two nodes of the beam element on which the moving mass m_c applies at time t.

Since it is difficult to find the damping matrices of the structural elements from the existing literature, the overall damping matrix $[C_b]$ of the inclined beam is determined by using the theory of Rayleigh damping [17]

$$[C_b] = a[\bar{M}_{(t)}] + b[\bar{K}_{(t)}] \tag{22a}$$

with

$$a = \frac{2\omega_i\omega_j(\xi_i\omega_j - \xi_j\omega_i)}{\omega_j^2 - \omega_i^2}, \tag{22b}$$

$$b = \frac{2(\xi_j\omega_j - \xi_i\omega_i)}{\omega_j^2 - \omega_i^2}, \tag{22c}$$

where $[\bar{M}_{(t)}]$ and $[\bar{K}_{(t)}]$ are respectively the overall mass and stiffness matrices given by Eqs. (19a) and (19b), while ξ_i and ξ_j are damping ratios corresponding to any two natural frequencies of the structure, ω_i and ω_j.

If the Coriolis force induced by the moving mass m_c is considered, one must add the contribution of the damping matrix of the moving mass element, $[\bar{c}]$, to the overall damping matrix of the inclined beam itself, $[C_b]$, to establish the *instantaneous* overall damping matrix, $[\bar{C}_{(t)}]$, i.e.,

$$[\bar{C}_{(t)}]_{n\times n} = [C_b]_{n\times n} + [\bar{c}]_{6\times 6}, \tag{23a}$$

where

$$\bar{C}_{ij} = C_{b,ij} \quad (i,j = 1,\ldots,n) \tag{23b}$$

except

$$\bar{C}_{s_i s_j} = C_{b,s_i s_j} + \bar{c}_{ij} \quad (i,j = 1,\ldots,6). \tag{23c}$$

3.2. Equivalent nodal forces and overall external force vector

The equivalent force vector $\mathbf{P}$ induced by the moving concentrated mass m_c at any time t is given by

$$\mathbf{P} = \mathbf{i}P_x + \mathbf{j}P_y, \tag{24}$$

where $\mathbf{i}$ and $\mathbf{j}$ are respectively the unit vectors in the local x and y directions (see Fig. 2), while P_x and P_y are the corresponding force components given by

$$P_x = -m_c g \sin\theta - F_f, \tag{25a}$$

$$P_y = -m_c g \cos\theta. \tag{25b}$$

In the last expressions, g is the acceleration of gravity and θ is the inclined angle of the beam. Besides, F_f is the frictional force at the contact point i between the moving load and the inclined beam. The force equilibrium along the inclined beam in the x direction requires that

$$F_f = \mu m_c g \cos\theta = m_c g \sin\theta. \tag{26a}$$

Thus,

$$\mu = \tan\theta, \tag{26b}$$

where μ is the friction coefficient. It is evident that Eq. (26a) must be satisfied then a load may move along an inclined beam with a constant speed V. Note that Eq. (26b) is obtained under the

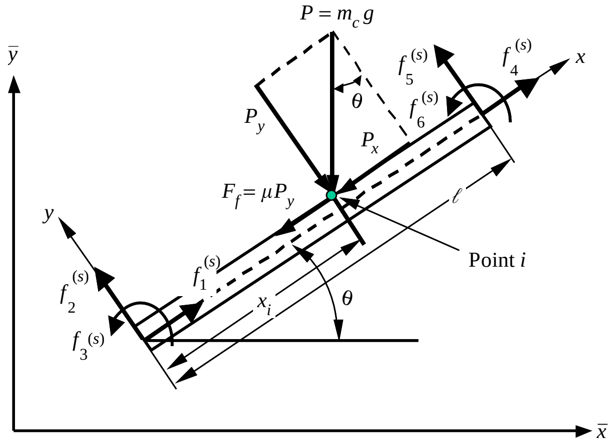


Fig. 2. The equivalent nodal forces ($f_1^{(s)}$–$f_6^{(s)}$) of an inclined beam element on which the moving load $P = m_c g$ applies.

assumption that there exists no sliding between the concentrated mass m_c and the inclined beam. For a horizontal beam (i.e., $\theta = 0°$), the effect of frictional force is usually neglected. This is equivalent to $\mu = \tan 0° = 0$. In other words, Eq. (26b) is available for either the inclined beam or horizontal beam.

The equivalent force vector **P** given by Eq. (24), with its components given by Eqs. (25) and (26), changes its location on the inclined beam from time to time. For convenience of the finite element analysis, it is replaced by an *equivalent nodal force vector*

$$\{f^{(s)}\} = [f_1^{(s)} \quad f_2^{(s)} \quad \cdots \quad f_6^{(s)}]^{\mathrm{T}} \tag{27a}$$

with

$$f_k^{(s)} = \phi_k P_x \quad (k = 1, 4), \tag{27b}$$

$$f_k^{(s)} = \phi_k P_y \quad (k = 2, 3, 5, 6), \tag{27c}$$

where the superscript s refers to the numbering of the beam element on which the moving mass m_c applies, while $\phi_k (k = 1, \ldots, 6)$ are the shape functions given by Eq. (3).

The equivalent nodal forces given by Eq. (27) are in the local xy coordinate of the beam element, they must be transformed into the global $\bar{x}\bar{y}$ coordinate of the entire vibrating system before assembling by using

$$\{\bar{f}^{(s)}\} = [T]^{\mathrm{T}}\{f^{(s)}\}, \tag{28a}$$

where

$$\{\bar{f}^{(s)}\} = [\bar{f}_1^{(s)} \quad \bar{f}_2^{(s)} \quad \cdots \quad \bar{f}_6^{(s)}]^{\mathrm{T}}. \tag{28b}$$

It is noted that Eq. (28a) is derived from Eq. (12) with $[T]^{-1} = [T]^{\mathrm{T}}$.

Since all the nodal forces of the entire vibrating system are equal to zero except those at the two nodes of the sth beam element on which the moving mass m_c applies, the overall external force vector $\{\bar{F}(t)\}$ in Eq. (18) takes the form:

$$\{\bar{F}(t)\} = [0 \quad \cdots \quad \bar{f}_1^{(s)} \quad \bar{f}_2^{(s)} \quad \bar{f}_3^{(s)} \quad \bar{f}_4^{(s)} \quad \bar{f}_5^{(s)} \quad \bar{f}_6^{(s)} \cdots 0]^{\mathrm{T}}, \tag{29}$$

where $\bar{f}_i^{(s)}$ $(i = 1, \ldots, 6)$ are the nodal forces equivalent to $\mathbf{P}$ in global $\bar{x}\bar{y}$ coordinate of the entire vibrating system and are determined by Eq. (28). They are the s_ith coefficients of $\{\bar{F}(t)\}$, where s_i, $i = 1, \ldots, 6$, represent the numberings for the 6 dof of the beam element on which the moving mass m_c applies at time t.

4. Dynamic responses of the inclined beam due to moving loads

If the effects of inertia force, Coriolis force and centrifugal force induced by the moving loads are considered, then the dynamic responses of the inclined beam subjected to the moving loads may be obtained with the following steps:

1. Determine the transformation matrix $[T]$ with Eq. (11).
2. Calculate the instantaneous mass, damping and stiffness matrices, $[\bar{m}]$, $[\bar{c}]$ and $[\bar{k}]$, of the moving mass element with Eq. (17).
3. Determine the *instantaneous* overall mass and stiffness matrices, $[\bar{M}_{(t)}]$ and $[\bar{K}_{(t)}]$, (see Eqs. (19)–(21)), the *instantaneous* overall damping matrix $[\bar{C}_{(t)}]$ (see Eqs. (22)–(23)) and the *instantaneous* overall force vector $\{\bar{F}(t)\}$ at time t (see Eqs. (24)–(29)).
4. Determine the dynamic responses of the inclined beam by solving for the equation of motion, Eq. (18), with the Newmark direct integration method [17].
5. Repeat steps 1–4 to obtain the dynamic responses of the inclined beam at time $t = t_r = t_{r-1} + \Delta t$ (with $r = 1, 2, \ldots$ and $t_0 = 0$), where Δt is the time interval.

5. Numerical results and discussions

The size and physical constants for the uniform undamped pinned–pinned beam studied in this paper are (cf. Figs. 3, 5, 13): the cross-section is rectangular with width $b = 0.018113\,\mathrm{m}$ and

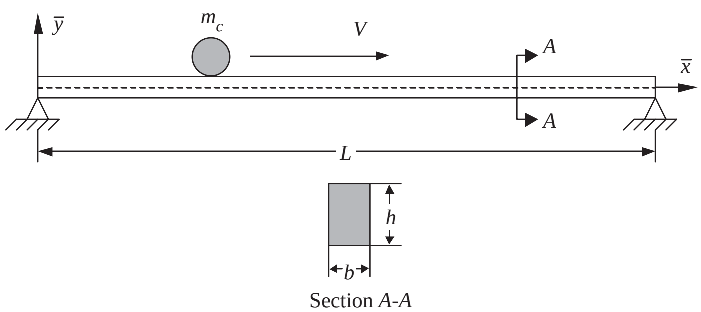


Fig. 3. A horizontal undamped pinned–pinned beam, with total length $L = 4.352\,\mathrm{m}$, width $b = 0.018113\,\mathrm{m}$ and thickness $h = 0.072322\,\mathrm{m}$, subjected to a concentrated mass $m_c = 21.8\,\mathrm{kg}$ moving from the left end to the right end of the beam with a constant speed $V = 27.49\,\mathrm{m/s}$.

thickness $h = 0.072322\,\text{m}$, the moment of inertia is $I = bh^3/12 = 5.71 \times 10^{-7}\,\text{m}^4$, total length is $L = 4.352\,\text{m}$, mass density is $\rho = 15267.1756\,\text{kg/m}^3$ and Young's modulus is $E = 205.9936 \times 10^8\,\text{kg/m}^2 = 2020.797216 \times 10^8\,\text{N/m}^2$. All the numerical results presented in this paper are obtained based on the acceleration of gravity $g = 9.81\,\text{m/s}^2$, the time interval $\Delta t = 0.001\,\text{s}$ and the friction coefficient $\mu = \tan\theta$ with θ representing the inclined angle of the beam. Besides, the overall damping matrix of the entire vibrating system, $[\bar{C}_{(t)}]$, is obtained based on the damping ratios $\xi_1 = \xi_2 = 0.005$ and the associated natural frequencies ω_1 and ω_2. For convenience of comparison, the dimensions and material properties of the beam studied in this paper are chosen to be identical to those of Ref. [6].

5.1. Validation

Although the presented technique is developed for the dynamic analysis of the *inclined* beam subjected to moving loads, it is also available for that of the *horizontal* beam if the inclined angle of the beam element, θ, in transformation matrix $[T]$ given by Eq. (11) is taken to be zero. In this subsection, the undamped pinned–pinned beam with inclined angle $\theta = 0^\circ$ (i.e., an horizontal beam) and subjected to a moving load with magnitude $m_c = 21.8\,\text{kg}$ and constant speed $V = 27.49\,\text{m/s}$ (see Fig. 3) is studied, and then the formulation presented and the computer programs developed for this paper are validated by comparing the dynamic responses of the beam with those of the existing literature [6]. The finite element model of the beam is composed of 14 identical beam elements and 15 nodes.

Fig. 4 shows the time histories for the vertical ($\bar{y}$) displacements of the contact point between the moving mass and the horizontal beam. In which, the solid curve with cross (—+—) represents the vertical ($\bar{y}$) displacements obtained by using the presented concept of *moving mass element*, while the dashed curve (- - - -) represents those obtained from Ref. [6]. It is seen that the differences between the last two curves are negligible.

5.2. Influence of moving-load speed

The present example is the same as the last one except that the inclined angle is $\theta = 30^\circ$ (see Fig. 5). The concentrated mass $m_c = 21.8\,\text{kg}$ is assumed to move from the lower end to the upper end of the beam with constant speeds $V = 5.0, 10.0$ and $20.0\,\text{m/s}$. For convenience, the moving mass m_c is assumed to be properly propelled, so that it can move along the inclined beam with a constant speed V.

Figs. 6(a) and (b) respectively show the time histories for the vertical ($\bar{y}$) and horizontal ($\bar{x}$) displacements of the center point of the inclined beam, where the solid curves with circles (–○–) represent the time histories with moving-load speed $V = 5.0\,\text{m/s}$, those with crosses (—+—) represent the ones with $V = 10.0\,\text{m/s}$ and those with triangles (–△–) represent the ones with $V = 20.0\,\text{m/s}$. From the last two figures, one sees that the larger the moving-load speed, the larger the maximum vertical ($\bar{y}$) and horizontal ($\bar{x}$) central displacements of the inclined beam. The last result is reasonable, because the dynamic response of the inclined beam increases when the moving-load speed approaches to the critical speed of the moving load, $V_{\text{cr}} = L/(\frac{1}{2}T) = 2\omega_1 L = 2 \times 3.58 \times 4.352 = 31.1603\,\text{m/s}$, where $\omega_1 = 3.58\,\text{Hz}$ is the fundamental natural frequency of the

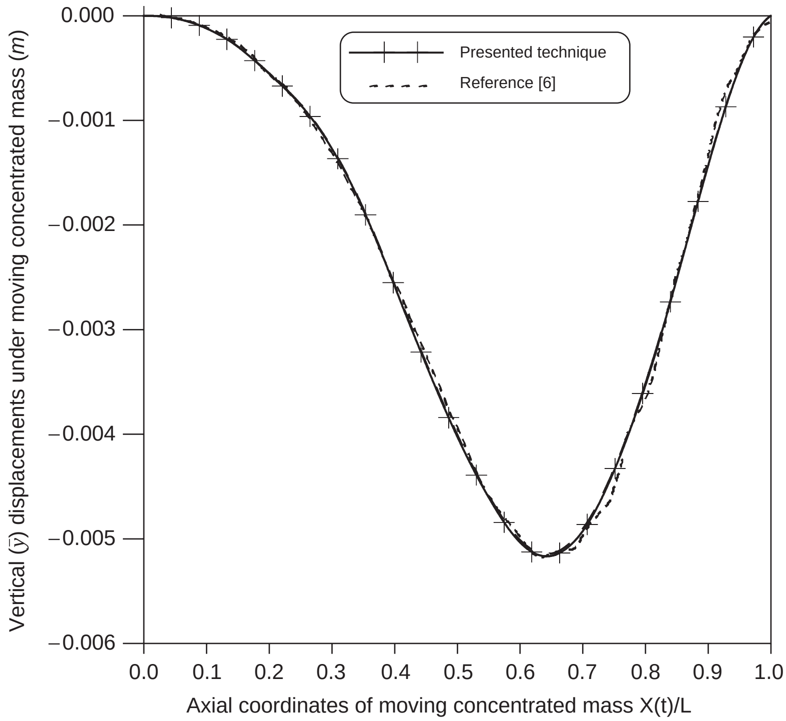


Fig. 4. Time histories for the vertical ($\bar{y}$) displacements under moving concentrated mass.

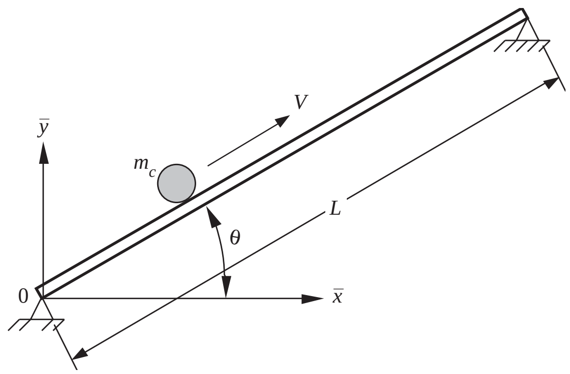
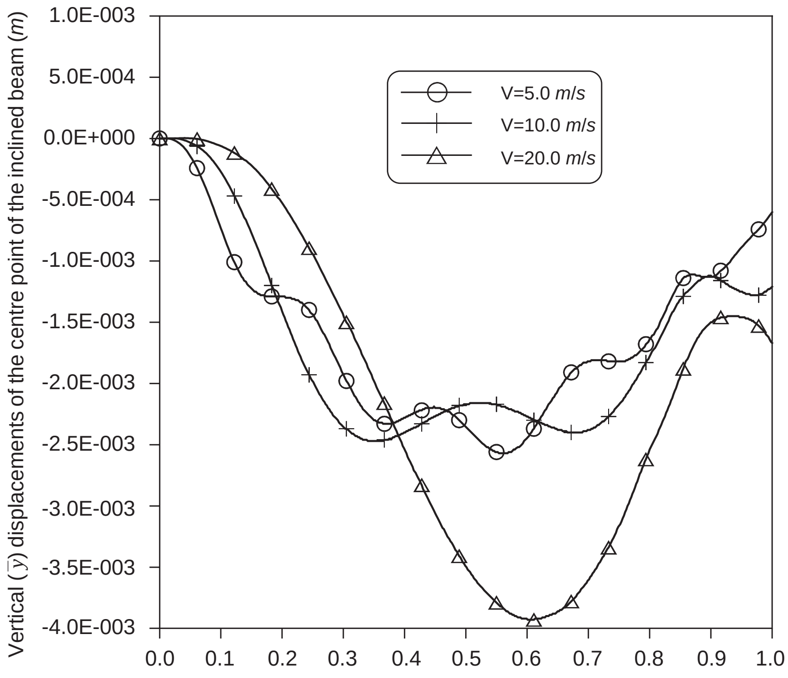
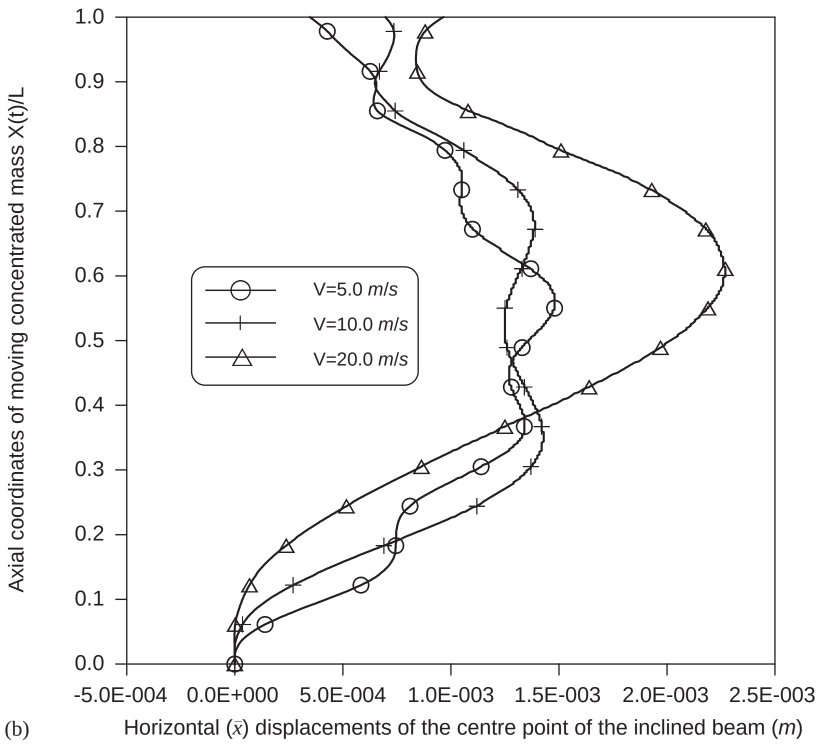


Fig. 5. An inclined pinned–pinned beam, with total length $L = 4.352$ m, subjected to a concentrated mass $m_c = 21.8$ kg moving from the lower end to the upper end of the beam with a constant speed V.

inclined beam studied. For convenience, in this paper, the moving speed is called the critical speed and denoted by V_{cr}, if the response amplitudes of the inclined beam increase linearly with time when load moves with speed $V = V_{cr}$. It is evident that the dynamic responses of the inclined beam will reach maximum when the moving speed of the concentrated mass m_c is equal to



(a)



(b)

Fig. 6. Time histories for the (a) vertical ($\bar{y}$) and (b) horizontal ($\bar{x}$) displacements of the center point of the inclined beam, with inclined angle $\theta = 30^\circ$, subjected to a moving load $m_c = 21.8$ kg with constant speeds $V = 5.0$, 10.0 and 20.0 m/s.

V_{cr}, i.e., $V = V_{cr}$. On the other hand, the dynamic responses of the inclined beam will be smaller than the last maximum responses, if $V > V_{cr}$ or $V < V_{cr}$. The fundamental frequency ω_1 of the inclined beam is independent of its inclined angle θ. Because the critical speed V_{cr} has close relationship with ω_1 and ω_1 changes with the total number of moving mass, the critical speed for the case of multiple moving masses will different from that for the case of single moving mass. However, the difference will be negligible if total magnitude of the multiple moving masses is equal to magnitude of the single moving mass.

5.3. Influence of Coriolis force

From the formulation of this paper, one can see that the effect of Coriolis force induced by the moving mass is to appear in the damping matrix $[c]$ of the moving mass element as one may see from Eq. (8b). Hence, if the damping matrix of the moving mass element is taken to be zero, i.e., $[c] = [0]$, then the effect of Coriolis force due to the moving mass will disappear.

The same beam as that of the last subsection is investigated and the vertical ($\bar{y}$) and horizontal ($\bar{x}$) displacements of the center point of the inclined beam are respectively shown in Figs. 7 and 8. Where the solid curves denote the dynamic responses of the beam with the Coriolis force considered, while the dashed curves denote those with Coriolis force neglected. Among the solid and dashed curves, those with circles, (—○— and - -○- -), crosses (—+— and - -+- -) and triangles (—△— and - -△- -) are for the cases with $V = 5.0$, 10.0 and 20.0 m/s, respectively. From the last figures, it can be seen that the influence of the Coriolis force on the vertical ($\bar{y}$) and horizontal ($\bar{x}$) central displacements of the inclined beam increases with increasing the moving-load speed. This is to be expected, because the magnitude of the Coriolis force appearing in the damping matrix $[c]$ of the moving mass element is proportional to the moving-load speed V, as one may see from Eq. (8b). Although the difference between the time histories with and without Coriolis force is small for the example studied in this subsection, it does not mean that the last effects may be neglected for other cases. Therefore, it may be better to consider the last effects in the analysis.

5.4. Influence of centrifugal force

Similarly, one can also ignore the effect of the centrifugal force due to moving mass by taking the stiffness matrix of the moving mass element to be zero, i.e., $[k] = [0]$. The same example as that of the last subsection is studied and the vertical ($\bar{y}$) and horizontal ($\bar{x}$) displacements of the center point of the inclined beam are respectively shown in Figs. 9 and 10. The legends for the curves in the last two figures are exactly the same as those in Figs. 7 and 8 except that the Coriolis force is replaced by the centrifugal force. From Figs. 9 and 10, one sees that the influence of the centrifugal force on the vertical ($\bar{y}$) and horizontal ($\bar{x}$) central displacements of the inclined beam also increase with increasing the moving-load speed. This is because the magnitude of the centrifugal force appearing in the stiffness matrix $[k]$ of the moving mass element is proportional to the square of the moving-load speed (see Eq. (8c)). For the current example, the difference between the time histories with and without centrifugal force is small. Nevertheless, the last effects may be significant for other cases. Therefore, it should be better to consider the last effects in the formulations.

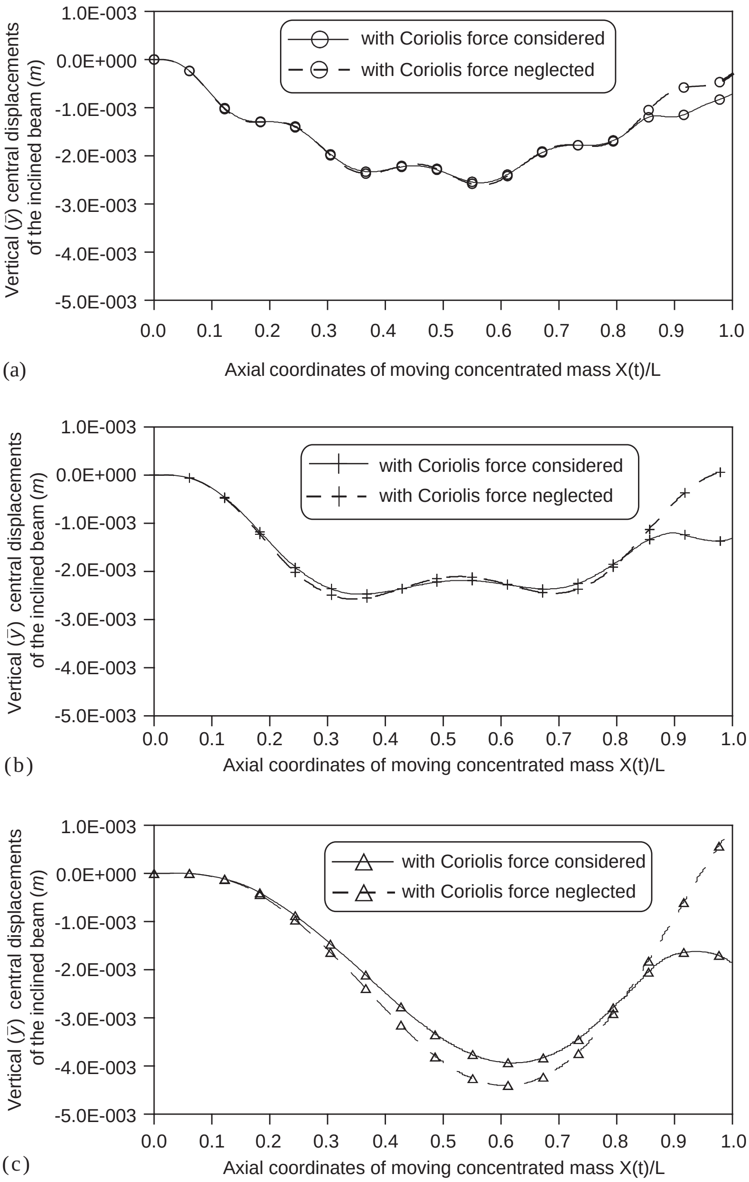


Fig. 7. Influences of Coriolis force on the time histories of the vertical ($\bar{y}$) central displacements of the inclined beam, with inclined angle $\theta = 30^{\circ}$, subjected to a moving load $m_c = 21.8\,\mathrm{kg}$ with constant speeds (a) $V = 5.0\,\mathrm{m/s}$, (b) $V = 10.0\,\mathrm{m/s}$ and (c) $V = 20.0\,\mathrm{m/s}$.

5.5. Influence of frictional force

The current example is the same as that of Section 5.2 except that the moving-load speed is $V = 10.0\,\mathrm{m/s}$. Figs. 11(a) and (b) show the vertical ($\bar{y}$) and horizontal ($\bar{x}$) displacements of the center point of the inclined beam, respectively. In which, the solid curves (—) are for the case with

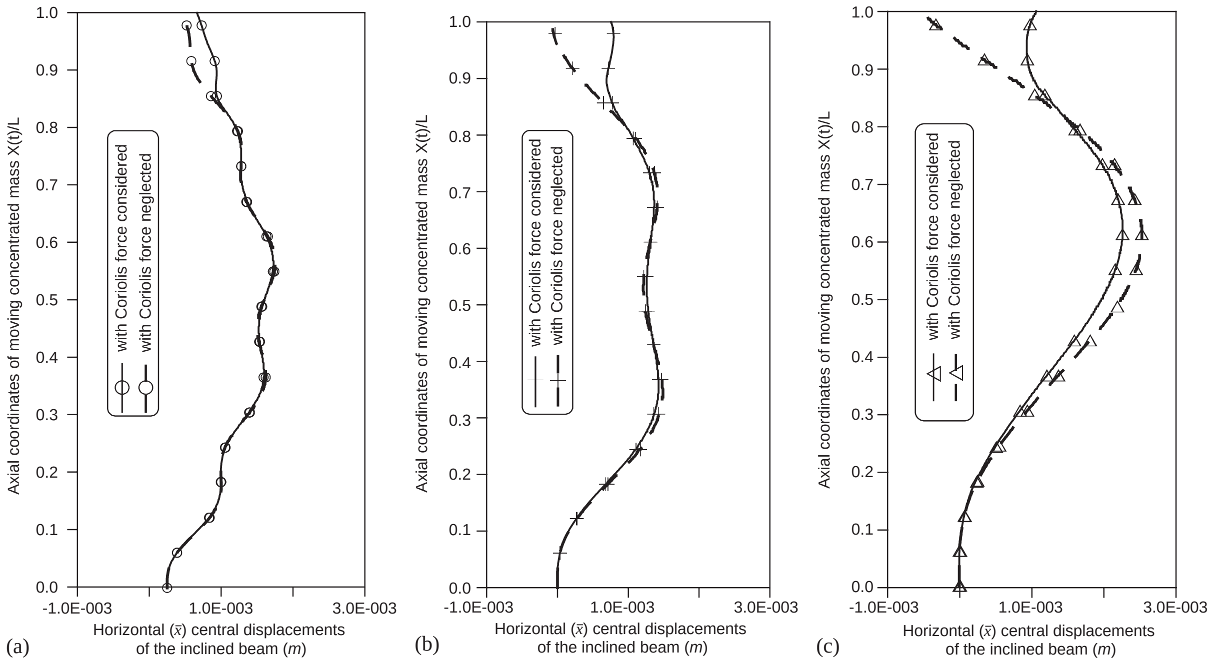


Fig. 8. Influences of Coriolis force on the time histories of the horizontal ($\bar{x}$) central displacements of the inclined beam, with inclined angle $\theta = 30°$, subjected to a moving load $m_c = 21.8$ kg with constant speeds (a) $V = 5.0$ m/s, (b) $V = 10.0$ m/s and (c) $V = 20.0$ m/s.

frictional force neglected, while the dashed curve with circles (- -○- -) are for the case with frictional force considered (with $\mu = \tan\theta = \tan 30° = 0.577$). From the figures, one sees that the influence of the frictional force on the vertical ($\bar{y}$) and horizontal ($\bar{x}$) central displacements of the inclined beam is negligible. This is because the frictional force is always along the axial (x) direction of the inclined beam and the transverse displacement at the central point of the beam induced by the frictional force is always zero (i.e., $u_y = 0$). Besides, because the axial stiffness of the inclined beam is much greater than its transverse bending stiffness, the axial central displacement (u_x) induced by the frictional force is negligible comparing with the transverse one induced by the inertial force, Coriolis force or centrifugal force, i.e., $u_x \approx 0$. It is evident that the substitution of the last frictional-force-induced axial central displacement ($u_x \approx 0$) and transverse one ($u_y = 0$) into Eqs. (1c) and (1d), one obtains $u_{\bar{x}} \approx 0$ and $u_{\bar{y}} \approx 0$. Note that if the mass traverses downward, the sign of$V$ may be kept unchangedbut the sign of μ is reversed.

5.6. Influence of inclined angle of the beam

In this subsection, the concentrated load $m_c = 21.8$ kg is assumed to move, with a constant speed $V = 10.0$ m/s, from the lower end to the upper end of the inclined beam with $\theta = 0°$, 15° and 30°.

The time histories for the vertical ($\bar{y}$) and horizontal ($\bar{x}$) displacements of the center point of the inclined beam are respectively shown in Figs. 12(a) and (b). In which, the solid curves with circles

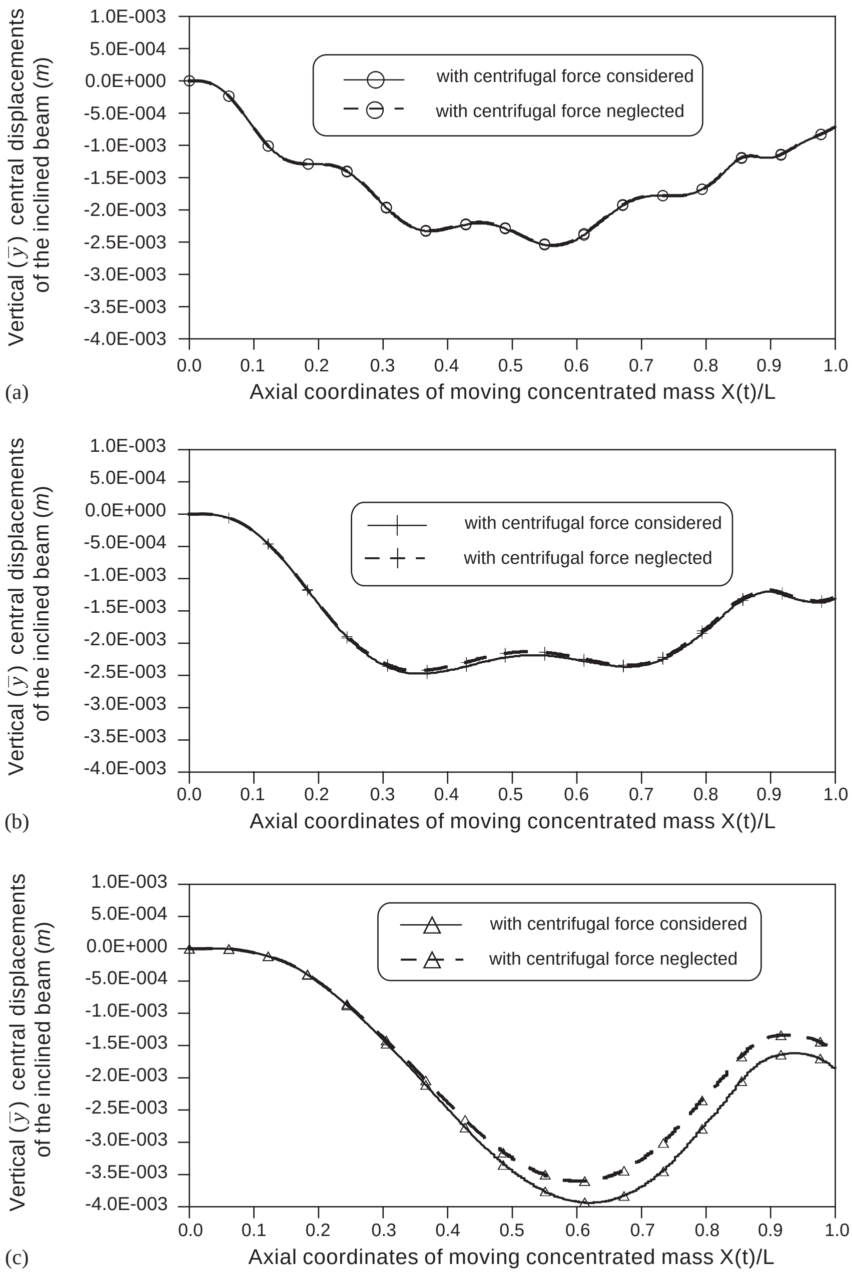


Fig. 9. Influences of centrifugal force on the time histories of the vertical ($\bar{y}$) central displacements of the inclined beam, with inclined angle $\theta = 30°$, subjected to a moving load $m_c = 21.8$ kg with constant speeds (a) $V = 5.0$ m/s, (b) $V = 10.0$ m/s and (c) $V = 20.0$ m/s.

(–○–) are for the case with $\theta = 0°$, those with crosses (—+—) are for the case with $\theta = 15°$ and those with triangles (–△–) are for the case with $\theta = 30°$. From the last two figures, one sees that the larger the inclined angle θ of the beam, the smaller the vertical ($\bar{y}$) central displacements and the

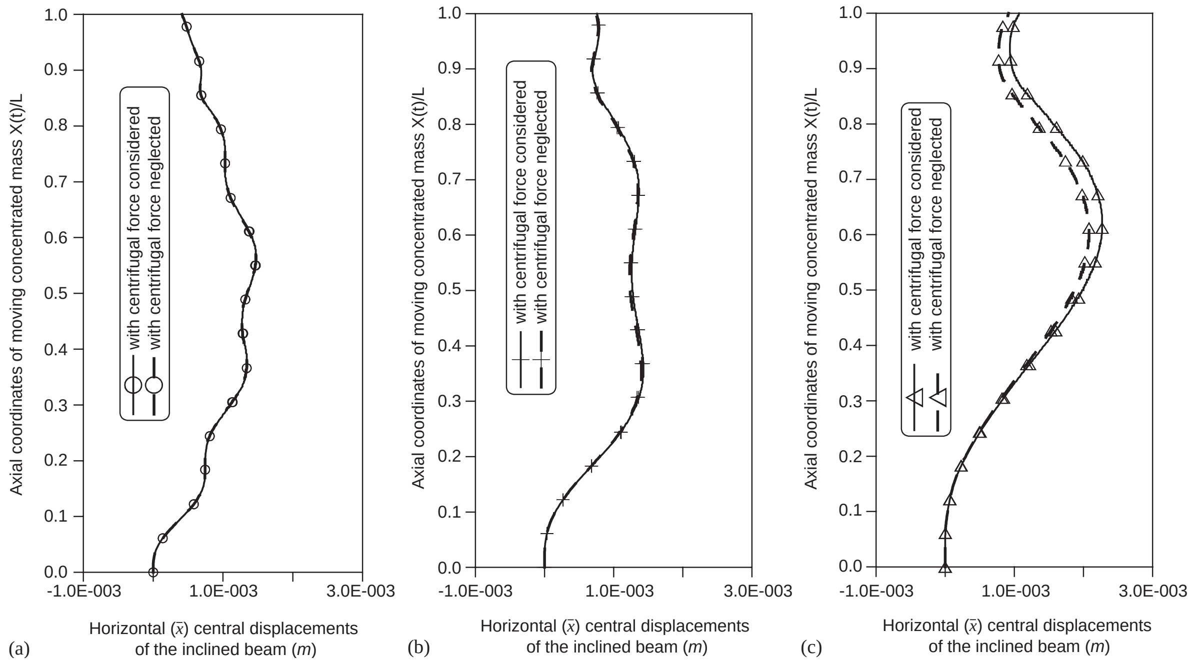
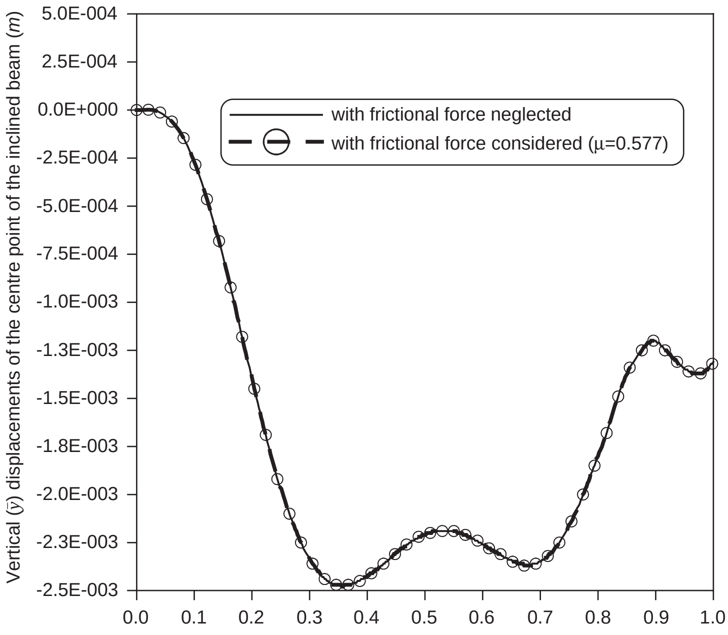


Fig. 10. Influences of centrifugal force on the time histories of the horizontal ($\bar{x}$) central displacements of the inclined beam, with inclined angle $\theta = 30^\circ$, subjected to a moving load $m_c = 21.8\,\text{kg}$ with constant speeds (a) $V = 5.0\,\text{m/s}$, (b) $V = 10.0\,\text{m/s}$ and (c) $V = 20.0\,\text{m/s}$.

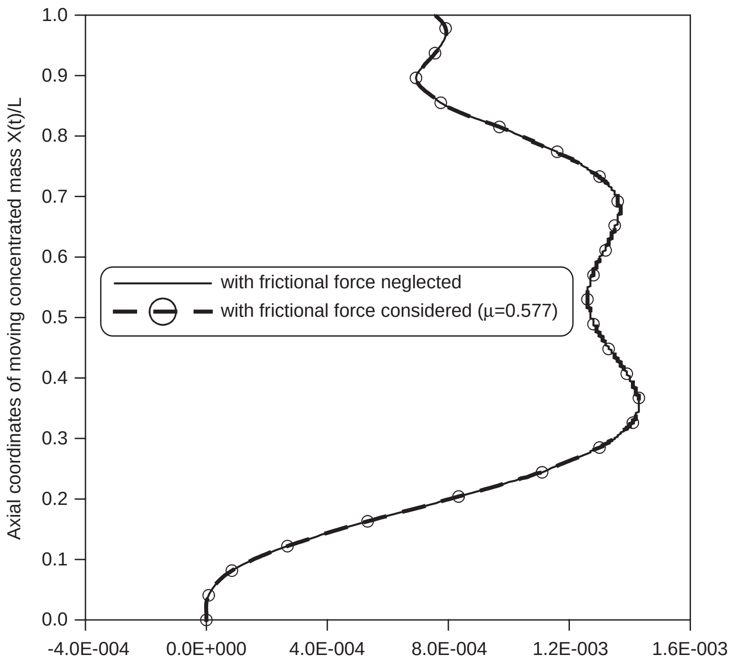
larger the horizontal ($\bar{x}$) ones. The reason for the last phenomenon is stated as follows. Since the axial stiffness of the inclined beam is much greater than its transverse stiffness, the axial central displacement (u_x) of the beam is negligible comparing with its transverse one (u_y). In such a case, its vertical central displacement ($u_{\bar{y}}$) and horizontal one ($u_{\bar{x}}$) are given by $u_{\bar{y}} \approx u_y \cos\theta$ and $u_{\bar{x}} \approx -u_y \sin\theta$ as one may see from Eqs. (1c) and (d). It is evident that the magnitude of $u_{\bar{y}}$ decreases and that of $u_{\bar{x}}$ increases when the inclined angle θ ($<45^\circ$) of the beam increases. Thus, if only the displacement in the transverse (y) direction of the inclined beam is interested, then increasing the inclined angle θ ($<45^\circ$) will be beneficial for reducing the transverse response of the beam, because the force component in the transverse (y) direction induced by the moving loads decreases when the inclined angle θ of the beam increases.

5.7. *Influence of total number of moving loads*

To show the applicability of the presented technique for the multiple moving loads, the dynamic analysis of the inclined beam with $\theta = 30^\circ$ subjected to two identical loads ($m_1 = m_2 = 21.8/2 = 10.9\,\text{kg}$) moving from the lower end to the upper end of the beam with the same constant speed $V = 10.0\,\text{m/s}$ is performed (see Fig. 13). For convenience, the spacing e between the last two moving loads is assumed to be a constant for each case. The forced vibration responses of the inclined beam are calculated for the period beginning from the instant that the first moving mass m_1 reaches the lower end of the inclined beam and ending at the instant that the second moving

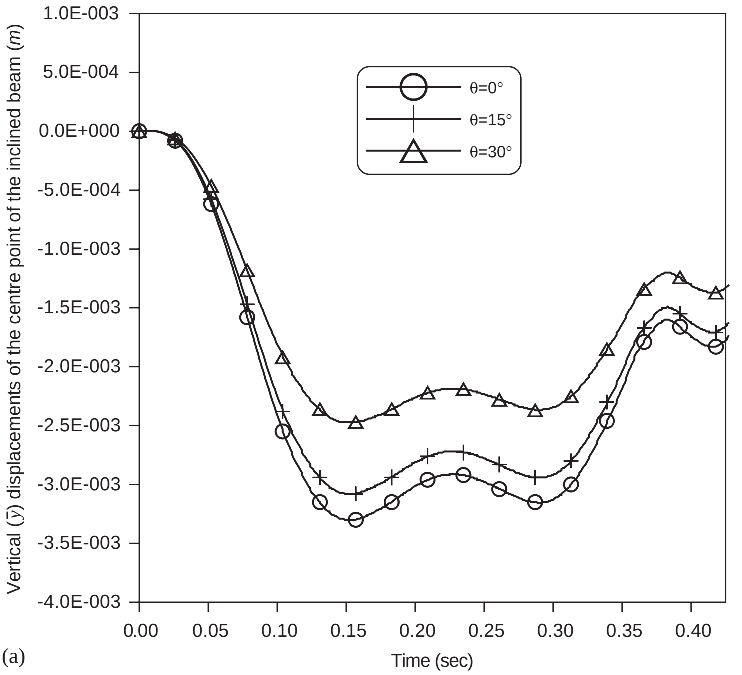


(a)

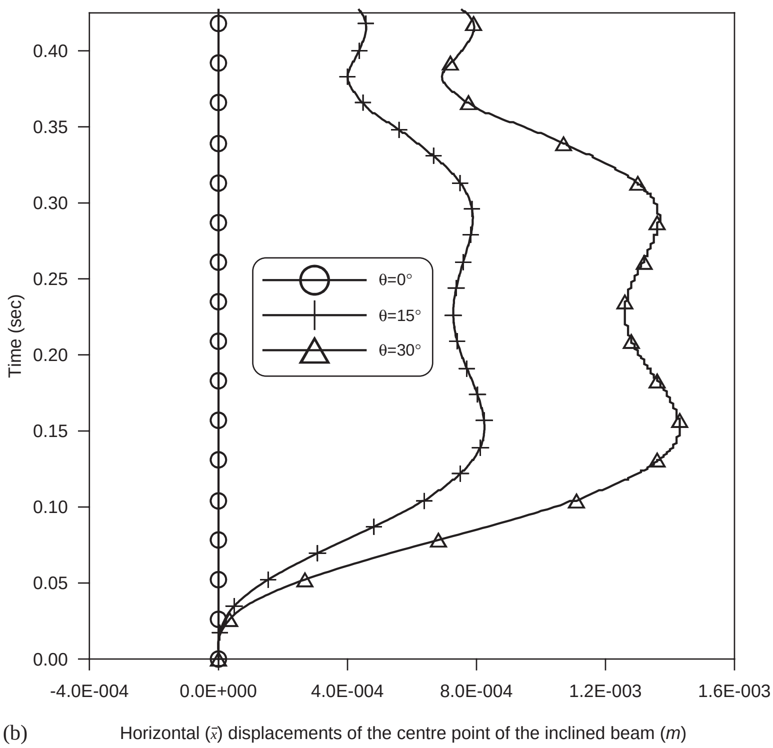


(b)

Fig. 11. Influences of frictional force on the time histories of the (a) vertical ($\bar{y}$) and (b) horizontal ($\bar{x}$) central displacements of the inclined beam, with inclined angle $\theta = 30^\circ$, subjected to a moving load $m_c = 21.8\,\text{kg}$ with a constant speed $V = 10.0\,\text{m/s}$.



(a)



(b)

Fig. 12. Time histories for the (a) vertical ($\bar{y}$) and (b) horizontal ($\bar{x}$) central displacements of the inclined beam subjected to a moving load $m_c = 21.8\,\text{kg}$ with a constant speed $V = 10.0\,\text{m/s}$ when the inclined angles of the beam are $\theta = 0^\circ$, $\theta = 15^\circ$ and $\theta = 30^\circ$.

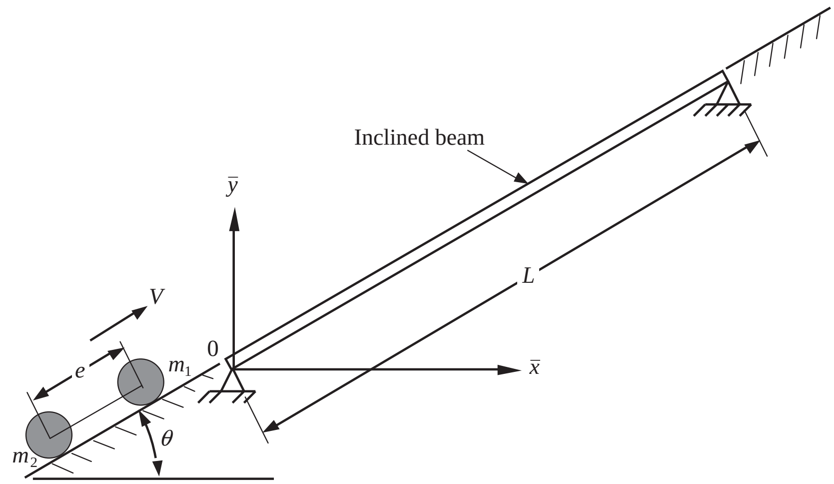


Fig. 13. An inclined pinned–pinned beam subjected to two concentrated masses $m_1 = m_2 = 21.8/2 = 10.9$ kg moving from lower end to upper end of the inclined beam with a constant speed $V = 10.0$ m/s ($\theta = 30°$, $e = 0.2$, 0.4 and 0.6 m, $L = 4.352$ m).

mass m_2 leaves the upper end of the beam. Four cases with spacings $e = 0.01$, 0.2, 0.4 and 0.6 m are studied (cf. Fig. 13).

Figs. 14(a) and (b) show the vertical ($\bar{y}$) and horizontal ($\bar{x}$) displacements of the center point of the inclined beam, respectively. In the last figures, the solid curves with triangles (–△–), the solid curves (—), the dashed curves (- - -) and the solid curves with crosses (—+—) are respectively for the case with spacings $e = 0.01$, 0.2, 0.4 and 0.6 m. From Fig. 14, one sees that the solid curves with triangles (–△–) are very close to those shown in Fig. 12. This is to be expected because the dynamic responses of the two moving masses, m_1 and m_2, will be very close to those of a single moving mass, m_c, if the spacing between m_1 and m_2 is small (i.e., $e \approx 0$) and $m_1 + m_2 = m_c$. In addition, it is also found that the maximum central displacement of the inclined beam decreases with increasing the spacing between the two moving masses. This is reasonable because the dynamic responses of the beam due to a moving *concentrated* load are larger than those of the same beam due to a moving *distributed* load if the magnitude of the *concentrated* load is equal to the combined magnitude of the *distributed* load. The dimensionless time histories corresponding to Figs. 14(a) and (b) are shown in Figs. 15(a) and (b), respectively. The same trend as Figs. 14(a) and (b) can be observed from Figs. 15(a) and (b).

6. Conclusions

1. The main difference between the inclined beam and the conventional horizontal beam is that both the frictional force between the moving load and the beam and the influence of inclined angle of the beam must be considered for the inclined beam and this is not true for the horizontal beam. Based on the theory of *moving mass element* presented in this paper, one may easily take account of the effects of inertia force, Coriolis force and centrifugal force induced by the moving loads in addition to the effects due to frictional force and inclined angle.

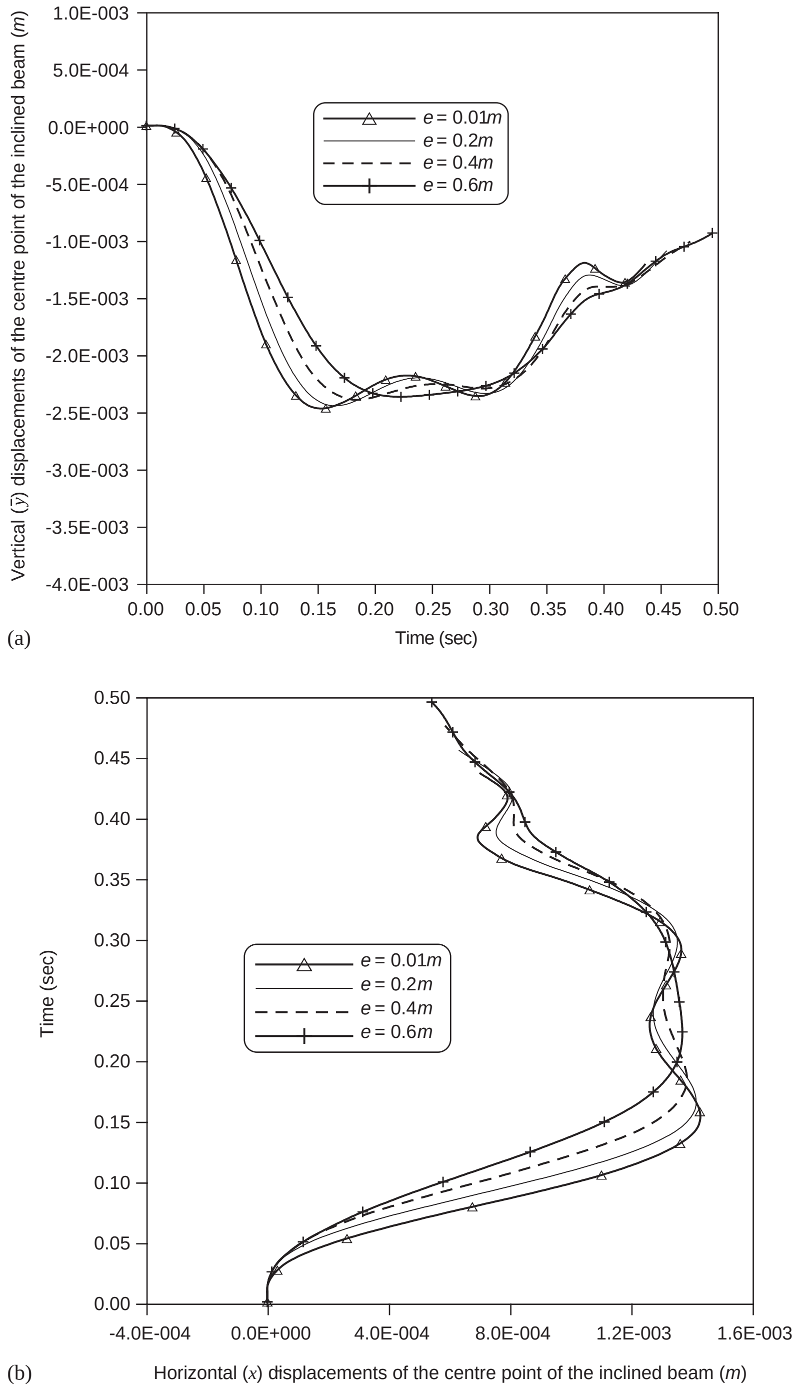
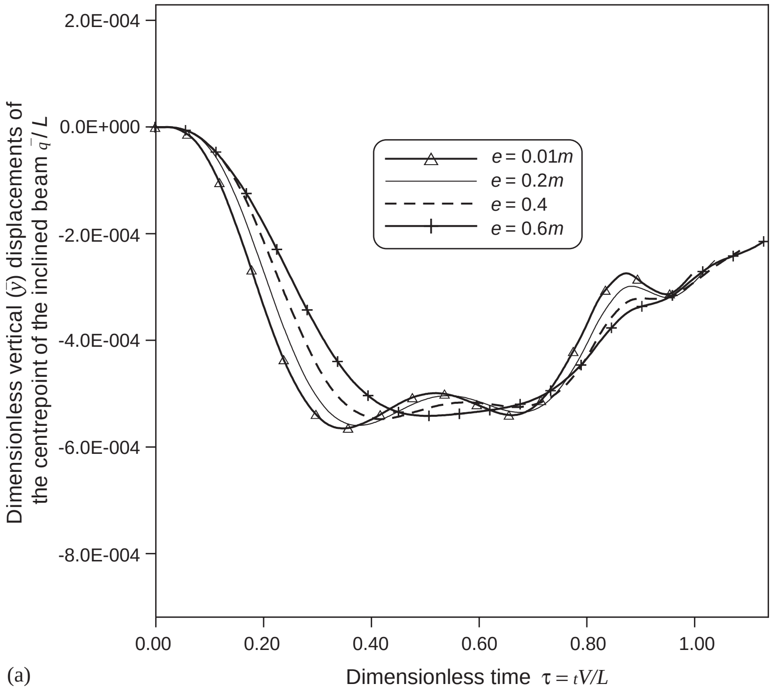
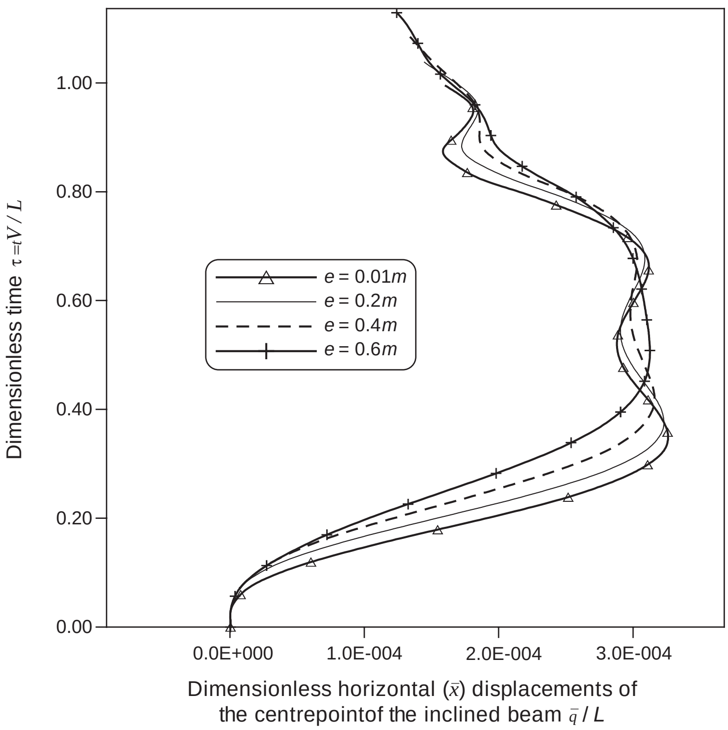


Fig. 14. Influences of spacing on the time histories for the (a) vertical ($\bar{y}$) and (b) horizontal ($\bar{x}$) central displacements of the inclined beam subjected to two moving loads $m_{c1} = m_{c2} = 21.8/2 = 10.9\,\text{kg}$ with a constant speed $V = 10.0\,\text{m/s}$ when the inclined angle of the beam is $\theta = 30^\circ$.



(a)



(b)

Fig. 15. Dimensionless time histories corresponding to (a) Fig. 14(a) and (b) Fig. 14(b), respectively.

2. If the moving-load speed is smaller than the critical speed, then the larger the moving-load speed, the larger the maximum vertical ($\bar{y}$) and horizontal ($\bar{x}$) displacements of the center point of the inclined beam.
3. Because the frictional forces induced by the moving loads on an inclined beam are always in the axial (x) direction of the beam and the axial stiffness of the beam is much greater than its transverse stiffness, the axial central displacement (u_x) due to the frictional forces is much smaller than its transverse one (u_y) due to the other force (such as inertial force, Coriolis force or centrifugal force). Therefore, the influence of the frictional forces on the dynamic behavior of the inclined beam is negligible.
4. In general, the axial stiffness of a beam is much greater than its transverse bending stiffness. Thus, for an inclined beam subjected to the moving loads, its axial displacement at the center point is negligible (i.e., $u_x = 0$). In other words, if only the displacement in the transverse (y) direction of the inclined beam is interested, then increasing the inclined angle θ ($<45^\circ$) will be beneficial for reducing the transverse response of the beam, because the force component in the transverse (y) direction induced by the moving loads decreases when the inclined angle θ of the beam increases.
5. It is well known that the dynamic responses of a beam due to a moving *concentrated* load are larger than those of the same beam due to a moving *distributed* load if the magnitude of the *concentrated* load is equal to the combined magnitude of the *distributed load*. Therefore, if an inclined beam is subjected to two identical moving masses, then the maximum displacement of the center point of the inclined beam decreases with increasing the spacing between the two identical moving masses.

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