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Journal of Sound and Vibration 288 (2005) 321–344
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Approximate solution for free vibrations of thin orthotropic rectangular plates

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Received 5 January 2004; received in revised form 7 December 2004; accepted 12 January 2005
Available online 13 March 2005

Abstract

In this paper the problem of approximate evaluation of the frequencies of orthotropic plates is investigated. Starting from a general approximate formula for the frequency, proposed by Hearmon, it is shown how to calculate the fundamental mode of an orthotropic rectangular plate with different restraint conditions using the proper coefficient values already available in the scientific literature. For the higher modal frequencies, a particular form of the Rayleigh method is proposed, leading to a simple procedure to calculate the fundamental frequency. In fact the frequency calculation is reduced to the evaluation of the fundamental frequency of a special plate associated to the real one.

In order to test the accuracy of the proposed method an extensive finite element investigation was carried out. The numerical results obtained show that the average difference between the values computed with the approximate new method and the finite element values is less than 0.74%.

1. Introduction

The use of orthotropic plates is common in all the fields of structural engineering: civil, mechanical, aerospace and naval. The orthotropic behaviour can derive not only from the use of anisotropic materials but also from the metallurgical process that can alter the isotropic characteristics along perpendicular directions, or from the use of stiffening beams coupled with an

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.01.005

Nomenclature

x, y, z	axis of the reference system
t	time
E_1	Young's modulus in bending for the x direction
E_2	Young's modulus in bending for the y direction
G_{12}	Shear modulus in bending for the xy plane
ν_{12}	Poisson's ratio corresponding to compressive strain in y direction due to extensional stress in x direction
ν_{21}	Poisson's ratio corresponding to compressive strain in x direction due to extensional stress in y direction
a	length of side parallel to x-axis
b	length of side parallel to y-axis
h	plate thickness
ω	circular frequency
ρ	mass density of the material
f	frequency $= \omega/2\pi$
W	transverse displacement of a point on the plate (along z direction)
i	number of half waves in x direction
j	number of half waves in y direction
M	bending moment reaction for clamped edges
R	restraint reaction for supported edges

isotropic plate. The wide use of such structures requires investigating the dynamic behaviour of orthotropic plates in order to develop accurate and reliable design. In this field, from an engineering point of view, the Finite Element Method (FEM) gives a complete solution to the problem of evaluating the modes and the dynamic response of an orthotropic plate when the material properties and the boundary conditions are known. However, in the preliminary design stage, when the dimensions and the properties of the materials must be selected, as well as in applying quality controls to the accuracy of the design by means of FE calculations, it is very useful to have a simplified method to compute all the modal frequencies of orthotropic rectangular plates.

The problem of free vibration of orthotropic rectangular plates has been intensively studied in the past 60 years by several researchers from different countries. Starting from the initial papers of Hearmon [1], Huffington and Hoppmann [2] and Gumenyuk [3], which examined some particular cases, it is possible to find a very large number of contributions for the general solution of the problem using different techniques and dedicated to plates with various edge conditions and shapes. Within the present limitation to thin plates, and citing for sake of completeness the work of Liew et al. [4], we mention below a list (not exhaustive) of significant authors and papers.

As it is possible to state theoretically, there are no closed form solutions for the calculation of frequencies at least that two opposite sides are not simply supported. For this reason many efforts were devoted to develop approximate solutions with a high level of accuracy.

A wide review of the literature up to 1990 is collected in the Leissa and Bert series of papers [5–13] on the dynamic behaviour of composite and sandwich plates while the use of FE modelling is described in Ref. [14].

A review of the scientific literature in this specific field reveals that, due to its high versatility and conceptual simplicity [15], one of the most popular methods to obtain approximate solutions for the frequencies of an orthotropic rectangular plate is the Rayleigh–Ritz method. Hearmon in 1959 [16] proposed an approximate general solution for the free vibrations of the orthotropic plates applying the Rayleigh method and using, for the deformed shape of the plate, characteristic beam functions appropriate for any combination of clamped or supported constraint conditions. His approach extended the treatment developed by Warburton [17] for all modes of vibration of isotropic plate with any combination of free, supported or clamped restraint conditions.

Among the papers where the Rayleigh–Ritz method is used, particularly interesting is the contribution of Marangoni et al. [18] that extended a method presented by Bazely et al. [19] to compute the lower bounds of the frequencies of a rectangular isotropic plate, evaluating upper and lower bounds of the natural frequencies of clamped rectangular plate formed by orthotropic materials. They used a combination of the Rayleigh–Ritz method for upper bound and the decomposition method proposed by Bazely for lower bound evaluation. This bracketing technique leads to a high degree of confidence for the approximate results of the analysis.

Recently Rossi et al. [20] extended the analysis to the more complicated case of plates with one or more free edge. The difficulty arises from the free edge constraint condition which prevents the use of the classic Ritz–Rayleigh formulation. Rossi et al. overcome the problem using an optimised formulation of the Rayleigh–Ritz method proposed by Laura et al. [21]. Obtained results show excellent agreement with the FEM calculation, including the case where a lumped mass is attached to the plate.

Gorman [22] applied the superposition method, developed for isotropic plate [23], to the case of clamped orthotropic plates showing the good accuracy of the method and including accurate eigenvalues tabulated for a wide range of plate geometries and orthotropic materials, ready for engineering use. The proposed method received much attention by several researchers due to the fact that the governing differential equation is satisfied exactly throughout the plate and the boundary conditions can also be satisfied to any desired accuracy. In fact Li [24] showed that it is suitable also to analyse the forced vibration of orthotropic rectangular plates, while Mossou and Nivoit [25] employed the method to provide a reliable non-destructive test method to determine the elastic constants of an orthotropic material forming a rectangular plate in free vibration condition.

In Ref. [26] a solution in a double trigonometric series is introduced to solve the problem of forced vibration of clamped orthotropic plates also using an iterative method for the analysis of free vibrations.

Sakata et al. [27] applied the method of iterative reduction of the plate partial differential equation, already studied for the isotropic case [28], to evaluate the natural frequencies of rectangular orthotropic plates, obtaining very accurate results. They emphasise that the method is simpler than others available in the literature such as, for example, the Rayleigh–Ritz method that requires a larger computing effort.

In Ref. [29] the possibility of using approximate solutions for the frequency calculation of anisotropic plates was investigated. In fact the authors compared the frequencies of simply supported anisotropic plates, computed by means of exact numerical analysis [30], and calculated by the exact solution of general simply supported orthotropic plates. The clamped edge plate problem was studied comparing the frequencies of the anisotropic plates with the values obtained using the Huber orthotropy and the isotropic plates formulas [31]. It was stated that, from an engineering point of view, these two approximations provide values with acceptable accuracy.

Finally another type of approach was used by Chen [32] to calculate the fundamental vibration frequency of an orthotropic plate. He used an iterative approach based on finite difference equations and numerical integration and demonstrated that, with a relatively small computing effort, it is possible to obtain a degree of accuracy comparable with that of other numerical techniques.

All the methods mentioned above involve large computation efforts and for this reason they are not suitable:

(a) to define a quick approximate method;
(b) to execute simple preliminary design considerations or fast final general checks of accuracy.

For these purposes in this paper a simplified method to evaluate the natural frequencies of an orthotropic plate is proposed. The method is based on an approximate formula elaborated in a general form, using and merging the results obtained by the researchers who have approached the problem in the last decades. Starting from a general expression of the fundamental frequency of a plate, suggested by Hearmon [1] and based on three coefficients that take into account the constraint conditions, it is possible to assess the higher modal frequencies, by means of a simple numerical procedure that uses a particular formulation of the Rayleigh method. This reduces the problem of evaluating any frequency of an orthotropic plate to calculate the fundamental frequency of an equivalent plate associated to the real one. A similar approach was used by Sakata [33] to study the higher-order natural frequencies, but he limited his analysis to the case of an isotropic rectangular plate simply supported along two opposite sides, parallel to y, and elastically restrained against rotation along the remaining sides parallel to x.

2. Background theory

An orthotropic material is characterised by the fact that the mechanical elastic properties have two perpendicular planes of symmetry. Due to this condition only four elastic constant are independent namely E_1, E_2, G_{12}, ν_{12}. The coefficient ν_{21} can be determined using the equation

$$\frac{E_1}{E_2} = \frac{\nu_{12}}{\nu_{21}}. \tag{1}$$

Introduced the parameters

$$D_1 = \frac{E_1 h^3}{12\mu}, \quad D_2 = \frac{E_2 h^3}{12\mu}, \quad D_{12} = \frac{G_{12} h^3}{12},$$

$$\mu = 1 - \nu_{12}\nu_{21}, \quad 2H = \nu_{21} D_1 + \nu_{12} D_2 + 4 D_{12} \tag{2}$$

and making use of the hypotheses of Love–Kirchoff, that neglect the effect of the shear forces and the rotational inertia, with reference to Fig. 1 [34], the equation of the motion follows

$$D_1 \frac{\partial^4 w}{\partial x^4}(x,y,t) + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2}(x,y,t) + D_1 \frac{\partial^4 w}{\partial y^4}(x,y,t) + \rho h \frac{\partial^2 w}{\partial t^2}(x,y,t) = 0. \tag{3}$$

Considering a solution with the general form

$$w = W(x,y)(A \cos \omega t + B \sin \omega t) \tag{4}$$

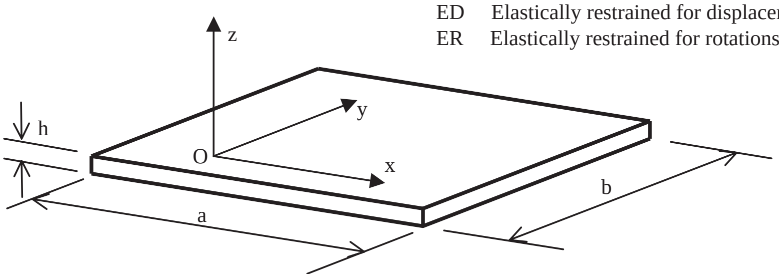


Fig. 1. Geometry and symbols used for the plate.

it is possible to obtain from Eq. (3) the following equation in terms of spatial variables only

$$D_1 \frac{\partial^4 W}{\partial x^4}(x,y) + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2}(x,y) + D_1 \frac{\partial^4 W}{\partial y^4}(x,y) + \Lambda^4 W = 0, \tag{5}$$

where

$$\Lambda^2 = \omega\sqrt{\rho h}. \tag{6}$$

Eq. (5) must be solved under the boundary conditions listed below:

$$\begin{aligned} &M = 0;\ R = 0 \quad \text{for a free edge,}\\ &M = 0;\ W = 0 \quad \text{for a supported edge,}\\ &W = 0;\ \frac{\partial W}{\partial x(or\ \partial y)} = 0 \quad \text{for a clamped edge.} \end{aligned} \tag{7}$$

For example, in a rectangular plate simply supported at the four edges, the function W can be expressed by means of a double trigonometric series [35]

$$W(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin(\alpha x)\sin(\beta y), \tag{8}$$

where

$$\alpha = \frac{m\pi}{a}, \quad \beta = \frac{n\pi}{b}$$

and a_{mn} is a mode shape factor. After manipulating Eq. (5) and substituting into Eq. (8), the eigenvalue equation is obtained

$$\alpha^4 D_1 + \beta^4 D_2 + 2H\alpha^2\beta^2 = \omega^2 \rho h. \tag{9}$$

Solving the eigenvalue equation, the expression of the frequencies for an orthotropic rectangular plate can be deduced

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{\rho h}}\sqrt{\alpha^4 D_1 + \beta^4 D_2 + 2\alpha^2\beta^2 H}. \tag{10}$$

This expression, for an isotropic plate where

$$D_1 = D_2 = H = D$$

takes the simplified form

$$f = \frac{1}{2\pi}\sqrt{\frac{D}{\rho h}}(\alpha^2 + \beta^2). \tag{11}$$

Once the frequencies are known it is possible to compute the corresponding mode shapes referring the mode shape factors to a proper normalising parameter. For example in Fig. 2 the mode shapes 1×1, 2, 1×2 and 2×2 are reported.

It is important to note, for the following development of the paper, that the nodal lines, where $W = 0$, are rectilinear, as shown in Fig. 3, where they are evidenced. This fact appears clear through the examination of the arguments of the functions in Eq. (8). On the contrary, for square isotropic plates, more complex nodal lines appear both for supported and fixed restraint conditions [36].

If the material has a special orthotropy, such as Huber [37] orthotropy, we have

$$H = \sqrt{D_1 D_2}$$

and the problem can be treated as an isotropic one using

$$a^* = a\sqrt[4]{\frac{D}{D_x}}, \quad b^* = b\sqrt[4]{\frac{D}{D_y}},$$

where D is arbitrary. If we use, for example, $D = D_x$ we have $a^* = a$.

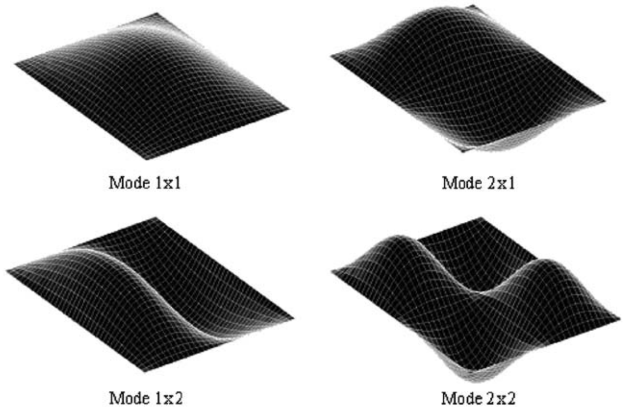


Fig. 2. First four modes for a rectangular plate (isometric view).

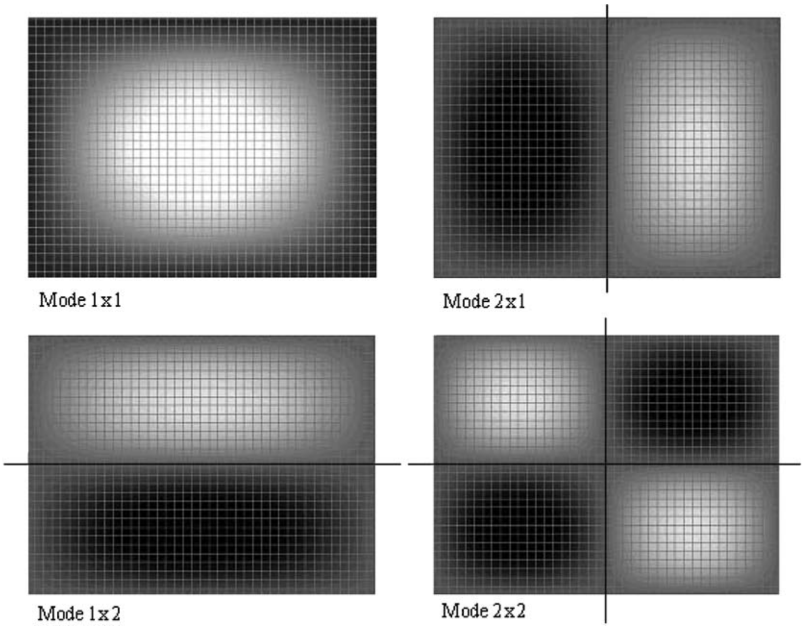


Fig. 3. First four modes for a rectangular plate (bottom view).

For the isotropic plates in literature [36,38] the formulas to calculate the first frequency with different boundary conditions are available and are summarised in Table 1. Therefore, in a general form, the parameter λ can be written as

$$\lambda = \sqrt{\frac{A}{a^4} + \frac{B}{b^4} + \frac{C}{a^2 b^2}} \tag{12}$$

and from this it is possible to derive a general expression for the first frequency of isotropic rectangular plate

$$f = \frac{\lambda}{2\pi} \sqrt{\frac{D}{\rho h}}. \tag{13}$$

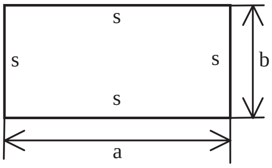
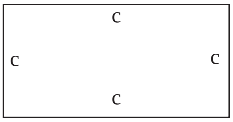
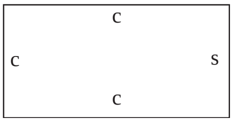
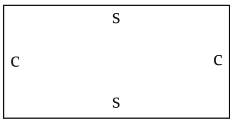
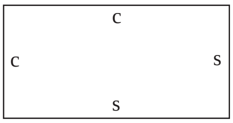
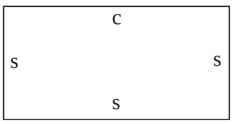
3. The new approximate method

3.1. The first frequency of orthotropic plates

For orthotropic rectangular plates Hearmon [1] proposed the following equation to calculate the first natural frequency:

$$f = \frac{1}{2\pi} \sqrt{\frac{1}{\rho h}} \sqrt{\left(\frac{D_1 A}{a^4} + \frac{D_2 B}{b^4} + \frac{HC}{a^2 b^2}\right)}. \tag{14}$$

Table 1
Formulas for the first frequency of isotropic rectangular plates

Restraint condition	λ
	$\lambda = \pi^2\sqrt{\frac{1}{a^4}+\frac{1}{b^4}+\frac{2}{a^2b^2}}$
	$\lambda = 4.730^2\sqrt{\frac{1}{a^4}+\frac{1}{b^4}+\frac{0.605}{a^2b^2}}$
	$\lambda = 4.730^2\sqrt{\frac{0.475}{a^4}+\frac{1}{b^4}+\frac{0.566}{a^2b^2}}$
	$\lambda = 4.730^2\sqrt{\frac{1}{a^4}+\frac{0.195}{b^4}+\frac{0.485}{a^2b^2}}$
	$\lambda = 3.927^2\sqrt{\frac{1}{a^4}+\frac{1}{b^4}+\frac{1.115}{a^2b^2}}$
	$\lambda = \pi^2\sqrt{\frac{1}{a^4}+\frac{2.441}{b^4}+\frac{2.333}{a^2b^2}]}$

The coefficients in Eq. (14) were calculated by Hearmon using the Rayleigh method, adopting for $W(x,y)$, the functions stated in Ref. [39]. It is possible to show that Eq. (14) is equivalent to Eq. (13) when the plate stiffness D is replaced by an equivalent D_{eq}:

$$D_{\mathrm{eq}} = \frac{AD_1b^4 + BD_2a^4 + HCa^2b^2}{Ab^4 + Ba^4 + Ca^2b^2}. \tag{15}$$

In fact, substituting Eq. (15) in Eq. (13) and taking into account the expression of λ shown in Eq. (12), we have

$$
\begin{aligned}
f &= \frac{1}{2\pi}\sqrt{\frac{D_{\mathrm{eq}}}{\rho h}}\sqrt{\frac{A}{a^4}+\frac{B}{b^4}+\frac{C}{a^2b^2}} \\
&= \frac{1}{2\pi}\sqrt{\frac{1}{\rho h}}\sqrt{\left(\frac{A}{a^4}+\frac{B}{b^4}+\frac{C}{a^2b^2}\right)\frac{AD_1b^4+BD_2a^4+HCa^2b^2}{Ab^4+Ba^4+Ca^2b^2}} \\
&= \frac{1}{2\pi}\sqrt{\frac{1}{\rho h}}\sqrt{\frac{Ab^4+Ba^4+Ca^2b^2}{a^4b^4}\,\frac{AD_1b^4+BD_2a^4+HCa^2b^2}{Ab^4+Ba^4+Ca^2b^2}} \qquad (16)\\
&= \frac{1}{2\pi}\sqrt{\frac{1}{\rho h}}\sqrt{\frac{AD_1b^4+BD_2a^4+HCa^2b^2}{a^4b^4}} = \frac{1}{2\pi}\sqrt{\frac{1}{\rho h}}\sqrt{\frac{D_1A}{a^4}+\frac{D_2B}{b^4}+\frac{HC}{a^2b^2}}. \qquad (17)
\end{aligned}
$$

Unfortunately the definition of an isotropic material, equivalent to the orthotropic one from the point of view of frequency in a rectangular plate, does not solve the problem of computing the frequencies in a general form. In fact D_{eq} depends on the values of the coefficients A, B, C. These depend not only from the constraint conditions but also from the order of the frequency to be calculated. Therefore, for each frequency, the values of A, B, C are to be determined simultaneously with D_{eq}.

Finally, comparing the coefficient A, B, C given by Hearmon for the first frequency with those shown in Table 1, it is shown that the numerical values are the same, except for some negligible differences due to the approximation of the calculations executed by Hearmon. However, using expression (14) and the coefficient listed in Table 1, the problem of the evaluation of the first natural frequency of an orthotropic rectangular plate with various restraint conditions is completely solved.

3.2. *The approximate calculation of the ith natural frequency of orthotropic plates*

To solve the problem of calculating the natural frequencies after the first one it is possible to use a particular formulation of Rayleigh method. For this it can be written

$$V' = T', \qquad (18)$$

where V' is the maximum potential energy of bending contained in the plate and T' is the corresponding maximum kinetic energy. The expression of the two types of energy for an orthotropic plate are

$$V' = \frac{1}{2}\int_0^a\int_0^b\left[D_1\left(\frac{\partial W}{\partial x^2}\right)^2 + D_2\left(\frac{\partial W}{\partial y^2}\right)^2 + 2H\left(\frac{\partial W}{\partial x \partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y,$$

$$T' = \frac{\rho h \omega^2}{2}\int_0^a\int_0^b W^2\,\mathrm{d}x\,\mathrm{d}y.$$

Using Eq. (18) it can be written

$$\omega^2 = \frac{V}{T}, \tag{19}$$

where

$$V = \int_0^a \int_0^b \left[D_1 \left(\frac{\partial W}{\partial x^2} \right)^2 + D_2 \left(\frac{\partial W}{\partial y^2} \right)^2 + 2H \left(\frac{\partial W}{\partial x \partial y} \right)^2 \right] \mathrm{d}x\,\mathrm{d}y, \tag{20}$$

$$T = \rho h \int_0^a \int_0^b W^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{21}$$

For the frequencies after the first one it was previously observed, for rectangular plates, that:

(a) the nodal lines are rectilinear and perpendicular to the edges;
(b) they divide the plate in rectangular parts that vibrate at the same value of ω.

For the mode $n \times m$ it is possible to write Eq. (19) considering the terms V and T as the sum of the contribution due to every rectangular part, that is

$$\omega^2 = \frac{V}{T} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} V_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{m} T_{ij}}, \tag{22}$$

where V_{ij} or T_{ij} are

$$V_{ij} = \int_{a_i}^{a_{i+1}} \int_{b_j}^{b_{j+1}} \left[D_1 \left(\frac{\partial W}{\partial x^2} \right)^2 + D_2 \left(\frac{\partial W}{\partial y^2} \right)^2 + 2H \left(\frac{\partial W}{\partial x \partial y} \right)^2 \right] \mathrm{d}x\,\mathrm{d}y,$$

$$T_{ij} = \rho h \int_{a_i}^{a_{i+1}} \int_{b_j}^{b_{j+1}} W^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Each part, defined by the nodal lines, can be regarded as a single plate and the fact that it belongs to the original plate requires

$$\omega^2 = \frac{V_{11}}{T_{11}} = \frac{V_{12}}{T_{12}} = \cdots = \frac{V_{ij}}{T_{ij}} = \cdots = \frac{V_{nm}}{T_{nm}}. \tag{23}$$

It is well known that Eq. (19) gives approximate values of ω^2 if approximate expression for $W(x, y)$ is used, whereas the proper restraint conditions are fulfilled.

Therefore, for each rectangular part forming the plate and defined by the nodal lines, one can use the expression of $W(x, y)$ referred to the fundamental frequency of a plate with the same size and the proper constraint conditions. If V_{ij}^* and T_{ij}^* are the corresponding terms of Eq. (19), one obtains

$$\omega^2 \approx \frac{V_{11}^*}{T_{11}^*} = \frac{V_{12}^*}{T_{12}^*} = \cdots = \frac{V_{ij}^*}{T_{ij}^*} = \cdots = \frac{V_{nm}^*}{T_{nm}^*}. \tag{24}$$

It can be demonstrated that, starting from Eq. (24), it is possible to obtain an equation equivalent to Eq. (22). In fact using the first of Eq. (24) one obtains

$$\frac{V_{11}^*}{T_{11}^*} = \frac{V_{12}^*}{T_{12}^*}$$

and then

$$\frac{V_{11}^*}{V_{12}^*} = \frac{T_{11}^*}{T_{12}^*}.$$

From this it can be deduced

$$\frac{V_{11}^* + V_{12}^*}{V_{12}^*} = \frac{T_{11}^* + T_{12}^*}{T_{12}^*}$$

and finally

$$\frac{V_{11}^*}{T_{11}^*} = \frac{V_{12}^*}{T_{12}^*} = \frac{V_{11}^* + V_{12}^*}{T_{11}^* + T_{12}^*} = \frac{V_{13}^*}{T_{13}^*} = \cdots .$$

Executing the same procedure of manipulation of the following $(n-2)(m-2)$ equations, one obtains

$$\omega^2 \approx \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} V_{ij}^*}{\sum_{i=1}^{n} \sum_{j=1}^{m} T_{ij}^*} = \frac{V_{11}^*}{T_{11}^*} = \frac{V_{12}^*}{T_{12}^*} = \cdots = \frac{V_{ij}^*}{T_{ij}^*} = \cdots = \frac{V_{nm}^*}{T_{nm}^*}. \tag{25}$$

Eq. (25) shows that it is possible to compute approximately the $n \times m$th frequency of an orthotropic plate, for which the modal shape divides the plate in $n \times m$ parts separated by rectilinear nodal lines, computing the fundamental frequency of any part using for it the proper constraint condition. To perform this calculation it is necessary to determine the values of a_i, b_i, that are the dimensions of each part. This can be done using the $(n-1)(m-1)$ independent equations (24) together with

$$\sum_{i=1}^{n} a_i = a, \quad \sum_{i=1}^{n} b_i = b.$$

All these form an equations system that can be easily solved. Moreover, the system can be simplified using, when available, the symmetry conditions.

3.3. Application of the new method

An example can clarify the described procedure. Let the orthotropic plate have the restraint conditions shown in Fig. 4, i.e. (SSCS). The frequencies 1×1, 2×1, 1×2, 2×2, 3×1, 3×2 are to be computed. For the frequency (1×1) the condition is shown in Fig. 4.

The value of the natural frequency can be calculated by means of the equation

$$f = \frac{\pi}{2} \sqrt{\frac{1}{\rho h}} \sqrt{\frac{D_1}{a^4} + \frac{2.441 D_2}{b^4} + \frac{2.333 H}{a^2 b^2}}.$$

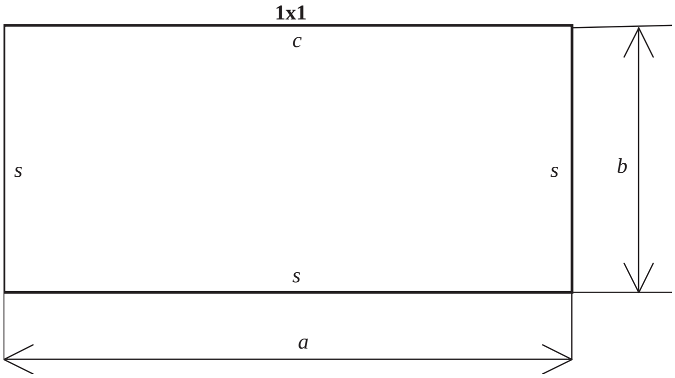


Fig. 4. 1 × 1 mode for SSCS constrained plate.

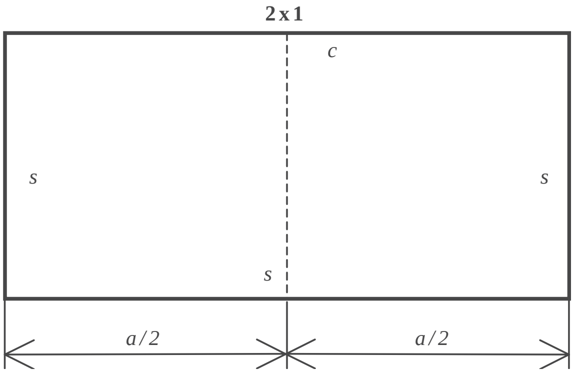


Fig. 5. 2 × 1 mode for SSCS constrained plate.

For the natural frequency (2 × 1) the nodal line pattern is shown in Fig. 5 where the location of the nodal line is symmetric due to the symmetry of the restraint condition.

For this the value of the frequency can be calculated by the equation

$$f = \frac{\pi}{2}\sqrt{\frac{1}{\rho h}}\sqrt{\frac{D_1}{(a/2)^4} + \frac{2.441 D_2}{b^4} + \frac{2.333 H}{(a/2)^2 b^2}}.$$

For the natural frequency (1 × 2) the condition is shown in Fig. 6 where the location of the nodal line is unsymmetrical due to the asymmetry of the restraint condition.

The value of b_1 can be calculated imposing the condition that the two parts of the plate vibrate at the same frequency, that is, from Eq. (14) and from the values listed in Table 1

$$\frac{D_1}{a^4} + \frac{2.441 D_2}{b_1^4} + \frac{2.333 H}{a^2 b_1^2} = \frac{D_1}{a^4} + \frac{D_2}{(b - b_1)^4} + \frac{2H}{a^2 (b - b_1)^2}.$$

Then the frequency is given by the equation

$$f = \frac{\pi}{2}\sqrt{\frac{1}{\rho h}}\sqrt{\frac{D_1}{a^4} + \frac{2.441 D_2}{b_1^4} + \frac{2.333 H}{a^2 b_1^2}}.$$

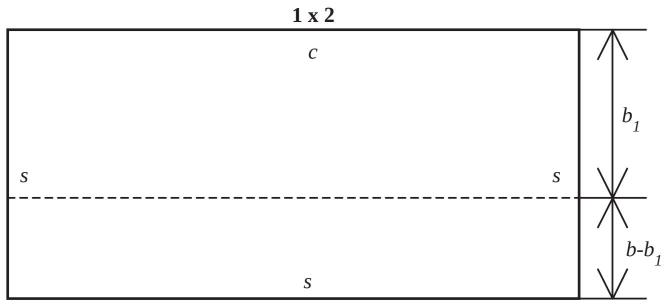


Fig. 6. 1×2 mode for SSCS constrained plate.

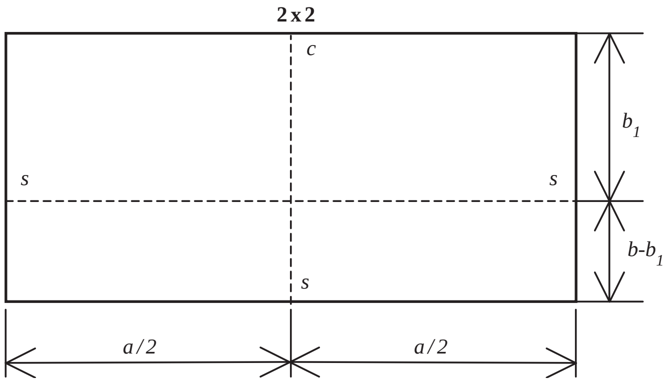


Fig. 7. 2×2 mode for SSCS constrained plate.

For the frequency (2×2) the condition is shown in Fig. 7 where the location of the two nodal lines are similar to those assumed for the previous cases.

The value of b_1 can be calculated imposing the condition that the two parts of the plate vibrate at the same frequency, that is

$$\frac{D_1}{(a/2)^4}+\frac{2.441D_2}{b_1^4}+\frac{2.333H}{(a/2)^2b_1^2}=\frac{D_1}{(a/2)^4}+\frac{D_2}{(b-b_1)^4}+\frac{2H}{(a/2)^2(b-b_1)^2}.$$

Then the frequency can be calculated by the equation

$$f=\frac{\pi}{2}\sqrt{\frac{1}{\rho h}}\sqrt{\frac{D_1}{(a/2)^4}+\frac{2.441D_2}{b_1^4}+\frac{2.333H}{(a/2)^2b_1^2}}.$$

The case of the frequency (3×1) can be solved considering (Fig. 8) the nodal lines are spaced by $a/3$ and the formula to be used is

$$f=\frac{\pi}{2}\sqrt{\frac{1}{\rho h}}\sqrt{\frac{D_1}{(a/3)^4}+\frac{2.441D_2}{b^4}+\frac{2.333H}{(a/3)^2b^2}}.$$

Finally, for the case of frequency (3×2) the pattern of the nodal lines are those shown in Fig. 9. The value of b_1 can be calculated imposing the condition of equal frequency for two adjacent

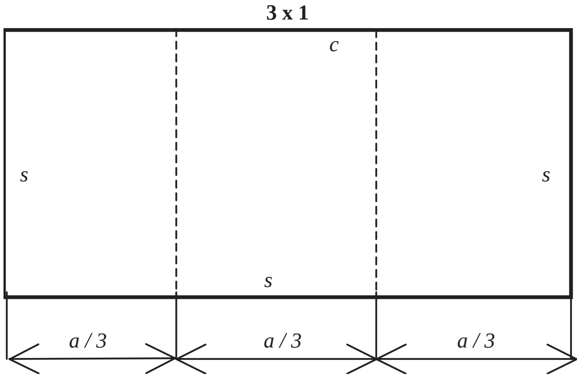


Fig. 8. 3×1 mode for SSCS constrained plate.

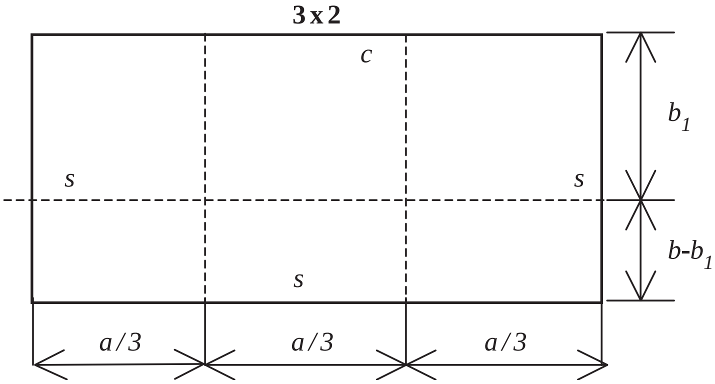


Fig. 9. 3×2 mode for SSCS constrained plate.

parts, that is

$$\frac{D_1}{(a/3)^4}+\frac{2.441D_2}{b_1^4}+\frac{2.333H}{(a/3)^2b_1^2}=\frac{D_1}{(a/3)^4}+\frac{D_2}{(b-b_1)^4}+\frac{2H}{(a/3)^2(b-b_1)^2}.$$

The formula to calculate the frequency value is therefore the following

$$f=\frac{\pi}{2}\sqrt{\frac{1}{\rho h}}\sqrt{\frac{D_1}{(a/3)^4}+\frac{2.441D_2}{b_1^4}+\frac{2.333H}{(a/3)^2b_1^2}}.$$

Since the procedure now introduced is general, it can be applied to any constraint condition shown in Table 1 and to any order of frequency, defined by the two indexes *i,j*.

4. Numerical results and discussion

In order to test the accuracy of the proposed procedure, an extended FE investigation was executed. The commercial program NASTRAN v.70.5 was used. The first step of the investigation was to define a reliable mesh to use in computing. To test the accuracy of the FE models the exact values of the first four frequencies for a simply supported isotropic rectangular plate were used comparing them with the corresponding values obtained with FEM models with 20 × 20, 40 × 40, 80 × 80 and 120 × 120 plate elements. The results obtained are summarised in Table 2. The error e is defined as

$$e = \frac{f_{\text{exact}} - f_{\text{FEM}}}{f_{\text{exact}}} \times 100,$$

where f_{exact} and f_{FEM} are, respectively, the values calculated with exact formula (11) and with the FE models for the modes 1 × 1, 2 × 1, 1 × 2 and 2 × 2.

The final choice was a mesh with 80 × 80 plate elements that it is a good compromise between the need for accuracy and that of limiting the computing time.

After this, the investigation of accuracy of the proposed method was performed using an isotropic material and three different orthotropic ones. Moreover the calculations were executed for three different ratios between a and b.

The properties of the material used are shown in Table 3.

It is important to note that the material named ORTHO1 is a Huber type material. Hence, given

$$D_1 = 850.340\,\text{N mm}, \quad D_2 = 425.170\,\text{N mm}, \quad D_{12} = 258.167\,\text{N mm}.$$

Table 2
Results obtained in the test of accuracy of finite element mesh

Mesh	$e = \frac{f_{\text{exact}} - f_{\text{FEM}}}{f_{\text{exact}}} \times 100$			
	1 × 1	2 × 1	1 × 2	2 × 2
20 × 20	0.44	0.69	0.45	1.21
40 × 40	0.22	0.21	0.11	0.38
80 × 80	0.11	0.05	0.04	0.11
120 × 120	0.11	0.05	0.00	0.05

Table 3
Properties of materials used

	Materials			
	ISO	ORTHO1	ORTHO2	ORTHO3
E_x (MPa)	1E+10	1E+10	1E+10	1E+10
E_y (MPa)	1E+10	5E+09	5E+09	6.67E+09
ν_{xy}	0.2	0.2	0.24	0.25
G_{xy} (MPa)	4.17E+09	3.1E+09	2.05E+09	3.04E+09
ρ (kg/m^3)	7800	7800	7800	7800

Table 4
Geometric properties of the plates

	Geometry		
	MOD1	MOD2	MOD3
a (m)	4	4	4
b (m)	3	3.2	2.667
h (m)	0.01	0.01	0.01

H respects the condition $H = \sqrt{D_1 D_2}$:

$$H = \nu_{21} D_1 + 2D_{12} = 601.367 = \sqrt{D_1 D_2}.$$

The geometric properties of the plates analysed are reported in Table 4.

In Table 5 the results obtained for the isotropic material are shown for the different constraint conditions defined in Table 1. In the table it is also shown the values of the relative difference between the FEM values and those computed with the new method, computed as

$$\Delta = \frac{f_{\mathrm{FEM}} - f_{\mathrm{NM}}}{f_{\mathrm{FEM}}} \times 100, \tag{26}$$

where f_{FEM} and f_{NM} are, respectively, the values calculated by the finite element models and those calculated with the new method.

In the Tables 6–8 the results obtained for orthotropic plates are shown respectively for the materials ORTHO1, ORTHO2 and ORTHO3 and reported in Table 3.

Comparing the values obtained by the FEM models and the new method we conclude that the maximum difference is equal to −0.74%, while the average difference is 0.37%. Numerical results obtained by means of the new method overestimate FEM results. For this reason all the differences, evaluated by means of expression (26) have negative algebraic signs.

5. Conclusions

A new method for computing the frequencies of orthotropic plates with any constraint condition was proposed. The method is based on the use of the Hearmon expression of the fundamental frequency and the calculation of the higher modal frequencies is executed considering the following main hypotheses:

- the nodal lines are rectilinear and divide the plate in different parts forming rectangular plates;
- all the resultant parts vibrate at the same frequency;
- the modal shape of each part is the same that part would have if separated from the original plate and restrained with the proper conditions.

Modal lines are determined by imposing the same modal frequency for each stand-alone part. Hence, the calculation of the frequency of any part directly leads to determine the frequency of the original plate.

Table 5
Isotropic plate: results of frequency calculation by the new method (f_{NM}) and comparison with the corresponding FEM values (Δ) defined by expression (26)

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.250$						
1 × 1	0.84	−0.04	1.56	−0.37	1.45	−0.42
1 × 2	2.37	−0.04	3.57	−0.48	3.51	−0.47
2 × 1	1.82	−0.06	2.74	−0.54	2.43	−0.57
2 × 2	3.36	−0.11	4.66	−0.71	4.45	−0.50
3 × 1	3.46	−0.06	4.65	−0.59	4.16	−0.69
3 × 2	5.00	−0.15	6.48	−0.73	6.09	−0.67
$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	1.11	−0.33	1.17	−0.63	1.07	−0.21
1 × 2	2.53	−0.73	2.95	−0.56	2.88	−0.08
2 × 1	2.45	−0.11	2.26	−0.65	1.98	−0.65
2 × 2	3.83	−0.53	3.97	−0.34	3.78	−0.25
3 × 1	4.44	−0.07	4.04	−0.62	3.58	−0.72
3 × 2	5.81	−0.30	5.70	−0.30	5.34	−0.41
$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.333$						
1 × 1	0.91	−0.08	1.70	−0.35	1.60	−0.41
1 × 2	2.66	−0.06	4.00	−0.46	3.95	−0.43
2 × 1	1.89	−0.07	2.86	−0.55	2.54	0.32
2 × 2	3.64	−0.11	5.07	−0.72	4.87	−0.49
3 × 1	3.53	−0.08	4.76	−0.61	4.27	−0.68
3 × 2	5.28	−0.16	6.87	−0.73	6.49	−0.65
$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	1.17	−0.41	1.27	−0.65	1.18	−0.22
1 × 2	2.81	−0.73	3.30	−0.56	3.23	−0.05
2 × 1	2.51	−0.15	2.35	−0.68	2.08	−0.64
2 × 2	4.09	−0.60	4.31	−0.33	4.13	−0.24
3 × 1	4.50	−0.10	4.13	−0.69	3.68	−0.74
3 × 2	6.07	−0.35	6.03	−0.29	5.69	−0.40

Table 5 (*continued*)

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.500$						
1×1	1.06	−0.04	2.02	−0.36	1.94	−0.38
1×2	3.27	−0.04	4.96	−0.43	4.91	−0.39
2×1	2.05	−0.07	3.13	−0.54	2.86	−0.51
2×2	4.26	−0.10	5.99	−0.69	5.81	−0.44
3×1	3.68	−0.07	4.99	−0.62	4.53	−0.67
3×2	5.89	−0.16	7.74	−0.70	7.39	−0.59
$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1×1	1.30	−0.47	1.50	−0.60	1.41	−0.14
1×2	3.41	−0.73	4.07	−0.37	4.01	−0.02
2×1	2.64	−0.18	2.55	−0.63	2.30	−0.55
2×2	4.68	−0.65	5.07	−0.33	4.90	−0.18
3×1	4.64	−0.10	4.32	−0.68	3.88	−0.74
3×2	6.65	−0.39	6.76	−0.25	6.44	−0.37

Table 6
Orthotropic plate with ORTHO1 material: results of frequency calculation by the new method (f_{NM}) and comparison with the corresponding FEM values (Δ) defined by expression (26)

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.250$						
1×1	0.68	−0.05	1.25	−0.38	1.12	−0.46
1×2	1.76	−0.05	2.63	−0.52	2.56	−0.53
2×1	1.66	−0.05	2.47	−0.53	2.15	−0.61
2×2	2.73	−0.11	3.76	−0.71	3.52	−0.57
3×1	3.28	−0.05	4.40	−0.55	3.89	−0.65
3×2	4.35	−0.14	5.63	−0.74	5.20	−0.72
$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1×1	0.98	−0.23	0.94	−0.64	0.83	−0.31
1×2	1.94	−0.69	2.17	−0.61	2.10	−0.09
2×1	2.31	−0.10	2.05	−0.65	1.75	−0.71
2×2	3.24	−0.44	3.22	−0.32	3.00	−0.31
3×1	4.27	−0.03	3.82	−0.57	3.36	−0.67
3×2	5.21	−0.23	4.96	−0.32	4.57	−0.45

Table 6 (*continued*)

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.333$						
1 × 1	0.73	−0.05	1.34	−0.38	1.22	−0.45
1 × 2	1.95	−0.05	2.93	−0.50	2.86	−0.51
2 × 1	1.70	−0.06	2.55	−0.54	2.23	−0.61
2 × 2	2.93	−0.11	4.04	−0.71	3.81	−0.54
3 × 1	3.33	−0.05	4.47	−0.61	3.97	−0.70
3 × 2	4.55	−0.15	5.88	−0.72	5.48	−0.73

$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	1.02	−0.26	1.01	−0.64	0.91	−0.27
1 × 2	2.13	−0.71	2.42	−0.58	2.35	−0.02
2 × 1	2.35	−0.11	2.11	−0.67	1.82	−0.69
2 × 2	3.42	−0.48	3.45	−0.33	3.25	−0.28
3 × 1	4.31	−0.12	3.88	−0.60	3.42	−0.69
3 × 2	5.39	−0.23	5.18	−0.31	4.81	−0.42

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.500$						
1 × 1	0.84	−0.05	1.56	−0.37	1.45	−0.42
1 × 2	2.39	−0.04	3.59	−0.48	3.53	−0.46
2 × 1	1.81	−0.06	2.72	−0.55	2.42	−0.57
2 × 2	3.36	−0.11	4.67	−0.70	4.46	−0.50
3 × 1	3.43	−0.06	4.62	−0.59	4.13	−0.68
3 × 2	4.98	−0.16	6.47	−0.72	6.08	−0.67

$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	1.11	−0.34	1.17	−0.63	1.07	−0.21
1 × 2	2.54	−0.73	2.96	−0.56	2.89	−0.09
2 × 1	2.44	−0.14	2.25	−0.65	1.97	−0.65
2 × 2	3.82	−0.56	3.97	−0.36	3.78	−0.18
3 × 1	4.40	−0.05	4.01	−0.62	3.56	−0.72
3 × 2	5.78	−0.32	5.68	−0.29	5.33	−0.41

Table 7
Orthotropic plate with ORTHO2 material: results of frequency calculation by the new method (f_{NM}) and comparison with the corresponding FEM values (Δ) defined by expression (26)

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.250$						
1 × 1	0.64	−0.04	1.22	−0.24	1.08	−0.29
1 × 2	1.69	−0.04	2.57	−0.34	2.50	−0.36
2 × 1	1.59	−0.04	2.41	−0.35	2.08	−0.41
2 × 2	2.55	−0.10	3.60	−0.49	3.36	−0.40
3 × 1	3.20	−0.04	4.33	−0.37	3.81	−0.44
3 × 2	4.10	−0.13	5.39	−0.51	4.95	−0.52

$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	0.94	−0.15	0.90	−0.42	0.79	−0.21
1 × 2	1.86	−0.48	2.10	−0.40	2.04	−0.05
2 × 1	2.25	−0.07	1.97	−0.43	1.68	−0.48
2 × 2	3.07	−0.31	3.04	−0.18	2.83	−0.18
3 × 1	4.21	−0.02	3.73	−0.38	3.27	−0.46
3 × 2	4.99	−0.19	4.70	−0.18	4.32	−0.34

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.333$						
1 × 1	0.69	0.03	1.31	−0.27	1.19	−0.33
1 × 2	1.89	−0.05	2.87	−0.35	2.81	−0.35
2 × 1	1.63	−0.03	2.48	−0.35	2.15	−0.43
2 × 2	2.74	−0.11	3.87	−0.49	3.64	−0.38
3 × 1	3.24	−0.05	4.38	−0.37	3.88	−0.45
3 × 2	4.28	−0.12	5.62	−0.50	5.20	−0.51

$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	0.98	−0.19	0.97	−0.44	0.86	−0.22
1 × 2	2.05	−0.47	2.35	−0.40	2.29	−0.03
2 × 1	2.29	−0.08	2.03	−0.45	1.74	−0.46
2 × 2	3.24	−0.35	3.27	−0.12	3.06	−0.15
3 × 1	4.25	−0.02	3.80	−0.40	3.32	−0.48
3 × 2	5.14	−0.209	4.91	−0.16	4.53	−0.32

Table 7 (*continued*)

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.500$						
1 × 1	0.79	−0.04	1.52	−0.24	1.41	−0.27
1 × 2	2.32	−0.03	3.53	−0.31	3.48	−0.31
2 × 1	1.72	−0.05	2.64	−0.36	2.33	−0.38
2 × 2	3.15	−0.10	4.47	−0.48	4.27	−0.35
3 × 1	3.33	−0.05	4.52	−0.39	4.02	−0.46
3 × 2	4.67	−0.14	6.16	−0.51	5.77	−0.47
$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	1.06	−0.23	1.12	−0.41	1.02	−0.14
1 × 2	2.47	−0.51	2.90	−0.37	2.84	0.00
2 × 1	2.36	−0.10	2.15	−0.44	1.87	−0.44
2 × 2	3.61	−0.40	3.77	−0.20	3.59	−0.22
3 × 1	4.32	−0.03	3.90	−0.42	3.44	−0.49
3 × 2	5.49	−0.24	5.37	−0.15	5.01	−0.27

Table 8
Orthotropic plate with ORTHO3 material: results of frequency calculation by the new method (f_{NM}) and comparison with the corresponding FEM values (Δ) defined by expression (26)

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.250$						
1 × 1	0.73	−0.05	1.36	−0.34	1.24	−0.40
1 × 2	1.98	−0.05	2.98	−0.45	2.91	−0.46
2 × 1	1.70	−0.06	2.56	−0.48	2.24	−0.54
2 × 2	2.92	−0.11	4.06	−0.65	3.83	−0.50
3 × 1	3.34	−0.05	4.49	−0.52	3.99	−0.60
3 × 2	4.54	−0.15	5.89	−0.67	5.48	−0.65
$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	1.02	−0.25	1.02	−0.57	0.91	−0.24
1 × 2	2.15	−0.65	2.45	−0.54	2.39	−0.07
2 × 1	2.36	−0.11	2.11	−0.59	1.82	−0.63
2 × 2	3.42	−0.45	3.46	−0.29	3.25	−0.20
3 × 1	4.34	−0.03	3.90	−0.54	3.43	−0.63
3 × 2	5.39	−0.26	5.18	−0.27	4.80	−0.39

Table 8 (*continued*)

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.333$						
1 × 1	0.79	−0.05	1.47	−0.34	1.36	−0.38
1 × 2	2.21	−0.04	3.33	−0.44	3.27	−0.43
2 × 1	1.76	−0.06	2.65	−0.49	2.34	−0.45
2 × 2	3.15	−0.11	4.39	−0.64	4.17	−0.40
3 × 1	3.39	−0.06	4.57	−0.53	4.07	−0.56
3 × 2	4.76	−0.15	6.19	−0.59	5.79	−0.51

$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	1.07	−0.28	1.10	−0.57	1.00	−0.21
1 × 2	2.37	−0.66	2.75	−0.79	2.68	−0.28
2 × 1	2.40	−0.12	2.18	−0.51	1.90	−0.61
2 × 2	3.63	−0.49	3.75	−0.64	3.55	−0.71
3 × 1	4.39	−0.09	3.96	−0.47	3.50	−0.65
3 × 2	5.58	−0.17	5.45	−0.44	5.09	−0.73

$i \times j$	SSSS		CCCC		CSCC	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
$a/b = 1.500$						
1 × 1	0.91	−0.05	1.73	−0.33	1.63	−0.36
1 × 2	2.71	−0.04	4.10	−0.41	4.05	−0.39
2 × 1	1.88	−0.07	2.86	−0.49	2.57	−0.49
2 × 2	3.65	−0.11	5.12	−0.63	4.93	−0.44
3 × 1	3.51	−0.07	4.75	−0.54	4.26	−0.61
3 × 2	5.25	−0.16	6.87	−0.65	6.50	−0.59

$i \times j$	SCSC		SSCC		SSCS	
	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)	f_{NM} (Hz)	Δ (%)
1 × 1	1.17	−0.36	1.28	−0.56	1.19	−0.16
1 × 2	2.86	−0.67	3.38	−0.49	3.31	−0.05
2 × 1	2.50	−0.16	2.34	−0.59	2.07	−0.56
2 × 2	4.09	−0.56	4.34	−0.30	4.16	−0.19
3 × 1	4.49	−0.06	4.11	−0.60	3.66	−0.67
3 × 2	6.04	−0.33	6.01	−0.25	5.67	−0.33

The new method follows from a particular application of the Rayleigh method in which the deformed shape $W(x, y)$ is assumed equal to the superposition of $W_i(x, y)$ due to the fundamental frequency of each part that originate from the nodal lines, provided that they are rectilinear and perpendicular to the edges. Therefore the proposed method cannot be applied to square isotropic plates with complex shapes of the nodal lines.

The procedure accuracy was tested comparing the natural frequencies obtained with the results of FEM analyses of the same plates. The comparison shows a very good agreement. The differences between proposed method results and FEM results, evaluated by expression (26), are less than 0.74%. Because of the high level of accuracy, the method is suitable for preliminary design calculation as well as for final check of accuracy in the quality control of the design.

Since the new procedure is quite general for rectangular plates, both isotropic and orthotropic, it seems possible to extend it to the cases of free boundaries and elastic restraints, that are not herein studied.

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