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Journal of Sound and Vibration 288 (2005) 33–42
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Vibration of elastically restrained cross-ply laminated plates with variable thickness

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Received 5 February 2004; received in revised form 6 December 2004; accepted 15 December 2004
Available online 17 February 2005

Abstract

The natural frequencies of symmetrically laminated plates of variable thickness are analyzed using the finite strip transition matrix technique. In this paper, the natural frequencies of such plates are determined for edges with being elastically restrained against both rotation and transition or both.

A successive conjunction of the classical finite strip method and the transition matrix method is applied to develop a new modification of the finite strip method to reduce the complexity of the problem. The displacement function is expressed as the product of a basic trigonometric series function in the longitudinal direction and an unknown function that has to be determined in the other direction. Using the new transition matrix, after necessary simplification and the satisfaction of the boundary conditions, yields a set of simultaneous equations that leads to the characteristic matrix of vibration.

The mode shapes and the frequency parameters for different combinations of elastic or translational restraint coefficients have been presented and compared with those available from other methods in the literature. Also, the effect of the tapered ratio and the aspect ratio on the natural frequencies and the mode shapes of the plates are presented. The good agreement with other methods demonstrates the validity and the reliability of the proposed method.

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2004.12.012

1. Introduction

Laminated plates are widely used in many engineering application. In some applications, the designer has to construct variable thickness plates to save material or to meet certain criteria. The vibration of uniform plates with elastically restrained boundary conditions has been investigated by many authors [1–10]. Rais-Rohani and Marcellier [11] presented approximate analytical solutions for the free vibration and buckling of rectangular anisotropic plates as well as rectangular sandwich plates with edge restrained against rotation.

In many applications, rectangular plates with edge beams, such as building or bridge slabs, can be modeled as elastically restrained plates [12]. The elastic restraints can also be used in wing structures [11], in modeling the cracks in beam or plates [13] and modeling step in a beam or plates [14]. Sonoda and Kobayashi [15] presented an exact solution for isotropic variable thickness plate in one direction with two opposite edges being simply supported and the other two edges being elastically restrained against rotation. Grossi and Bhat [16] used Rayleigh–Ritz method with the boundary characteristics orthogonal polynomials as shape functions and Rayleigh Schmidt method to find the natural frequencies of isotropic tapered rectangular plates with edges elastically restrained against rotation and translation. Gutierrez and Laura [17] used the differential quadrature method to determine the fundamental frequencies of rectangular plates with linearly varying thickness and non-uniform boundary conditions. Filipich et al. [18] used the Galerkin method to obtain an approximate solution to the vibration of isotropic rectangular plates of variable thickness with two opposite edges simply supported and very general boundary conditions on the other two edges. Li [19] used an analytical approach to determining natural frequencies and mode shape of non-uniform flexural-shear plates. Ashour [20] used the finite strip transition matrix method to investigate the vibration of isotropic variable thickness plates with edges restrained against both rotation and translation.

To the best of the author's knowledge there is no publication available in the open literature on this problem. The main objective of this paper is to determine the natural frequencies of cross-ply symmetrically laminated plates of variable thickness subjected to elastically restrained boundary conditions against both rotation and translation in the variable thickness direction and any combination of clamped or simply supported boundary conditions in the other direction.

2. Governing equations of elastic restrained plates

Consider a cross-ply symmetrically laminated rectangular plate of variable thickness h, length a, width b, density ρ and with elastically restrained boundary conditions at $y = 0$ and b as shown in Fig. 1.

The governing equation can be written as

$$\bar{D}_{11}\,\frac{h^3(y)}{h_0^3}\,W_{xxxx} + 2(\bar{D}_{12} + 2\bar{D}_{66})\left[\frac{h^3(y)}{h_0^3}\,W_{xxyy} + \frac{1}{h_0^3}\,\frac{\partial h^3(y)}{\partial y}\,W_{xxy}\right] + \nu_{12}\bar{D}_{22}\,\frac{1}{h_0^3}\,\frac{\partial^2 h^3(y)}{\partial y^2}\,W_{xx}$$
$$+\frac{\bar{D}_{22}}{h_0^3}\left(h^3(y)W_{yyyyy} + 2\,\frac{\partial h^3(y)}{\partial y}\,W_{yyy} + \frac{\partial^2 h^3(y)}{\partial y^2}\,W_{yy}\right) = -\bar{m}_0\,\frac{h(y)}{h_0}\,W_{tt}, \qquad (1)$$

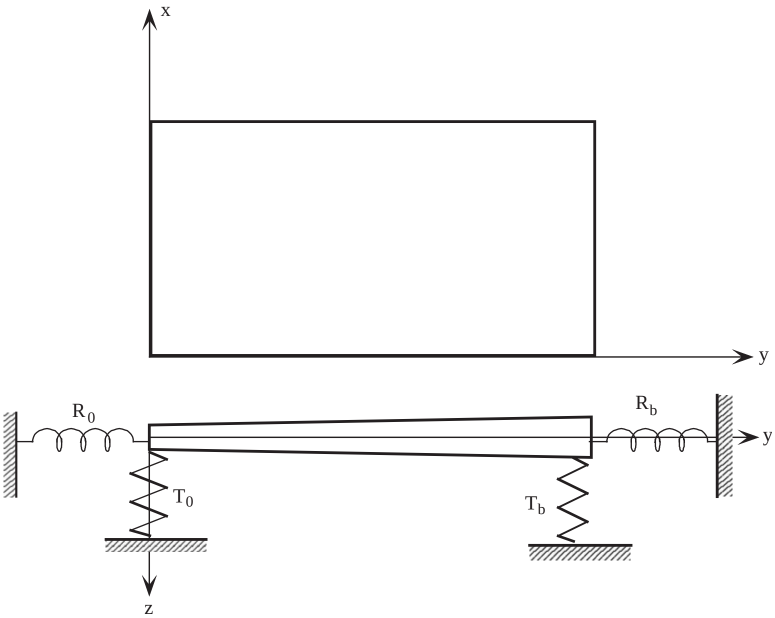

Fig. 1. An elastically restrained laminated plate with variable thickness.

where W is the flexural displacement, $\bar{D}_{ij} = D_{ij}(h^3/h_0^3)$ are the bending rigidities, h_0 is the plate height at $y = 0$,

$$D_{ij} = \sum_{k=1}^{N} \int_{-h/2}^{h/2} Q_{ij}^{(k)} z^2 \, \mathrm{d}z, \quad i,j = 1, 2, 6 \tag{2}$$

and $\bar{Q}_{ij}^k$ are the plane stress transformed reduced stiffness coefficients of the lamina in the laminate coordinate system $oxyz$. They are related to the reduced stiffness coefficients of the lamina in the material axes of the lamina Q_{ij}^k by proper coordinate relationships which are available in many texts and can be expressed in terms of Engineering notations as

$$Q_{11} = \frac{E_{11}}{(1 - v_{12}v_{21})}, \quad Q_{22} = \frac{E_{22}}{(1 - v_{21}v_{12})}, \quad Q_{12} = \frac{v_{21}E_{11}}{(1 - v_{12}v_{21})}, \quad Q_{21} = Q_{12}, \quad Q_{66} = G_{12}. \tag{3}$$

Here E_{11}, E_{22} are the longitudinal and transverse plate moduli, respectively, and G_{12} is the in-plane shear modulus, and v_{12} and v_{21} are the Poisson's ratios.

2.1. Boundary conditions

The considered boundary conditions along the y direction are elastically restrained against both rotation and translation. At $y = 0$, the boundary conditions for this case are

$$R_0 \frac{\partial W}{\partial y} - D_{22_0} \left\{ \frac{\partial^2 W}{\partial y^2} + v_{12} \frac{\partial^2 W}{\partial x^2} \right\} = 0, \tag{4}$$

$$T_0 W + D_{22_0} \frac{\partial^3 W}{\partial y^3} + (2D_{3_0} - \nu_{12} D_{22_0}) \frac{\partial^3 W}{\partial x^2 y} = 0, \tag{5}$$

where $D_3 = D_{12} + 2D_{66}$, the suffixes "0, b" means the rigidities are calculated at $y = 0$ and b, respectively. The boundary conditions on the other elastically restrained end $y = b$ are

$$R_b \frac{\partial W}{\partial y} + D_{22_b} \left\{ \frac{\partial^2 W}{\partial y^2} + \nu_{12} \frac{\partial^2 W}{\partial x^2} \right\} = 0, \tag{6}$$

$$T_b W - D_{22_b} \frac{\partial^3 W}{\partial y^3} + (2D_{3_b} - \nu_{12} D_{22_b}) \frac{\partial^3 W}{\partial x^2 y} = 0. \tag{7}$$

The boundary conditions at the other two edges $x = 0$ and a can be any combination of the classical boundary conditions.

2.2. Method of solution and the eigenvalue problem

Assuming a solution of the form

$$W = \sum_{m=1}^{M} X_m(x) y_m(y), \tag{8}$$

where X_m are the beam functions that satisfy the boundary conditions at $x = 0$ and a. The governing equation can be transformed into $4M$ number of first-order differential equations in terms of the normalized coordinates which can be solved as in Ref. [21]. In many papers, e.g. Refs. [1,3], the boundary conditions are solved approximately by neglecting some of the terms in Eqs. (4)–(7). In this paper, the elastically restrained boundary conditions are solved exactly. Using Eq. (8), the boundary conditions at the normalized coordinates $\eta = 0$ are

$$\frac{d^2 Y_j}{d\eta^2} + \frac{\nu_{xy}}{\beta^2} \sum_{i=1}^{M} \frac{c_{ij}}{a_{ij}} Y_i = \phi_{R_0} \frac{dY_i}{d\eta}, \tag{9}$$

$$\frac{d^3 Y_j}{d\eta^3} + \frac{1}{\beta^2} \left(\frac{2D_{3_0}}{D_{22_0}} - \nu_{xy} \right) \sum_{i=1}^{M} \frac{c_{ij}}{a_{jj}} \frac{dY_i}{d\eta} = -\phi_{T_0} Y_i, \quad j = 1, 2, \ldots, M, \tag{10}$$

where $\phi_{T_0} = b^3 T_0 / D_{22_0}$ and $\phi_{R_0} = bR_0 / D_{22_0}$ and at $\eta = 1$,

$$\frac{d^2 Y_j}{d\eta^2} + \frac{\nu_{xy}}{\beta^2} \sum_{i=1}^{M} \frac{c_{ij}}{a_{ij}} Y_i = -\phi_{R_b} \frac{dY_i}{d\eta}, \tag{11}$$

$$\frac{d^3 Y_j}{d\eta^3} + \frac{1}{\beta^2} \left(\frac{2D_{3_b}}{D_{22_b}} - \nu_{xy} \right) \sum_{i=1}^{M} \frac{c_{ij}}{a_{jj}} \frac{dY_i}{d\eta} = \varphi_{T_b} Y_i, \quad j = 1, 2, \ldots, M, \tag{12}$$

where $\phi_{T_b} = b^3 T_b / D_{22_b}$ and $\phi_{R_b} = bR_b / D_{22_b}$.

In the next sections, the symbol S–C–S–ER for example means, that the edges $x = 0$, $y = 0$, $x = a$, $y = b$ are simply supported, clamped, simply supported and elastically restrained, respectively. A linearly tapered plate is used to illustrate the above technique with the following non-dimensional variable thickness $h(\eta) = 1 + \delta\eta$, where δ is called the taper ratio.

3. Convergence and comparison investigation

3.1. Convergence analysis

Since no solution exists for the above problem, one has to carry out several convergence studies. First, a convergence investigation is carried out to examine the effect of number of terms of the power series M used in the solution. The results are shown in Table 1 for uniform three-layer cross-ply symmetrically laminated plates (90/0/90). The material properties used in this case are given by $E_{22}/E_{11} = 40$, $G_{12}/E_{11} = 0.6$, $v_{21} = 0.25$, the aspect ratio $\beta = 1$, and the frequency parameter λ is $\lambda = \omega a^2/h\sqrt{(\rho/E_{11})}$. From Table 1, it is very clear that the method converge very rapidly. Then, convergence studies of the frequency parameter for three-ply (90,0,90) C–ER–C–ER plate with uniform and varying thickness plates are carried out for equally elastically restrained plate ($\phi_{T_a} = \phi_{T_b} = \phi_{R_a} = \phi_{R_b} = 100$). The results are shown in Table 2, it is clear the method converges much faster, only after few terms.

3.2. Comparison analysis

The first four frequency parameters Ω for uniform laminated plates subject to classical boundary conditions are presented and compared with results from Ref. [23] in Table 3. It can be

Table 1
Convergence investigations of the frequency parameters λ for three-ply 90,0,90 simply supported SSSS and SCSC plates ($\beta = 1$)

	M	λ_1	λ_2	λ_3	λ_4
SSSS	1	18.8913	26.9401	46.2420	
	2	18.8913	26.9401	46.2420	71.6202
	3	18.8913	26.9401	46.2420	71.6202
	4	18.8913	26.9401	46.2420	71.6202
	5	18.8913	26.9401	46.2420	71.6202
	6	18.8913	26.9401	46.2420	71.6202
(Exact [22])		18.891			
CSCS	1	40.7435	45.2365	59.0590	84.8359
	2	40.7434	45.2344	59.0511	84.8192
	3	40.7434	45.2344	59.0511	84.8192
	4	40.7434	45.2342	59.0503	84.8174
	5	40.7434	45.2342	59.0503	84.8174
(Exact [22])		40.743			

Table 2
Convergence investigation of frequency parameters λ for three-ply 90,0,90 C–ER–C–ER-plate with uniform and varying thickness plates

	δ	M	λ_1	λ_2	λ_3	λ_4
ER = 100	0.0	2	40.5422	41.6811	46.0965	59.3214
	0.0	3	40.5421	41.6809	46.0945	59.3136
	0.0	4	40.5421	41.6809	46.0945	59.3136
	0.0	5	40.5421	41.6809	46.0943	59.3128
	0.4	2	44.3975	51.4047	56.5283	70.5018
	0.4	3	44.3973	51.4036	56.5267	70.4925
	0.4	4	44.3973	51.4036	56.5267	70.4925
	0.4	5	44.3973	51.4035	56.5265	70.4915

Table 3
The frequencies parameters Ω_i of three-ply uniform laminated plates 90,0,90 for different combinations of boundary conditions with some comparison ($N = 6$)

Boundary condition	Reference	Ω_1	Ω_2	Ω_3	Ω_4
CCCC	Present	14.6680	17.6187	24.5396	35.6562
	[23]	14.666	17.614	24.511	35.532
CFCF	Present	14.0758	14.2218	15.1026	18.2373
	[23]	14.072	14.199	15.037	18.136
SSSS	Present	6.6254	9.4482	16.2177	25.1181
	[23]	6.625	9.447	16.205	25.115
SCSC	Present	7.3962	12.1480	20.8719	25.3685
	[23]	7.396	12.144	20.841	25.365
SFSF	Present	6.2093	6.4730	8.0746	12.8690
	[23]	6.208	6.436	7.975	12.752

seen that the results agree very well with Ref. [23]. The frequency parameter Ω is obtained from λ using the relation $\Omega_1 = \lambda/\pi^2 * \sqrt{12[1 - \nu_{12}\nu_{21}]}$.

Since no results are available in the literature for elastically retrained laminated plates with variable thickness, we compare the results obtained from the limiting cases of the elastic restrained boundary conditions (i.e. $T, R \to \infty$ or 0). Table 4 shows the frequency parameters Ω_1' for elastically restrained single layer uniform and variable thickness laminated plates compared with some available results in the literature. Three cases have been considered for equally elastically restrained boundary conditions at $y = 0$ and b,

I. $R_{1,2} \to \infty$ and $T_{1,2} \to \infty$, the clamped–clamped case,
II. $R_{1,2} \to 0$ and $T_{1,2} \to \infty$, the simply supported–simply supported case,
III. $R_{1,2} \to 0$ and $T_{1,2} \to 0$, the free–free case.

Table 4
Comparison of frequency parameters Ω'_i of elastically retrained uniform and variable thickness plates ($N = 6$)

BC	ϕ_r	ϕ_i	δ	Ω'_1	Ω'_2	Ω'_3	Ω'_4
SSSS			0	15.19479	33.29969	44.42631	60.78307
SERSER	0.00E+00	1.00E+07	0	15.19477	33.29963	44.42608	60.78303
[24]				15.1946	33.2996	44.4188	60.7787
CCCC			0	29.10724	50.83234	67.32315	87.15097
CERCER	1.00E+07	1.00E+07	0	29.10721	50.8323	67.32296	87.15091
SSSS			0.4	18.17806	39.78196	53.04357	72.65364
SERSER	0.00E+00	1.00E+07	0.4	18.17801	39.78184	53.04317	72.65298
[24]			0.4	18.17794	39.78182	53.0339	72.6440
CCCC			0.4	34.72033	60.57577	80.27028	103.4959
CERCER	1.00E+07	1.00E+07	0.4	34.72027	60.57569	80.26991	102.3028
SERSER	1.00E−07	1.00E−07	0.4	7.54536	13.79558	29.49693	36.71928
SFSF			0.4	7.54536	13.79558	29.49693	36.71928
[24]				7.54537	13.79558	29.49693	36.71695

The frequency parameter Ω'_1 is defined by $\Omega'_1 = \lambda\sqrt{E_{11}/E_{22}} * \sqrt{12[1 - v_{12}v_{21}]}$. The material properties used in this case are those used by Liew [23], $E_{11} = 24.8$, $E_{22} = 60.7$, $G_{12} = 12$, $v_{21} = 0.23$ and $v_{12} = 0.09397$, the results agree very well with other results in the literature.

4. Numerical results

A parametric investigation is carried out to study the effect of elastic restraint coefficients on both the frequency parameters and the modes shapes for uniform and non-uniform laminated plates. The normalized frequency parameter λ defined earlier, $m = 4$ and $N = 20$ is used in all calculations. Fig. 2 shows the first three frequency parameters and the mode shapes for different elastic restraint coefficients. In this case, a square laminated plate (with the same laminated material as in Table 1) with simply supported at $x = 0$ and a is considered. The elastically restrained coefficients at the boundary $y = 0$ and b are considered equal ($\phi_{T_a} = \phi_{T_b} = \phi_t = \phi_{R_a} = \phi_{R_b} = \phi_r$). From the figures, one can deduce that the frequency parameters and the mode shapes are affected drastically in certain range for uniform plates $1 < \phi < 1000$, the upper limit of this range increases with the increase of the taper ratio δ and also increases for higher modes.

The effect of the taper ratio (δ) on the fundamental frequency parameter and mode shape is presented in Fig. 3 for $\delta = 0, 0.2$ and 0.4 and for different elastic restraint coefficients for the same laminated material as in Fig. 2. It can be seen that the effect of the elastic restraint coefficient is significant only in the range $1 < \phi < 1000$. This effect increases as the taper ratio δ increases. Also, it can be seen that the side with small thickness has been affected more than the other side of the variable thickness plate.

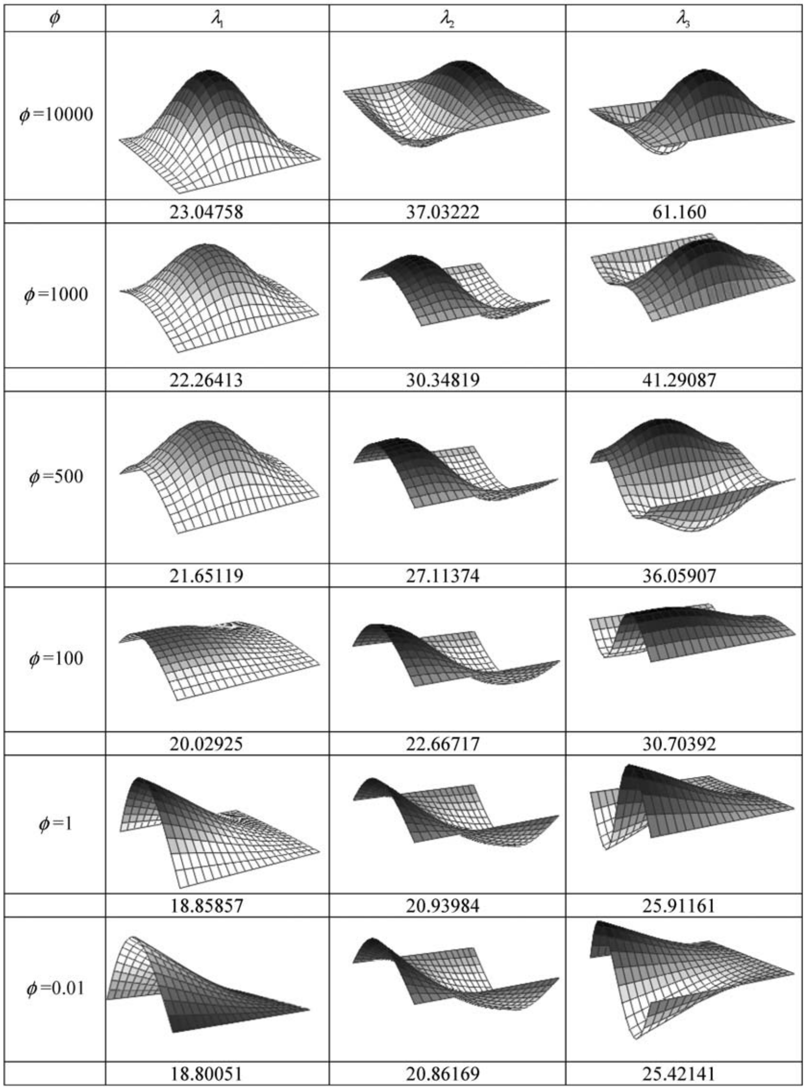

Fig. 2. The first three frequencies and modes shape of elastically restrained variable thickness laminated plates ($\delta = 0.2$).

5. Conclusion

The finite strip transition matrix method is utilized to investigate the vibration of laminated plates of variable thickness. The lamination used is limited to symmetric cross-ply. The boundary conditions considered in this paper are any combination of elastic restraints against translation, rotation or both in the variable direction and any combination of classical boundary condition in the other side. The effect of the elastic restraint coefficients on the vibration frequency parameters and mode shapes are investigated and presented.

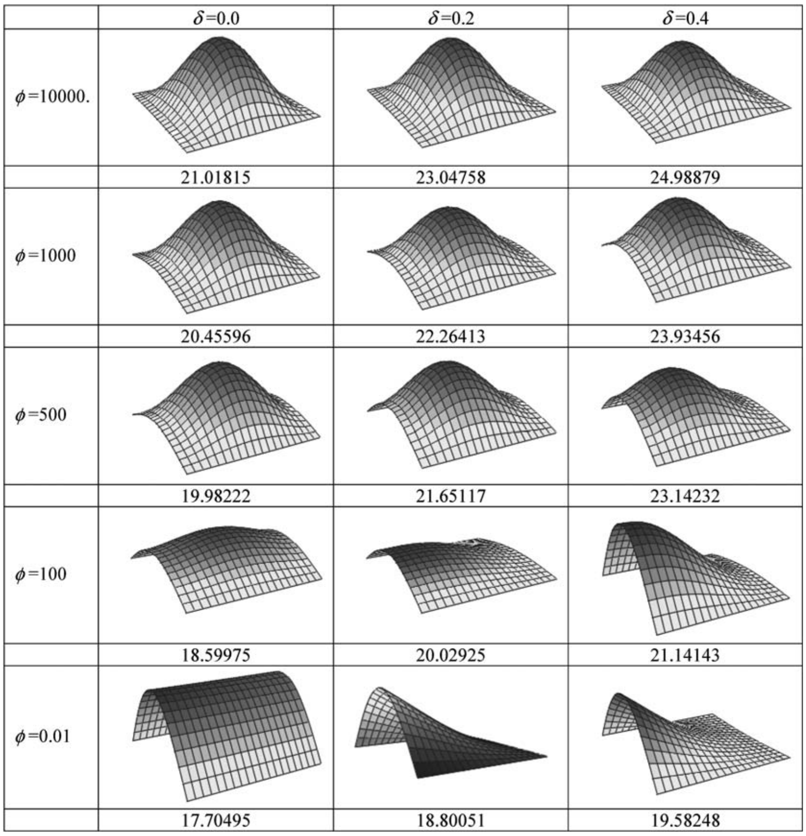

Fig. 3. The fundamental frequency parameter λ and mode for laminated plates of variable thickness ($\delta = 0, 0.2, 0.4$) for different elastic restraint coefficients.

The results presented in this paper for elastically restrained laminated plated can be considered as new in the literature and can be useful for designers and engineers.

Acknowledgements

This research was supported by the Research Center, International Islamic University Malaysia.

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