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Short Communication

Vibration analysis of a magneto-elastic beam with general boundary conditions subjected to axial load and external force

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Abstract

In this study, the interactive behaviors among transverse magnetic fields, axial loads and external force of a magneto-elastic beam with general boundary conditions are investigated. The axial force and transverse magnetic force are assumed to be periodic with respect to time and two specified frequencies, one for axial force and the other for oscillating transverse magnetic field, are applied to the whole system. The equation of motion for the physical model is derived by using the Hamilton's principle and the vibration analysis is performed by employing the characteristic orthogonal polynomials as well as the Galerkin's method. The displacement of the beam with the effect of the magnetic force, axial force and spring force are determined from the modal equations by using the Runge–Kutta method. Based on the present study, we can conclude that the effect of the magnetic field not only reduces the deflection but also decreases the natural frequencies of the system, also it should be noted that the specified beam model can be adopted to simulate several structures in mechanical, civil and electronic engineering.

1. Introduction

The electromagnetic phenomena that arose from the electrical machinery, communications equipment and computers chips has received attention in the past years because of its important

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effect on human life as well as its significant role in providing a real understanding of electricity and magnetism. In this study, the interactive behaviors among transverse magnetic fields, axial loads and external transverse forces of a beam subjected to general boundary conditions are investigated. The fundamental concepts and relations of the electromagnetic theory adopted in this paper are given in Ref. [1]. Further systematic references for the theory of magneto-elastic solid mechanics are described in Ref. [2].

Moon and Pao [3] have proposed a mathematical model for the buckling problem of a cantilever beam-plate in a transverse magnetic field with distributed magnetic forces and torques. Wallerstein and Peach [4] have studied the magneto-elastic buckling of beams and plates with magnetically soft material. Miya et al. [5] have investigated the magneto-elastic buckling of a cantilevered beam-plate using experimental and finite element methods. Moon and Pao [6] have presented the vibration and parametric instability of a cantilevered beam-plate in a transverse magnetic field and also provided the theoretical and experimental results. In the Moon–Pao theoretical analysis, the magnetic torque without axial load was considered, but the axial pulsating load was not included in their discussion. Kojima and Nagana [7] have investigated the parameter nonlinear forced vibrations of a beam with a tip mass subjected to alternating electromagnetic forces acting on the tip mass. Shin et al. [8] have studied the transient vibrations of a simply supported beam with axial loads and transverse magnetic fields.

The influence of axial pulsating load, magnetic force and magnetic couples is complicated and important for the dynamical analysis of structural instability. A beam system involving axial load, transverse magnetic field, external transverse force, spring force and damping effect is considered in this paper. The axial and transverse magnetic forces are assumed to be periodic functions with respect to time and two specified frequencies, one for axial force and the other for oscillating transverse magnetic field. The equation of motion for the physical model is derived using Hamilton's principle and the vibration analysis is performed using characteristic orthogonal polynomials and Galerkin's method. The displacement of the beam with the effect of the magnetic force and axial force is determined from the modal equations by using the Runge–Kutta method.

2. Theoretical analysis

2.1. Statement of problem

In this paper, the physical model of a beam system with general boundary conditions in a magnetic field as shown in Fig. 1 is investigated. The beam is of linearly elastic material with width d, depth h, length L and is subjected to an oscillating axial force $\mathbf{P} = (P_0 + P_1 \cos \Omega_2 t)\mathbf{i}$ in the x-direction and a transverse alternating magnetic field $\mathbf{B}_0 = (B_m \cos \Omega_1 t)\mathbf{j}$, transverse external force, linear viscous damper in the y-direction, and attached by the linear springs with constant k.

The following assumptions are made in order to simplify the analysis of the beam problem mentioned above.

(1) The length is much larger than the depth ($L \ll h$).

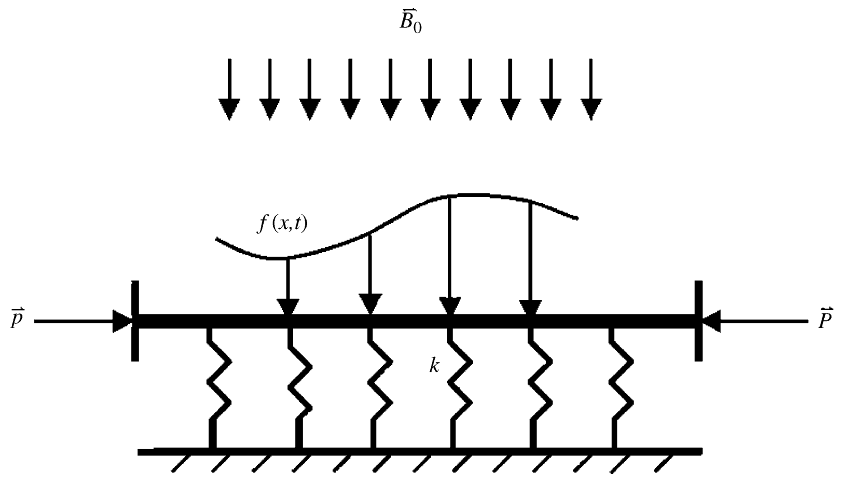

Fig. 1. The model of a beam with axial load, spring foundation and magnetic force.

(2) The cross section of the beam is symmetric, and the depth is much larger than the width ($h \gg d$).
(3) Cross sections of the beam remain to be plane and normal to longitudinal lines within the beam after deformation.
(4) The material obeys the Hooke's law.
(5) The deflection of the beam is small.

2.2. Mathematical modeling

Hamilton's principle [9] is adopted to derive the equation of motion [8] of the beam as follows:

$$m\frac{\partial^2 y}{\partial t^2} + C_d\frac{\partial y}{\partial t} + EI\frac{\partial^4 y}{\partial x^4} + ky + P\frac{\partial^2 y}{\partial x^2} = f(x,t) + \frac{\partial c}{\partial x} + \frac{\partial}{\partial x}\left[\left(\int_0^x p\,\mathrm{d}\xi\right)\frac{\partial y}{\partial x}\right], \tag{1}$$

where $y(x,t)$ is the displacement function, m is the mass per unit length, C_d is damping coefficient, E is the Young's modulus, I is the moment of inertia of the cross section, p is the body force of the beam per unit length in the x-direction, $\mathrm{d}\xi$ denotes the infinitesimal partial length of the beam, k is the constant of spring, $f(x,t)$ is the distributed force in the y-direction and c is the component of $\mathbf{c}$ in the y-direction.

The $\mathbf{c}$ is the induced couple per unit length caused by the magnetic field and can be calculated as follows:

$$\mathbf{c} = \int (\mathbf{M} \times \mathbf{B}_0)\,\mathrm{d}V, \tag{2}$$

where $\mathbf{M} = (\chi_m/\mu_0\mu_r)|\mathbf{B}_0|\mathbf{n}$ is the magnetic dipole moment per unit volume, or simply called the magnetization and $\mathbf{B}_0$ is called the induced magnetic field, μ_0 is the permeability of the vacuum, μ_r is the relative permeability, and χ_m is the magnetic susceptibility (Fig. 2).

Fig. 2. The relationship between $\mathbf{M}$ and $\mathbf{B}_0$ in magnetic field.

2.3. Generation of characteristic orthogonal polynomials

Given a polynomial $\phi_1(x)$, an orthogonal set of polynomials in the interval $a \leqslant x \leqslant b$ can be generated by using a Gram–Schmidt process stated below [10,11]:

$$\phi_2(x) = (x - B_2)\phi_1(x), \tag{3}$$

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x), \tag{4}$$

$$B_k = \left[\int_0^L x w(x)\phi_{k-1}^2(x)\,\mathrm{d}x\right] \Big/ \left[\int_0^L w(x)\phi_{k-1}^2(x)\,\mathrm{d}x\right], \tag{5}$$

$$C_k = \left[\int_0^L x w(x)\phi_{k-1}(x)\phi_{k-2}(x)\,\mathrm{d}x\right] \Big/ \left[\int_0^L w(x)\phi_{k-1}^2(x)\,\mathrm{d}x\right], \tag{6}$$

where $w(x)$ is the weighting function and the polynomials satisfy the following orthogonality condition:

$$\int_0^L w(x)\phi_k(x)\phi_l(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } k \neq l, \\ \alpha_{kl} & \text{if } k = l. \end{cases} \tag{7}$$

Construction of the first member is carried out so as to satisfy all the boundary conditions of the beam problems. Even though $\phi_1(x)$ satisfies all the boundary conditions, both geometric and natural, however, the other members of the orthogonal set satisfy only geometric boundary conditions, which can be easily checked from the way the set is constructed by using Eqs. (3)–(6).

2.4. Analytical procedure

The deflection of a magneto-elastic beam undergoing an alternative transverse magnetic field can be expressed in terms of the characteristic orthogonal polynomial in the x-direction as

$$y(x,t) = \sum_n q_n(t)\phi_n(x), \quad 0 \leqslant x \leqslant L, \tag{8}$$

where $\phi_n(x)$ is the characteristic polynomial that satisfies the boundary conditions. Since the middle plane of the beam is assumed to remain unstrained while bending, the length of the beam will be constant, therefore, we have

$$\int_0^x [1 + y'^2(\xi,t)]^{1/2}\,\mathrm{d}\xi = \ell. \tag{9}$$

Differentiating Eq. (9) with respect to t yields

$$\int_0^x \left[\frac{y'\dot{y}'}{(1+y'^2)^{1/2}}\right] \mathrm{d}\xi + (1+y'^2)^{1/2}\dot{x} = 0. \tag{10}$$

According to the small deflection theory, the velocity term, $\dot{x}$, in the above equation can be simplified to

$$\begin{aligned} \dot{x} = \mathrm{d}x/\mathrm{d}t &\approx -\int_0^x y'\dot{y}'\,\mathrm{d}\xi \\ &= \sum_m \sum_n q_m(t)\dot{q}_n(t) \int_0^x \phi'_m(\xi)\phi'_n(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{11}$$

Also, the component of the body force $\mathbf{p}$ contributed by the magnetic force $\mathbf{B}_0$ can be shown as below

$$\begin{aligned} p &= \text{component} \int \sigma(\dot{\mathbf{r}} \times \mathbf{B}_0) \times \mathbf{B}_0\,\mathrm{d}V \\ &= \sigma\,h\,d(B_m \cos\,\Omega_1 t)^2 \sum_m \sum_n q_m(t)\dot{q}_n(t) \int_0^x \phi'_m(\xi)\phi'_n(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{12}$$

Meanwhile, the component of the couple $\mathbf{c}$ per unit length induced by the magnetic force can be written as follows:

$$c = \text{component} \int \mathbf{M} \times \mathbf{B}_0\,\mathrm{d}V = MB_0hd \sum_n q_n(t)\phi'_n. \tag{13}$$

2.5. Galerkin's method

Substituting Eqs. (9)–(14) into Eq. (1) will lead to a nonlinear residual operator of the governing equation, $R(q, x)$, which can be derived to be in the following form:

$$R(q,x) = \sum_l \left\{ \begin{array}{c} m\ddot{q}_l\phi_l(x) + C_d\dot{q}_l\phi_l(x) - EIq_i\phi_l^{(iv)}(x) \\ +kq_l\phi_l(x) + (p_0 + p_1\cos\,\Omega_2 t)q_l\phi''_l \\ -\dfrac{\chi_m}{\mu_0\mu_r}B_0^2hdq_l\phi''_l(x) \\ -\sigma hdB_0^2q_l\left[\phi''_l \cdot \sum\limits_m\sum\limits_n q_m\dot{q}_nB_{mn}(x) + \phi''_l \cdot \sum\limits_m\sum\limits_n q_m\dot{q}_nA_{mn}(x)\right] \end{array} \right\}, \tag{14}$$

where

$$A_{mn}(x) = \int_0^x \phi'_m(\xi)\phi'_n(\xi)\,\mathrm{d}\xi, \tag{15}$$

$$B_{mn}(x) = \int_0^x \int_0^{\xi_2} \phi'_m(\xi_1)\phi'_n(\xi_1)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = \int_0^x A_{mn}(\xi)\,\mathrm{d}\xi. \tag{16}$$

By adopting the Galerkin's techniques, the weighting function is chosen to be the same as the approximating function $\phi_k(x)$, and it is required that

$$\int_0^L R(q,x)\phi_k(x)\,\mathrm{d}x = 0 \quad \forall k = 1,2,3,\ldots \tag{17}$$

A set of time-dependent differential equations can be derived from the previous equation, which will be of the following form:

$$\begin{aligned} & m a_k \ddot{q}_k + C_d a_k \dot{q}_k + EI \sum_l b_{kl} q_l + k a_k q_k \\ & + (P_0 + P_1 \cos \Omega_2 t) \sum_l c_{kl} q_l - \frac{\chi_m}{\mu_0 \mu_r} B_0^2 h d \sum_l c_{kl} q_l \quad \forall k = 1,2,3,\ldots \\ & - \sigma h d (B_m \cos \Omega_1 t)^2 \sum_l \sum_m \sum_n q_l q_m \dot{q}_n (d_{klmn} + e_{klmn}) = 0, \end{aligned} \tag{18}$$

where

$$a_k = \sum_l \int_0^L \phi_l(x)\phi_k(x)\,\mathrm{d}x = \int_0^L \phi_k^2(x)\,\mathrm{d}x, \tag{19}$$

$$b_{kl} = \int_0^L \phi_l^{(iv)}(x)\phi_k(x)\,\mathrm{d}x, \tag{20}$$

$$c_{kl} = \int_0^L \phi''_l(x)\phi_k(x)\,\mathrm{d}x, \tag{21}$$

$$d_{klmn} = \int_0^L \phi''_l(x)\phi_k(x)B_{mn}(x)\,\mathrm{d}x, \tag{22}$$

$$e_{klmn} = \int_0^L \phi'_l(x)\phi_k(x)A_{mn}(x)\,\mathrm{d}x. \tag{23}$$

By carrying out the required mathematical operations in Eq. (18), a set of second-order nonlinear differential equations can be obtained, which are generally composed of the term $\ddot{q}_k$, $\dot{q}_k$, q_k, $\sum_l b_{kl} q_l$, $\sum_l c_{kl} q_l$ and $\sum_l \sum_m \sum_n q_l q_m \dot{q}_n$. This kind of nonlinear ODE set can be solved by using the Runge–Kutta method and thus the free vibration analysis can be accomplished.

For the transient vibration, assuming $f(x,t) = \sum_{n=1,2,\ldots} \sin(t)\phi_n(x)$ in Eq. (1) to be an external force acting on the beam system, the displacement at a transient time can be determined. Also it is necessary to assume a set of initial conditions in this model while the transverse force is equal to zero. The fourth-order Runge–Kutta method can also be applied to solve the nonlinear ODE set for the forced vibration part.

3. Numerical results and discussions

A low-carbon steel is considered in this study, where $E = 194\,\text{Gpa}$, $m = 0.03965\,\text{kg/m}$, $L = 0.5\,\text{m}$, $h = 0.005\,\text{m}$, $d = 0.001\,\text{m}$, $\mu_r = 1.00001$, $\mu_0 = 1.26 \times 10^{-6}\,\text{H/m}^1$, $\sigma = 10^7\,\text{V/m}^1$, $\Omega_1 = \Omega_2 = 12\,\text{rad/s}$, $k = 1.0\,\text{N/m}$, $C_d = 0.1$. Initial conditions $q_k(0)/h = 1.0$ and $\dot{q}_k(0)/h = 0$ are chosen for all free vibration discussions in this study.

Fig. 3 presents the free vibration behaviors of a S–S beam subjected to the effect of different damping ratio, axial force ($P_0 = P_1 = 1$), spring foundation and magnetic force for the first mode ($N = 1$). In the case of undamped and damped free vibration with no external forces imposed, we can find that the fundamental natural frequency of the present system is about $\omega_1 = 9\,\text{cycles/s} = 18\pi\,\text{rad/s}$. After applying the axial force ($P_0 = P_1 = 1$), the natural frequency of the system is obviously reduced to $\omega_1 = 7.5\,\text{cycles/s} = 15\pi\,\text{rad/s}$, thus we find that the existence of axial force will result in the reduction of the natural frequency. Also we see that the behavior of the peaks are not consistent, i.e. do not have similar shape, because the pulsating axial force has its own driving frequency, $\Omega_2 = 12\,\text{rad/s} \approx 1.91\text{cycles/s}$, and the behavior of the peaks will be controlled by the external frequency, it is about 2 cycles in 1 s duration. For the case with spring foundation only ($k = 1.0$), the natural frequency of the system obviously increased to near $\omega_1 = 11\,\text{cycles/s} = 22\pi\,\text{rad/s}$, which means the effect of spring foundation play the role of stabilizing the whole system. The relation between amplitude and time of the simply-supported beam with only the magnetic effect is depicted in the bottom of Fig. 3, in this case, we can easily observe that the magnetic force plays a stabilizing role just as the damping effect.

Fig. 4 is the damped free vibrations at the middle point of a S–S beam under various interactive effects of axial loads ($P_0 = P_1 = 1$), spring foundation ($k = 1.0$) and magnetic field ($B_m = 3$) for the first three modes. In the case of first mode ($N = 1$), the deflection of a beam subjected to various external loads becomes a little irregular, that is, the oscillation is not consistent in each cycle. The fundamental natural frequency here is about $\omega_1 = 10\,\text{cycles/s} = 20\pi\,\text{rad/s}$. The oscillating behaviors in the second ($N = 2$) and third mode ($N = 3$) cases are apparently consistent in comparison with the first one, and the deflection for the third mode seems to oscillate along an invisible skeleton curve. At the bottom of this figure, the total deflection for the first three modes are summed up, and its shape is quite similar to that of the first mode, which can be concluded that the first mode is dominant, i.e. low-frequency is more important. In this figure, the effect of magnetic force is not obvious since the magnitude of B_m is quite small so that its effect is overwhelmed by the other external forces.

Fig. 5 demonstrates the interactive behavior between axial force ($P_0 = P_1 = 1$) and the magnetic field ($B_m = 3$) as well as spring foundation ($k = 1.0$) and magnetic field ($B_m = 3$) at the middle point of a clamped–clamped beam for the first three modes. The same phenomena on

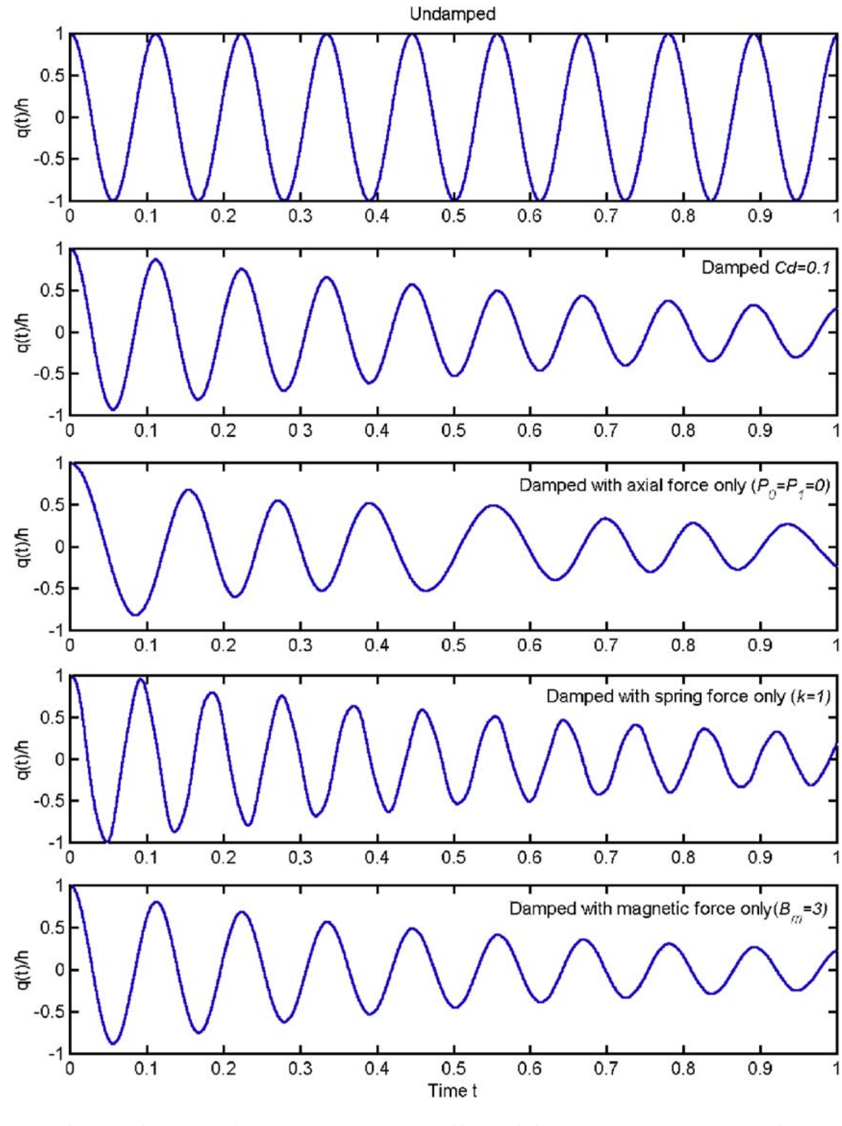

Fig. 3. Free vibration of a S–S beam with respect to various effect of damping ratio, axial loads, spring foundation and magnetic force for the first mode.

insignificant magnetic effect can be drawn as in Fig. 4 due to the relative small value of magnetic field.

Fig. 6 pictures the relationship between amplitude and time of the simply supported beam with only the effect of magnetic field for damped free vibration. In this figure, there are four different kinds of magnetic field ($B_m = 0.03$, 0.3, 3, 30) on the system, the smaller magnetic field seems to have insignificant effect on the deflection, as the magnetic field is getting larger, it plays a dramatically damping effect even heavier than the structural damping ratio. If we look at the response for 30 T case very carefully, we can surprisingly find that the fundamental frequency is

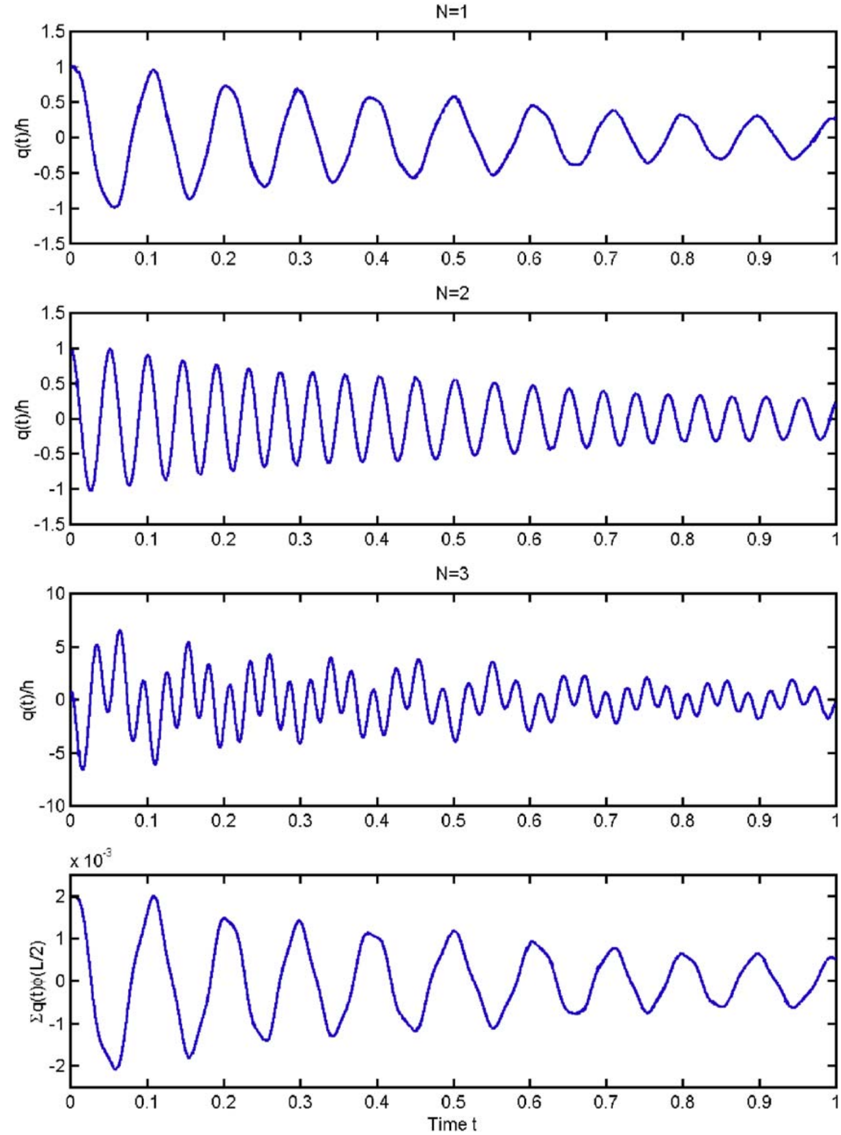

Fig. 4. Damped free vibrations of a S–S beam under various interactive effects of axial loads, spring foundation and magnetic force.

slightly reduced to 8.2 cycles/s $\approx 16.4\pi$ rad/s. This kind of phenomena can be explained reasonably by reviewing Eq. (18), in which two terms are related to the parameter B_m, one is $-\sum_l c_{kl} q_l$ and the other is $\sum_l \sum_m \sum_n q_l q_m \dot{q}_n (d_{\mathrm{klmn}} + e_{klmn})$. The first term is obviously effective to weaken the structural rigidity and the second term is equivalent to enhance the damping coefficient.

Fig. 7 is the damped vibration of a clamped–clamped beam under different magnetic effect. Similar results can be concluded as in Fig. 6 except that the deflection of the beam gets smaller and the natural frequency of the beam gets larger.

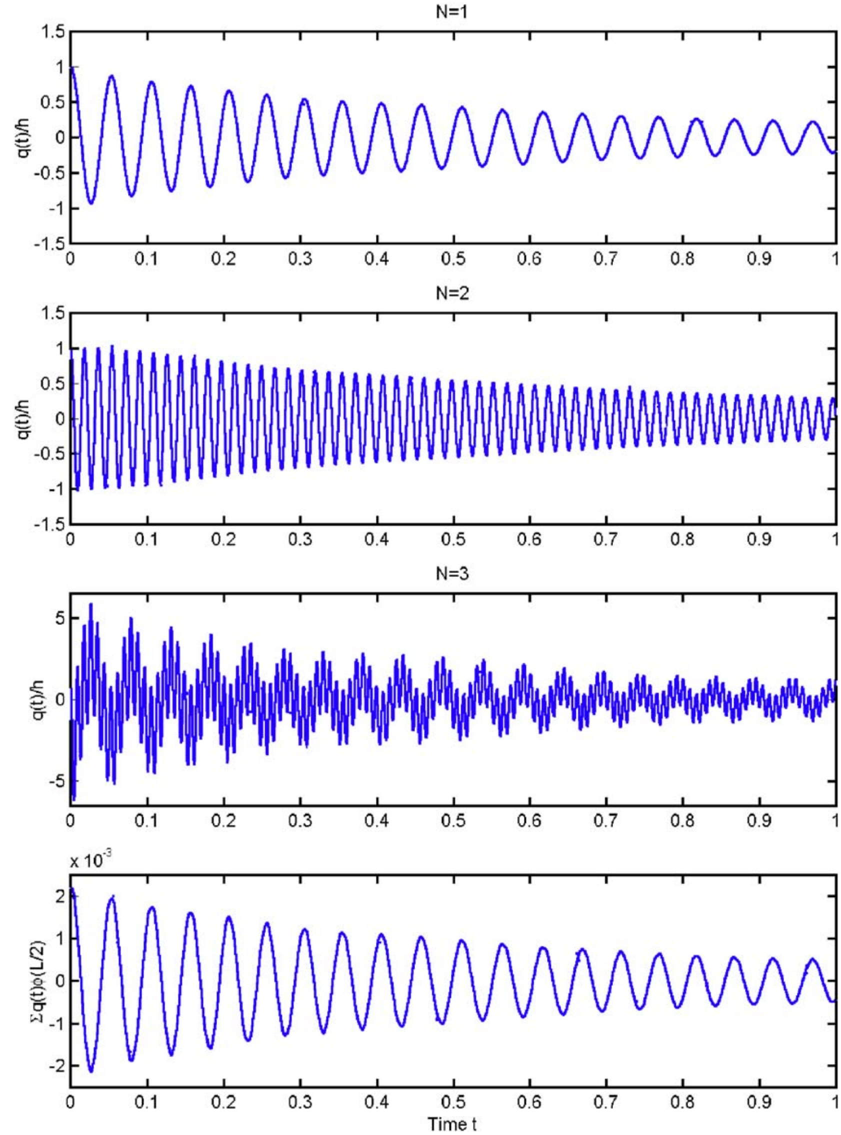

Fig. 5. Damped free vibrations of a C–C beam under various interactive effects of axial loads, spring foundation and magnetic force.

4. Conclusions

Based on the assumption of the inextensible beam, the motion of the beam in the transverse magnetic field leads to a nonlinear damping effect proportional to the square of the amplitude, and a slight decrement on the system's rigidity, which is linearly related to the amplitude. So the effects of the magnetic field not only reduce the deflection but also reduce the natural frequencies

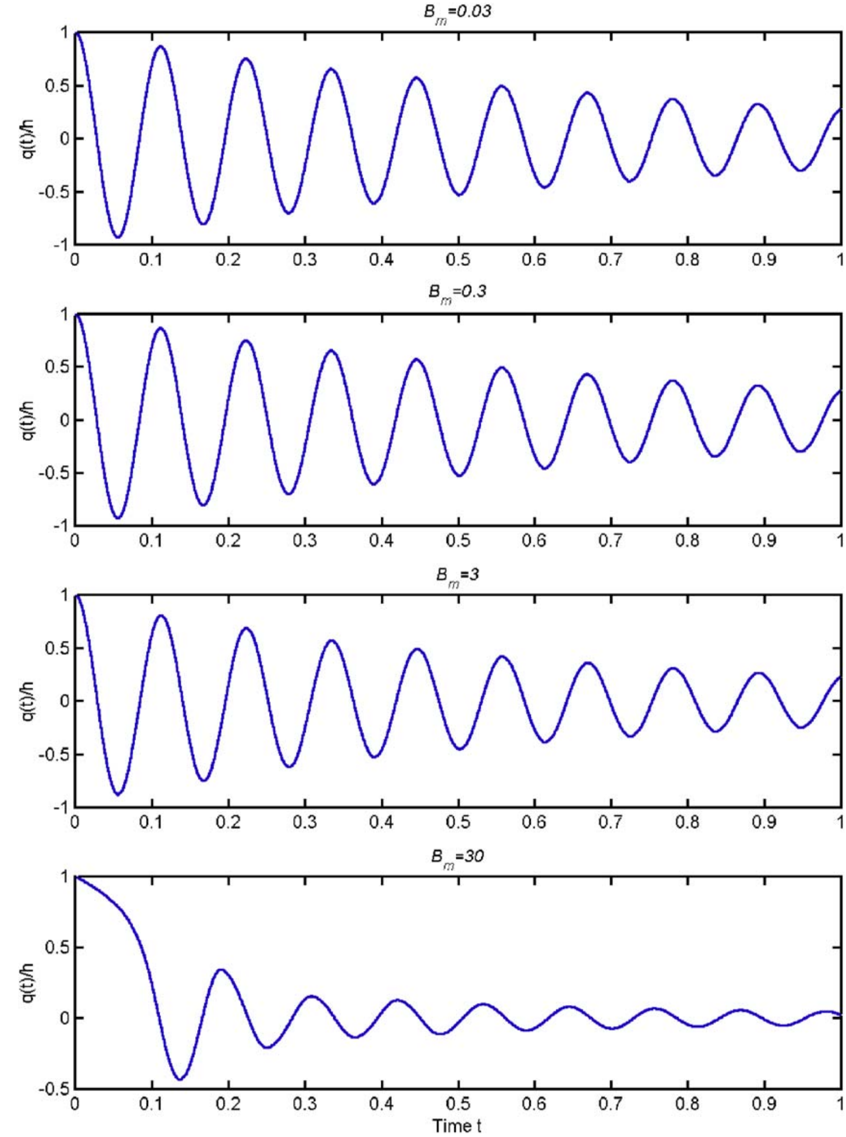

Fig. 6. Damped free vibrations of a S–S beam under different values of imposed magnetic fields.

of the system. It is noticed that under stable situations, namely, when no resonant parameters are chosen, the more the transverse magnetic field increases, the more the displacement and the natural frequency of the beam system decrease. Meanwhile, as the axial force gets larger, the displacement becomes greater but the natural frequency decreases. In addition, the effect of spring foundation in this study plays a stabilizing role to the whole system, which is quite rational as we expected. As long as the parameters of the beam system are practical and reasonable, we can also

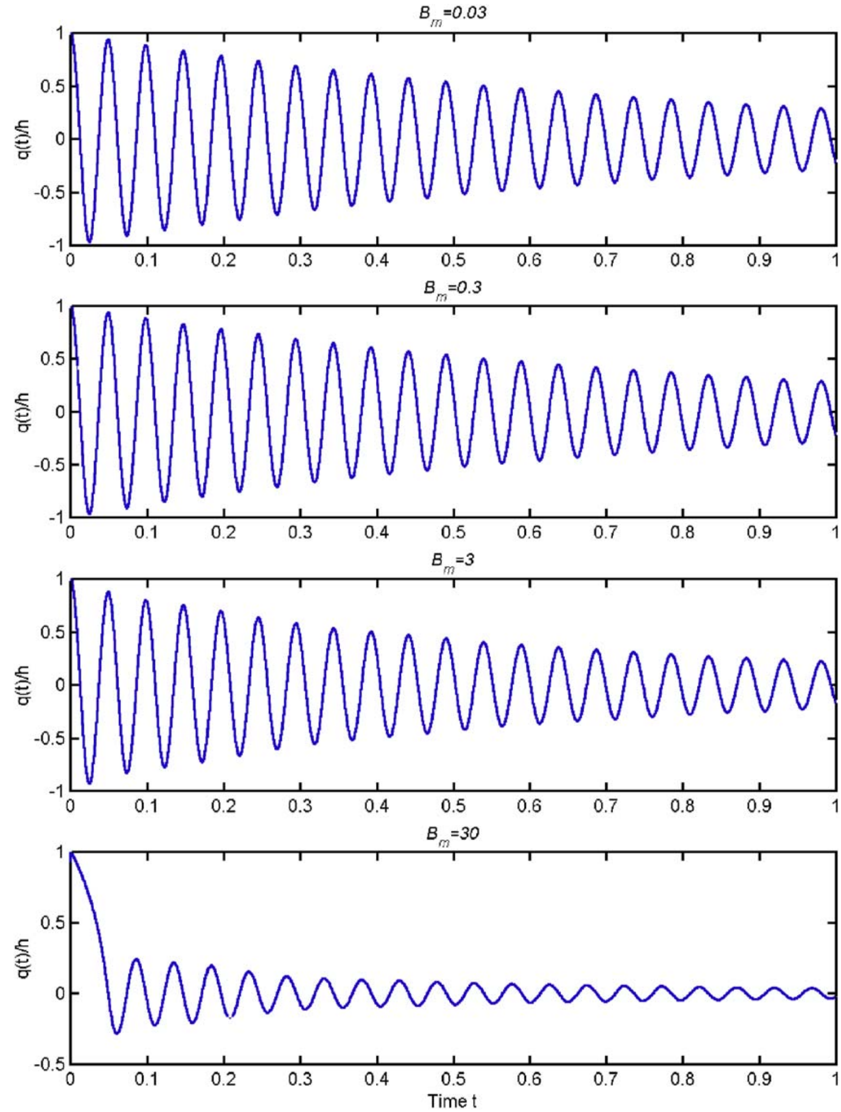

Fig. 7. Damped free vibrations of a C–C beam under different values of imposed magnetic fields.

investigate the behavior of the micro beam system under magnetic field by the technique proposed in the present study.

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