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Uncertainty identification by the maximum likelihood method

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Abstract

To incorporate uncertainty in structural analysis, a knowledge of the uncertainty in the model parameters is required. This paper describes efficient techniques to identify and quantify variability in the parameters from experimental data by maximising the likelihood of the measurements, using the well-established Monte Carlo or perturbation methods for the likelihood computation. These techniques are validated numerically and experimentally on a cantilever beam with a point mass at an uncertain location. Results show that sufficient accuracy is attainable without a prohibitive computational effort. The perturbation approach requires less computation but is less accurate when the response is a highly nonlinear function of the parameters.

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1. Introduction

Many applications are concerned with a response that depends on parameter values of the system. This may be formally described by the equation

$$\mathbf{y} = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = [x_1\, x_2\, \ldots\, x_n]^{\mathrm{T}}$ are parameters of the structure, for example thickness or material properties, and $\mathbf{y} = [y_1\, y_2\, \ldots\, y_m]^{\mathrm{T}}$ are the desired response quantities, for example natural frequencies or point frequency-response functions (FRFs). Usually the parameters are assumed to be known for the forward problem, or unknown but fixed for the inverse or model updating problem [1]. In this paper, the parameters are assumed to follow a particular robability distribution X, and hence the response will also follow a probability distribution, denoted Y.

This paper will be concerned exclusively with probabilistic models of uncertainty. Other uncertainty models [2], such as evidence theory, possibility theory, fuzzy logic, interval methods or convex models, are equally valid, and the inverse problem based on measured data may also be posed for these uncertainty models. Generally, probabilistic approaches perform well when there is sufficient information available to define the underlying input distributions. It is precisely this information that this paper seeks, and hence the proposed methods are valuable in the derivation of probability density functions. However, similar approaches could be proposed based on other models of uncertainty.

In most applications the probability distribution of the parameters is known while the probability distribution of the response (or its low-order statistics) is sought. Well-established methods are available to address this problem: the Monte Carlo method samples points of the function $\mathbf{f}$ according to the distribution X; the perturbation method approximates $\mathbf{f}$ by a low-order polynomial centred on the mean value of $\mathbf{x}$.

The knowledge of the probability distribution of the parameters is a precondition to apply any uncertainty propagation method. However, there is very little literature concerned with the estimation of the parameter probability density function. In some cases it may be possible to directly measure samples of the parameters but often it is easier to measure the response. For example it is easier to measure the global natural frequencies or FRFs than to measure localised material properties such as densities, thicknesses or equivalent joint stiffnesses. The statistics of the parameters may be inferred from the measurements, and this knowledge could then be applied to new problems. The inverse problem of estimating the distribution of the parameters from that of the response measurements is called uncertainty identification or quantification, and is the subject of this paper.

Bayesian model updating [3–5] is a well-established procedure for refining parameter uncertainty using experimental data (for example, to update the predicted reliability index of a single structure), but no such procedure is widely available for quantifying *irreducible* uncertainty, for example, to quantify the variability in a structure due to the uncertainties introduced by the manufacturing process. Attempting to fill the gap, this paper develops an algorithm that quantifies the parameter uncertainty by maximising the likelihood of the experimental data. This algorithm is both reasonably efficient and accurate, while relying on the existing uncertainty propagation methods. Mares et al. [6,7] recently developed a similar but different procedure,

called the *stochastic model updating* method, where an experimental data cloud is converged upon by a simulated data cloud generated by the Monte Carlo method.

Statistical methods have been used in model updating for many years. Usually the estimated variance of the measurements and parameters is used to weight the different terms in a least-squares procedure [1,8]. This is taken a stage further in the minimum variance estimation methods, where the parameters that have the minimum variance are estimated [9,10]. It should also be emphasised that this paper is not concerned with the choice of parameters to update or with regularisation. This has been the subject of significant research [1,11], and most of the issues that are important in standard model updating will be equally important for uncertainty quantification.

2. Maximum likelihood estimation

To solve the inverse uncertainty propagation problem one might be tempted to invert Eq. (1) as

$$\mathbf{x} = \mathbf{f}^{-1}(\mathbf{y}) \tag{2}$$

and use the standard uncertainty propagation methods. The difficulty with this approach is determining $\mathbf{f}^{-1}$, since the inversion is usually ill-conditioned or even impossible. A better alternative is to use maximum likelihood estimation, which also allows one to use the existing uncertainty propagation methods, and do not require inverting $\mathbf{f}$.

For estimation purposes it is assumed that the parameters follow a certain probability distribution, X, belonging to a probability distribution family, denoted by $\sim$, and written as

$$X \sim D(\boldsymbol{\theta}_x), \tag{3}$$

where $\boldsymbol{\theta}_x$ are the parameters of the family to be estimated. For a multivariate normal distribution, the parameters would be the mean vector $\boldsymbol{\mu}_x$ and covariance matrix $\boldsymbol{\Sigma}_x$. For a given $\boldsymbol{\theta}_x$ the response probability density function $f(\mathbf{y}|\boldsymbol{\theta}_x)$ can be approximated by using one of the uncertainty propagation methods.

Let $\mathbf{Y}'$ be a set of N response measurements $[\mathbf{y}'_{(1)}\,\mathbf{y}'_{(2)}\,\ldots\,\mathbf{y}'_{(N)}]$. The measurements are assumed to be independent; therefore, the measurement likelihood is

$$L(\boldsymbol{\theta}_x) = f(\mathbf{y}'_{(1)}, \mathbf{y}'_{(2)}, \ldots, \mathbf{y}'_{(N)}|\boldsymbol{\theta}_x) = \prod_{i=1}^{N} f(\mathbf{y}'_{(i)}|\boldsymbol{\theta}_x). \tag{4}$$

The log likelihood is more tractable and given by

$$l(\boldsymbol{\theta}_x) = \log\, L(\boldsymbol{\theta}_x) = \sum_{i=1}^{N} \log f(\mathbf{y}'_{(i)}|\boldsymbol{\theta}_x). \tag{5}$$

The maximum likelihood estimator $\hat{\boldsymbol{\theta}}_x$ is the value of $\boldsymbol{\theta}_x$ for which $l(\boldsymbol{\theta}_x)$ attains a maximum. A non-gradient-based optimisation method such as the simplex method can be employed for the maximisation, allowing the use of standard uncertainty propagation methods without alteration.

The drawback of this approach is its iterative nature. The uncertainty propagation methods are by themselves computationally intensive and to repeatedly execute these methods in an iterative optimisation loop would be prohibitive for most interesting applications. Ways to efficiently

integrate the maximum likelihood estimation with the two most common propagation methods is the purpose of this paper.

2.1. Perturbation method

It is assumed that the uncertain parameters follow a multivariate normal distribution

$$X \sim N_n(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x), \tag{6}$$

where n is the number of variables, $\boldsymbol{\mu}_x$ is the mean vector and $\boldsymbol{\Sigma}_x$ is the covariance matrix. In theory this incurs no loss in generality since random variables may be transformed into uncorrelated Gaussian variables exactly using the Rosenblatt transformation, or approximately using the Nataf transformation [2,12]. The case where all of the elements of $\boldsymbol{\mu}_x$ and $\boldsymbol{\Sigma}_x$ are independent will be considered. However, there are examples where this assumption is not valid, such as random fields [13] where there is a dependency structure. In a spatial AR(1) random field, $\boldsymbol{\mu}_x$ and $\boldsymbol{\Sigma}_x$ could be fully described by three scalars, namely the mean μ, variance σ and correlation length L.

For the perturbation method Eq. (1) is first expanded as

$$\mathbf{y} = \mathbf{f}(\mathbf{x}^0) + \sum_{i=1}^{n} \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}^0) \cdot (x_i - x_i^0) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \mathbf{f}}{\partial x_i \partial x_j}(\mathbf{x}^0) \cdot (x_i - x_i^0) \cdot (x_j - x_j^0) + \cdots \tag{7}$$

around the point $\mathbf{x}^0 = [x_0^0\, x_1^0 \ldots x_n^0]^{\mathrm{T}}$, which is assumed to be in the vicinity of $\boldsymbol{\mu}_x$. The effect of the choice of $\mathbf{x}^0$ will be considered in more detail later. Taking only the first-order terms, Eq. (7) can be rewritten as

$$\mathbf{y} = \mathbf{f}^0 + \mathbf{J}^0(\mathbf{x} - \mathbf{x}^0), \tag{8}$$

where $\mathbf{f}^0$ and $\mathbf{J}^0$ are the function and its Jacobian, respectively, evaluated at the point $\mathbf{x}^0$. Refs. [14,15] give more detail on how to evaluate the Jacobian for natural frequencies. From Eqs. (6) and (8) the probability density function of $\mathbf{y}$ may be approximated by

$$Y \sim N_m(\boldsymbol{\mu}_y = \mathbf{f}^0 + \mathbf{J}^0(\boldsymbol{\mu}_x - \mathbf{x}^0), \boldsymbol{\Sigma}_y = \mathbf{J}^{0\mathrm{T}} \boldsymbol{\Sigma}_x \mathbf{J}^0) \tag{9}$$

and its probability density function is

$$\hat{f}(\mathbf{y}|\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x) = (2\pi)^{-m/2} |\boldsymbol{\Sigma}_y|^{-1/2} \mathrm{e}^{-(\mathbf{y}-\boldsymbol{\mu}_y)^{\mathrm{T}} \boldsymbol{\Sigma}_y^{-1} (\mathbf{y}-\boldsymbol{\mu}_y)/2}. \tag{10}$$

A detailed study of the distribution of eigenvalues can be found in Ref. [16]. Replacing $f(\mathbf{y}'_{(i)}|\boldsymbol{\theta}_x)$ in Eq. (5) by the approximation given in Eq. (10) yields

$$l(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x) = -\frac{1}{2} \left(Nm \log 2\pi + N \log |\boldsymbol{\Sigma}_y| + \sum_{i=1}^{N} (\mathbf{y}'_{(i)} - \boldsymbol{\mu}_y)^{\mathrm{T}} \boldsymbol{\Sigma}_y^{-1} (\mathbf{y}'_{(i)} - \boldsymbol{\mu}_y) \right). \tag{11}$$

Ideally the linearisation point $\mathbf{x}^0$ would be equal to the mean value $\boldsymbol{\mu}_x$ but since the latter is not known a priori a guess must be made for its initial value. Depending how far this initial guess is from the estimated $\hat{\boldsymbol{\mu}}_x$, it may be necessary to recompute $\mathbf{f}^0$ and $\mathbf{J}^0$, to more accurately describe the response surface near $\boldsymbol{\mu}_x$. It is unnecessary, however, to perform the computation at every

evaluation of Eq. (11). For most applications, approximate knowledge of the mean value is available, reducing the need for such recalculation.

2.2. The Monte Carlo simulation method

Let $\mathbf{X}''$ be a set of M samples of the parameters $[\mathbf{x}''_{(1)}\,\mathbf{x}''_{(2)}\,\ldots\,\mathbf{x}''_{(M)}]$, and $\mathbf{Y}''$ the respective response set $[\mathbf{y}''_{(1)}\,\mathbf{y}''_{(2)}\,\ldots\,\mathbf{y}''_{(M)}]$. If the uncertain parameters are sampled according to *their* probability density function $f(\mathbf{x})$ then the response probability density function can be estimated by using a kernel density estimator [17] as

$$\hat{f}(\mathbf{y}|\boldsymbol{\theta}_x) = \frac{1}{M}\sum_{j=1}^{M} \kappa_{\mathbf{H}}(\mathbf{y} - \mathbf{y}''_{(j)}), \tag{12}$$

where $\kappa_{\mathbf{H}}$ is the kernel function with a bandwidth matrix $\mathbf{H}$. If the parameters are sampled according to a *different* probability density function $g(\mathbf{x})$ then the probability density function of the response may be estimated by

$$\hat{f}(y|\boldsymbol{\theta}_x) = \frac{1}{M}\sum_{j=1}^{M} \frac{f(\mathbf{x}''_{(j)}|\boldsymbol{\theta}_x)}{g(\mathbf{x}''_{(j)})}\kappa_{\mathbf{H}}(\mathbf{y} - \mathbf{y}''_{(j)}). \tag{13}$$

The suitability of different probability density functions of the parameters may be tested using Eq. (13) without re-sampling $\mathbf{X}''$. If the function $g(\mathbf{x})$ is close to $f(\mathbf{x})$ then a smaller number of samples, M, would be required. However, the only requirements for $\mathbf{X}''$ are that $g(\mathbf{x}) \gg 0$ in the same region where $f(\mathbf{x}) \gg 0$ and that a sufficiently high number of samples M is generated. In practice uniform or Latin hypercube sampling of the parameters over the likely parameter subspace is sufficient. Further time-savings can be achieved by only evaluating Eq. (1) for a small fraction of M and then interpolating for the remaining samples.

Replacing $f(\mathbf{y}'_{(i)}|\boldsymbol{\theta}_x)$ in Eq. (5) by the approximation given in Eq. (13) yields

$$l(\boldsymbol{\theta}_x) = -N\,\log\,M + \sum_{i=1}^{N}\log\sum_{j=1}^{M}\exp[\log f(\mathbf{x}''_{(j)}|\boldsymbol{\theta}_x) - \log\,g(\mathbf{x}''_{(j)}) + \log\,\kappa_{\mathbf{H}}(\mathbf{y}'_{(i)} - \mathbf{y}''_{(j)})]. \tag{14}$$

Note that the only term in Eq. (14) that depends on $\boldsymbol{\theta}_x$ is the probability density function of the parameters, $f(\mathbf{x}''_{(j)}|\boldsymbol{\theta}_x)$. All of the other terms can be pre-calculated before entering the optimisation loop. Also note that the sum $j = 1\ldots M$ does not need to be calculated for every j (see Appendix A for detailed explanation).

For the kernel density estimation a multivariate normal kernel was used, given by

$$\kappa_{\mathbf{H}}(\mathbf{y}) = |\mathbf{H}|^{-1}\kappa(\mathbf{H}^{-1}\mathbf{y}), \tag{15}$$

where

$$\kappa(\mathbf{y}) = (2\pi)^{-m/2}\mathrm{e}^{-\mathbf{y}^{\mathrm{T}}\mathbf{y}/2}. \tag{16}$$

A careful choice for the bandwidth matrix must be made to ensure accurate estimation. Satisfactory results have been obtained using a variation of the rule of thumb for multivariate

kernel density estimation presented in Ref. [17], where the bandwidth matrix is given by

$$\mathbf{H} = M^{-1/(d+4)}\boldsymbol{\Sigma}_y^{1/2}, \tag{17}$$

where $d = \min(n, m)$ is the number of dof in the response.

3. Application to a cantilever beam

3.1. Simulated example

The simulated example is a cantilever beam with a point mass at an uncertain position along the beam length, shown schematically in Fig. 1. The beam has length $l = 1\,\text{m}$, a rectangular section of $100 \times 10\,\text{mm}^2$ and is made of steel with Young's modulus $E = 210\,\text{GPa}$ and density $\rho = 7800\,\text{kg/m}^3$. The discrete mass is $m = 0.100\,\text{kg}$ and its position x follows a normal distribution $X \sim N(\mu = 0.75\,\text{m}, \sigma = 0.05\,\text{m})$.

If the x variation is small enough, the natural frequencies vary almost linearly and the perturbation approach becomes attractive because of its computational efficiency. Fig. 2 shows the variation of the natural frequencies with the position of the discrete mass and demonstrates that the perturbation approach is only suitable for the lower natural frequencies and for small position variations.

Figs. 3 and 4 show the log-likelihood given by Eq. (5) for the simulated cantilever beam for the perturbation and Monte Carlo approaches. In both cases the function is very steep for mean values away from the real mean and for low variance, but the function is very flat for large variances. More importantly the log-likelihood only has one maximum, which is very close to the real parameter values, and to which the estimation procedure will converge for any initial set of parameters.

For the analysis of the results, two (relative) errors are defined. The *real* error is the error between the estimates and the *population* statistics

$$\varepsilon_r = \frac{\hat{\mu}_x - \mu_x}{\mu_x} \quad \text{or} \quad \varepsilon_r = \frac{\hat{\sigma}_x - \sigma_x}{\sigma_x}. \tag{18}$$

The *effective* error is the error between the estimates and *sample* statistics

$$\varepsilon_e = \frac{\hat{\mu}_x - \bar{x}}{\bar{x}} \quad \text{or} \quad \varepsilon_e = \frac{\hat{\sigma}_x - s_x}{s_x}. \tag{19}$$

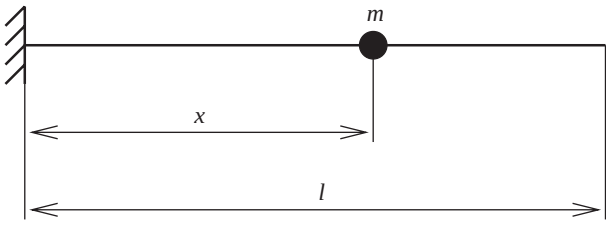

Fig. 1. The simulated cantilever beam with a discrete mass at an uncertain position.

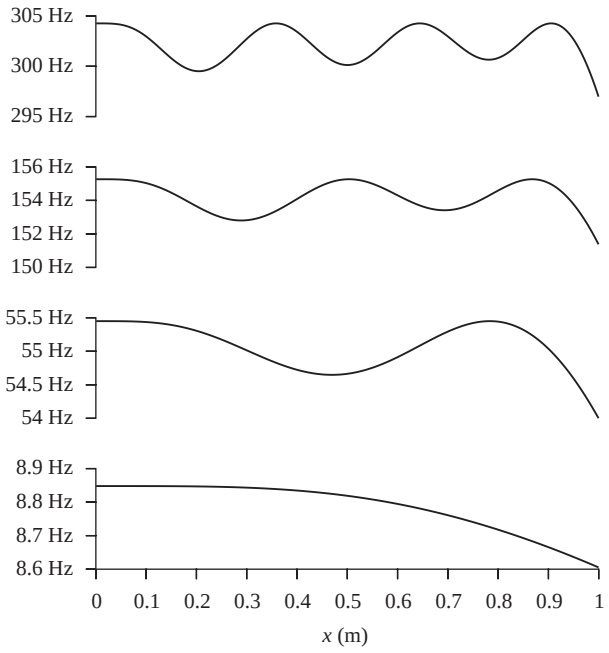

Fig. 2. The variation of the first four natural frequencies with the position of the lumped mass, x.

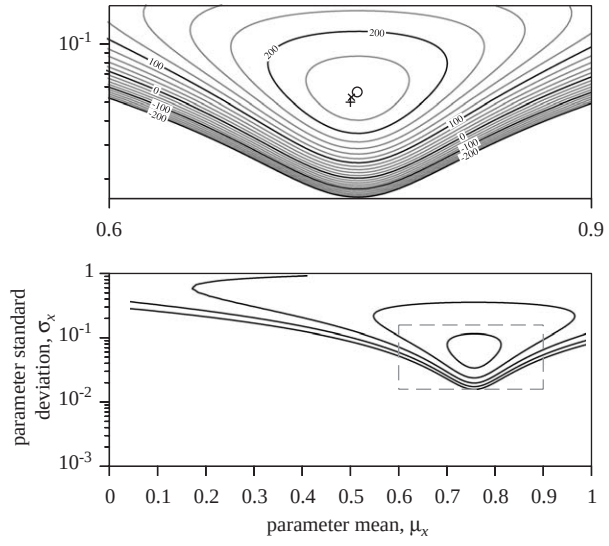

Fig. 3. The log-likelihood given by Eq. (11) for the perturbation method. +, real; ×, effective; ○, estimate.

The effective error is often the most relevant error, since for a fixed set of N measurements the sample statistics are the best one can really ever hope to know.

As mentioned above, the best results for the perturbation approach are obtained when the linearisation is centred on the mean value of the parameters. However, this point is not known

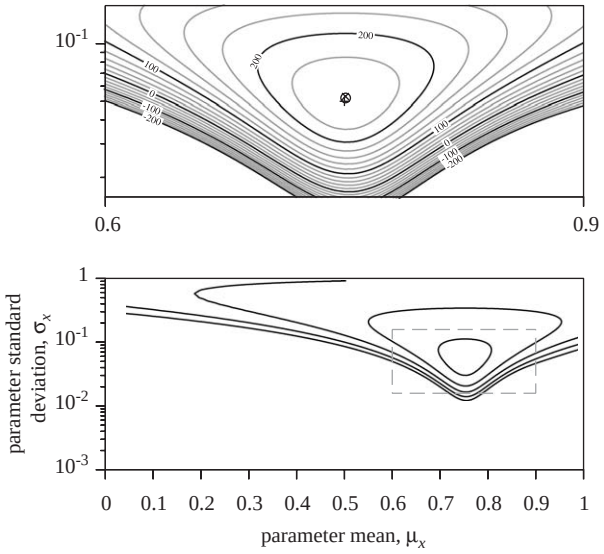

Fig. 4. The log-likelihood given by Eq. (14) for the Monte Carlo method. +, real; ×, effective; ○, estimate.

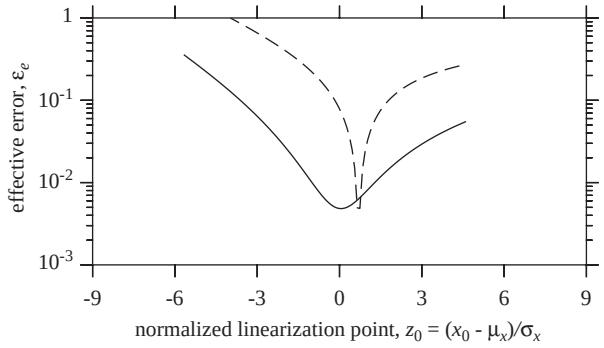

Fig. 5. The influence of the linearisation point on the effective estimation error. ——— μ_x; – – – – σ_x.

beforehand and a guess must be made. Fig. 5 shows the effect performing the linearisation away from the real mean has on the estimation error. The error in the mean does not change significantly if x_0 is within $\mu_x \pm 3\sigma_x$. We can see that the minimum error is not necessarily obtained at the real mean.

The accuracy of the Monte Carlo approach depends on the number of samples, with an error estimate that decreases as $M^{1/2}$, as evidenced by Fig. 6 for this example.

In this particular example there is only a single parameter, and therefore measuring a single natural frequency (for example the first) would enable the parameter to be estimated. Fig. 7 shows the effect of using more than one natural frequency to estimate the discrete mass position. The extra information available using more natural frequencies should allow more averaging of the measured data and therefore more accurate estimates. However, this example highlights an undesirable property of the perturbation approach, where adding more redundant information

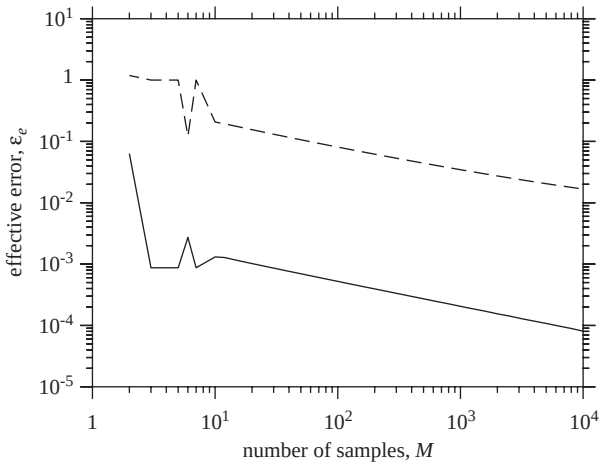

Fig. 6. The influence of the number of Monte Carlo samples on the effective estimation error. ——— μ_x; – – – – σ_x.

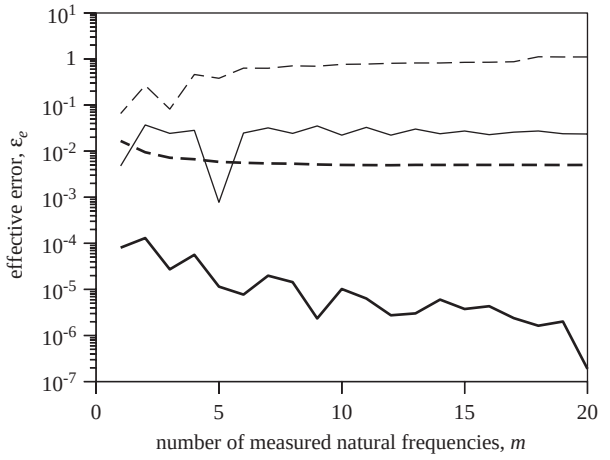

Fig. 7. The effect of using higher natural frequencies in the estimation. ——— μ_x perturbation; – – – – σ_x perturbation; ——— μ_x Monte Carlo; – – – – σ_x Monte Carlo.

(in the sense of adding new natural frequencies rather than more samples) can make the estimates worse. The problem is caused by the loss in accuracy of the linearised solution for the higher natural frequencies, and Fig. 2 has already demonstrated that the higher natural frequencies vary more with mass position than the lower frequencies.

As more measurements are taken the estimates obtained by both approaches generally improve, as shown in Fig. 8. The convergence of perturbation approach is limited by the response nonlinearities as mentioned above.

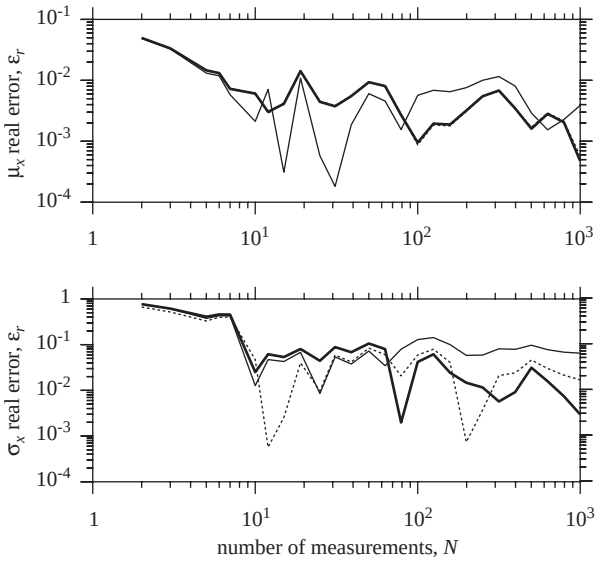

Fig. 8. The influence of the number of measurements on the real estimation error. ——— perturbation; ━━━ Monte Carlo; ····· effective.

3.2. Experimental validation

For experimental validation a similar system to the one analysed in Section 3.1 was created, and is shown in Fig. 9. The model was modified to account for the accelerometer (with mass $m_a = 34.1$ g positioned at $x_a = 20$ mm from the beam free end) and to allow for some translational and rotational clamping flexibility (K and K_t), as illustrated in Fig. 10. The beam has length $l = 60$ cm, a rectangular section of $70 \times 12\,\text{mm}^2$ and is made of steel with Young's modulus $E = 210$ GPa and density $\rho = 7800\,\text{kg/m}^3$. The discrete mass is $m = 93.6$ g and its position x follows a normal distribution $X \sim N(\mu = 15\,\text{cm}, \sigma = 5\,\text{cm})$.

The clamping stiffnesses were determined by model updating of the beam without the mass, by minimising the relative error in the first three natural frequencies. The estimated stiffnesses were $K = 91.466 \times 10^6$ N/m and $K_t = 109.825 \times 10^3$ N/rad, and the first four measured and updated natural frequencies are given in Table 1. The resulting model is clearly excellent, although with two unknown parameters and three natural frequencies some residual error will exist, as shown in Table 1.

Fifty samples of x were generated and rounded to the nearest mm. The discrete mass was positioned accordingly and the measurements of the first natural frequency taken. Fig. 11 shows these measurements and compares them to the response of the model. Even after model updating there is a small bias Δ between the measured natural frequencies and the ones derived by the model. To reduce these modelling errors (which would otherwise yield an offset in the mean estimate) a bias is introduced in Eq. (1) as

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) + \Delta. \tag{20}$$

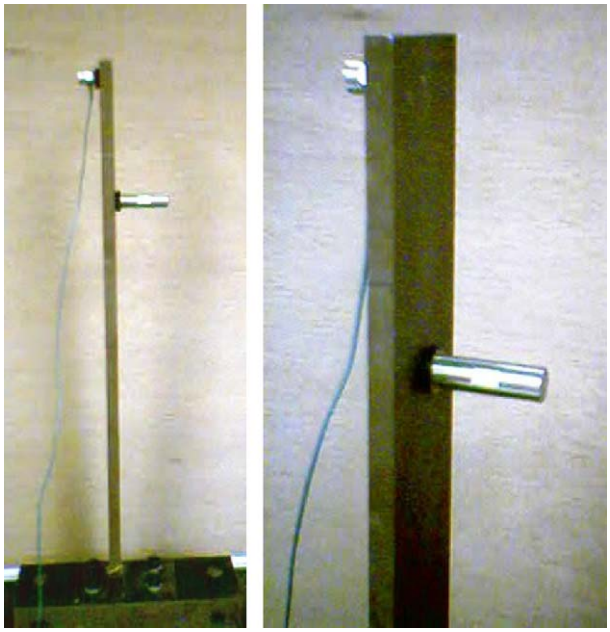

Fig. 9. Experimental set-up of the cantilever beam with a lumped mass at an uncertain position.

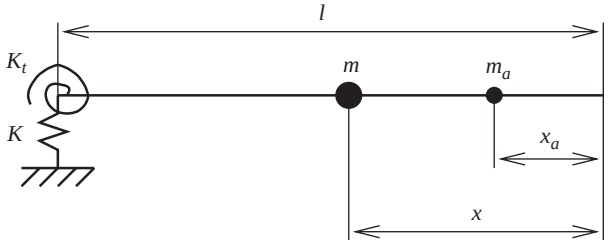

Fig. 10. The model of the experimental cantilever beam system.

Table 1
Model updating of the clamping stiffnesses of the beam without the discrete mass

Mode	Measured (Hz)	Updated (Hz)	Δ (Hz)
1	25.9049	25.8906	−0.0143
2	162.9649	163.1180	+0.1532
3	456.7434	456.5961	−0.1473
4	890.0572	889.5894	−0.4678

Table 2 shows the estimates of μ_x and σ_x compared with their real and effective counterparts. The estimates obtained by the Monte Carlo method are only slightly closer to the effective values than those obtained by the perturbation method. In this case the perturbation method performs well, and this is because variation of the first natural frequency is almost linear.

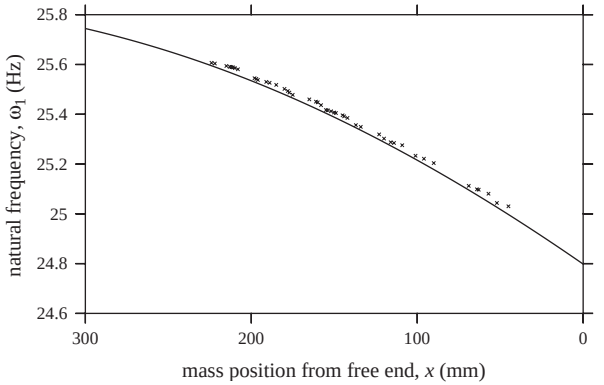

Fig. 11. Natural frequency response (experimental vs. model). ×, experimental; ——— numerical.

Table 2
Estimated mean and variance for the experimental example

	μ_x (mm)	σ_x (mm)
Real	150.0	50.0
Effective	151.5	49.5
Perturbation estimate	149.5	50.4
Monte Carlo estimate	153.4	48.5

4. Conclusion

This paper has described methods to quantify the parameter uncertainty from experimental data based on the Monte Carlo and perturbation approaches. These methods were validated using numerical and experimental examples. Although very efficient, the perturbation approach has problems when the linear approximation to the response is poor, and can lead to more information giving higher parameter estimation errors. The Monte Carlo approach works well, but care must be taken to ensure that the computational effort is realistic.

The next stage is to extend the methods to applications with several parameters and random fields, and to better deal with modelling and/or measurement errors.

Appendix. Sum of numbers in logarithmic representation

Consider the expression

$$b = \log \sum_{j=1}^{M} \exp\, a_j. \tag{A.1}$$

Let k be the index of the maximum a_j, then Eq. (A.1) can be rewritten as

$$b = a_k + \log\left(1 + \sum_{j \neq k} \exp(a_j - a_k)\right), \tag{A.2}$$

where $a_j - a_k \leqslant 0$. All terms where $a_j - a_k \ll 0$ are negligible since $\exp(a_j - a_k) \ll 1$. b can then be approximated within a tolerance ε by

$$b = a_k + \log\left(\sum_{a_j - a_k \geqslant \log\, \varepsilon/M} \exp(a_j - a_k)\right). \tag{A.3}$$

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