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Approximate eigensolutions for arbitrarily damped nearly proportional systems

Philip D. Cha*

Department of Engineering, Harvey Mudd College, 301 E. 12th street, Claremont, CA 91711-5990, USA

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Abstract

Two efficient methods for determining the approximate eigenvalues and eigenvectors for arbitrarily damped nearly proportional systems are developed. Both approaches are formulated by means of a first-order perturbation technique, whereby the real modes of vibration of the undamped system are used to derive approximate expressions for the complex eigenvalues and eigenvectors of a nearly proportionally damped system. Using either approach, the unperturbed configuration corresponds to a damped one whose damping matrix can be diagonalized by the same transformation that uncouples the undamped system, and the perturbation consists of the deviation of this diagonalizable damping matrix from the actual damping matrix. The proposed approaches are easy to code, implement and solve, and do not require forming state equations. Numerical examples are presented to validate the effectiveness of the current methods.

1. Introduction

Dynamic analysis of a discrete vibratory system typically begins with an evaluation of their eigensolutions. To meet new performance specifications, one is often required to make design modifications after an initial analysis has been completed, and to determine the resulting changes in the eigensolutions. Clearly, if these modifications are large, then a new analysis and computational cycle are necessary in order to compute the new eigendata. However, if the changes

*Tel.: +1 909 607 4102; fax: +1 909 621 8967.
E-mail address: philip-cha@hmc.edu.

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made are small, then the perturbation theory can be applied whereby the initial modal characteristics are used as a basis to extract the new eigensolution of the modified system without performing a new and possibly costly analysis. Over the years the perturbation theory has been used in the solution of many different problems [1–6].

The perturbation approach can also be used to study the effects of light damping. For an undamped system, the eigensolutions are real. These eigensolutions, also known as the modes of vibration, are characterized by the natural frequencies and the mode shapes of the system. For a damped system, the eigensolutions are typically complex. To obtain the eigensolutions exactly, state equations are used, resulting in a generalized eigenvalue problem with complex eigenvalues and eigenvectors that require extensive computations. Many efficient methods have been proposed over the years to determine the complex eigenvalues and eigenvectors of lightly damped systems by means of a perturbation technique [7–11]. If the amount of damping in the system is very small, then the eigensolutions of the lightly damped system differ only slightly from those of the undamped configuration. Hence, the eigensolutions of the lightly damped system can be approximated in terms of a power series expanded from the eigensolutions of the undamped system. Knowing the eigensolutions of the undamped system, the higher order terms of this expansion which reflect the effects of damping can be easily obtained.

In this paper, two approaches are introduced that can be used to determine the first-order eigenvalues and eigenvectors of an arbitrarily damped but nearly proportional or weakly non-proportional system. Both approaches are developed by means of the perturbation theory. Using the proposed methods, the eigensolutions of the unperturbed or proportionally damped system will be used to extract the eigensolutions of the arbitrarily damped, weakly non-proportional system. The unperturbed configuration corresponds to one whose damping matrix can be diagonalized by the modal matrix of the undamped system. Thus, the eigensolutions of the unperturbed system depend only on the undamped natural frequencies and mode shapes, which are strictly real quantities. Both approaches rely on the perturbation technique, but they are completely different in the ways in which their diagonalizable damping matrices are formulated. The benefits of the current methods will be discussed and highlighted, and numerical experiments will be presented to show the effectiveness of the proposed approaches.

2. Theory

In the following sections, the eigensolutions of an undamped, generally damped and proportionally damped systems are first introduced. These results are well known and they are presented for completeness, and more importantly, for notational purposes.

2.1. Undamped systems

Consider an undamped N-degree of freedom (dof) system whose equations of motion are given by

$$[M]\underline{\ddot{q}}_0 + [K]\underline{q}_0 = \underline{0}, \tag{1}$$

where $[M]$ and $[K]$ are the symmetric mass and stiffness matrices of the system. The modes of vibration of the undamped system correspond to the eigensolution of the following generalized eigenvalue problem:

$$[K]\underline{u}_0 = \omega^2[M]\underline{u}_0, \tag{2}$$

where ω denotes the natural frequency of the system and $\underline{u}_0$ is its mode shape. Assume the N natural frequencies are all distinct and the mode shapes are properly normalized, then they satisfy the following orthogonality conditions:

$$\underline{u}_{0j}^{\mathrm{T}}[M]\underline{u}_{0i} = \delta_i^j \quad \text{and} \quad \underline{u}_{0j}^{\mathrm{T}}[K]\underline{u}_{0i} = \omega_j^2\delta_i^j \quad \text{for } i,j = 1,\ldots,N, \tag{3}$$

where δ_i^j is the Kronecker delta. Eq. (3) can also be expressed compactly in matrix form as

$$[U_0]^{\mathrm{T}}[M][U_0] = [I] \quad \text{and} \quad [U_0]^{\mathrm{T}}[K][U_0] = [\Lambda], \tag{4}$$

where $[I]$ is the identity matrix, $[\Lambda]$ is a diagonal matrix whose ith element is simply ω_i^2, and $[U_0]$ is the modal matrix of the system whose columns are the normalized mode shapes, i.e., $[U_0] = [\underline{u}_{01}\ \underline{u}_{02}\ \cdots\ \underline{u}_{0N}]$.

2.2. Generally damped systems

The governing equations for a generally damped N-dof system can be expressed as

$$[M]\,\underline{\ddot{q}} + [C]\,\underline{\dot{q}} + [K]\,\underline{q} = \underline{0}, \tag{5}$$

where $[M]$, $[C]$ and $[K]$ are the symmetric mass, damping and stiffness matrices, respectively, of the system. The vector of generalized displacements, $\underline{q}$, has a solution given by the following exponential form:

$$\underline{q}(t) = \underline{u}\,\mathrm{e}^{\lambda t}, \tag{6}$$

where λ is a constant scalar and $\underline{u}$ is a constant vector, and they are known as the eigenvalue and eigenvector, respectively. Collectively, they form the eigensolution of the system. Inserting Eq. (6) into Eq. (5) and noting that an exponential can never be zero, one obtains

$$\{\lambda^2[M] + \lambda[C] + [K]\}\,\underline{u} = \underline{0}. \tag{7}$$

To have a non-trivial solution for $\underline{u}$, the determinant of the coefficient matrix of $\underline{u}$ must vanish. Expanding the resulting determinant leads to a $2N$-order polynomial in λ, the solution of which can be readily solved using any prepackaged code such as *rpzero* in CMLIB [12] or *roots* in MATLAB. Once the eigenvalues are known, the corresponding eigenvectors are obtained by solving Eq. (7) using Gaussian elimination.

Alternatively, the eigensolutions, λ and $\underline{u}$, of Eq. (6) can also be determined by using a state matrix approach, which effectively replaces the N coupled second-order differential equations of Eq. (5) by $2N$ coupled first-order ordinary differential equations as follows [13]. A state vector of length $2N$ is introduced,

$$\underline{y} = \begin{bmatrix} \underline{\dot{q}} \\ \underline{q} \end{bmatrix} = \begin{bmatrix} \lambda\,\underline{u} \\ \underline{u} \end{bmatrix}\mathrm{e}^{\lambda t} = \underline{z}\,\mathrm{e}^{\lambda t}, \tag{8}$$

such that Eq. (5) can be rewritten in a form that consists of $2N$ simultaneous first-order ordinary differential equations

$$[A]\underline{\dot{y}} - [B]\underline{y} = \underline{0}, \tag{9}$$

where matrices $[A]$ and $[B]$ are both symmetric and are given by

$$[A] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \quad \text{and} \quad [B] = \begin{bmatrix} [M] & [0] \\ [0] & -[K] \end{bmatrix}. \tag{10}$$

Substituting Eq. (8) into Eq. (9) yields the following $2N \times 2N$ generalized eigenvalue problem:

$$[B]\underline{z} = \lambda[A]\underline{z}. \tag{11}$$

Assume the $2N$ eigenvalues are distinct and all the eigenvectors are properly normalized. Then the orthogonality properties of the eigenvalues and eigenvectors can be expressed as [13]

$$[Z]^{\mathrm{T}}[A][Z] = [I] \quad \text{and} \quad [Z]^{\mathrm{T}}[B][Z] = [\Lambda], \tag{12}$$

where the modal matrix $[Z] = [\underline{z}_1 \quad \underline{z}_2 \quad \cdots \quad \underline{z}_{2N}]$, $[I]$ is the identity matrix, and $[\Lambda]$ here is a $2N \times 2N$ diagonal matrix whose ith elements is λ_i. Using a state vector formulation, the λ_i's and the corresponding $\underline{z}_i$'s (and hence the $\underline{u}_i$'s), can be readily solved by using any existing prepackaged code such as *rsg* in EISPACK [14] or *eig* in MATLAB. When N is large, the eigensolution of the generalized eigenvalue problem of Eq. (11) may be computationally intensive.

2.3. Proportionally damped systems

Consider now the special case of a proportionally damped system, whose eigensolutions can be determined exactly using only the modes of vibration of the undamped configuration. For a proportionally damped system, its equations of motion are given by

$$[M]\underline{\ddot{q}}_p + [C_p]\underline{\dot{q}}_p + [K]\underline{q}_p = \underline{0}, \tag{13}$$

where $\underline{q}_p$ is the vector of generalized coordinates for the proportionally damped system, whose damping matrix $[C_p]$ can be expressed as a linear combination of the mass and stiffness matrices as

$$[C_p] = \alpha[M] + \beta[K]. \tag{14}$$

The parameters α and β are real constants, and $[C_p]$ can be diagonalized by the same transformation that was used earlier to decouple the equations of motion for the undamped configuration, namely

$$\underline{q}_p = [U_0]\underline{\eta}_p, \tag{15}$$

where $\underline{\eta}_p$ represents the vector of normal coordinates for the proportionally damped system. Substituting Eq. (15) into Eq. (13), premultiplying by $[U_0]^{\mathrm{T}}$ and utilizing the orthogonality properties of the undamped mode shapes, the equations of motion become decoupled in the normal coordinates

$$\ddot{\eta}_{pj} + 2\zeta_j\omega_j\dot{\eta}_{pj} + \omega_j^2\eta_{pj} = 0 \quad \text{for } j = 1, \ldots, N, \tag{16}$$

where the jth damping factor is given by

$$\zeta_j = \frac{\alpha + \beta\omega_j^2}{2\omega_j}. \tag{17}$$

The solution of Eq. (16) is given by

$$\eta_{pj} = \bar{\eta}_{pj} e^{\lambda_{pj} t}, \tag{18}$$

where λ_{pj}, the eigenvalues, are found to be

$$\lambda_{pj} = \left(-\zeta_j \pm \sqrt{\zeta_j^2 - 1}\right)\omega_j \quad \text{for } j = 1, \ldots, N, \tag{19}$$

where ζ_j and ω_j are the jth damping factor and the jth undamped natural frequency, respectively. For a given ζ_j and ω_j, two eigenvalues λ_{pj}^+ and λ_{pj}^- are possible, one corresponding to the positive square root and the other to the negative square root of Eq. (19). If $0 \leqslant \zeta_j < 1$, the eigenvalues λ_{pj}^+ and λ_{pj}^- are complex conjugates. For $\zeta_j = 1$, the eigenvalues are real, negative and identical. For $\zeta_j > 1$, the eigenvalues are real, negative and distinct. Regardless, Rayleigh [15] showed that if the damping matrix is a linear combination of the mass and stiffness matrices, then the damped system will have the same normal modes as the undamped system. Thus, the eigenvectors associated with both eigenvalues are given by $\underline{u}_{0j}$.

2.4. *Nearly proportionally damped systems*

In this paper, two alternative means to determine the eigenvalues λ_i and the eigenvectors $\underline{u}_i$ are proposed for the special case of nearly proportionally damped systems. In particular, the eigensolutions or the modes of vibration of the undamped configuration will be used as basis to find the approximate eigensolutions for an arbitrarily damped but weakly non-proportional system without resorting to state form.

2.4.1. *Least squares approach*

For a damped weakly non-proportional system, its damping matrix can be expressed as

$$[C] = [C_p] + [\delta C], \tag{20}$$

where $[\delta C]$ is the deviation from a proportionally damped matrix. Because the system is weakly non-proportionally damped, $[\delta C]$ represents a first-order damping matrix. The constants α and β of $[C_p]$ can be obtained by using a least squares formulation such that the norm of $||[C] - [C_p]||$, defined as

$$f_n = \sum_{i=1}^{N}\sum_{j=1}^{N}[C_{ij} - (\alpha M_{ij} + \beta K_{ij})]^2, \tag{21}$$

is minimized, where C_{ij}, M_{ij} and K_{ij} are the (i,j)th element of the damping, mass and stiffness matrices, respectively. Setting the partial derivatives of f_n with respect to α and β equal to zero yields the matrix equation

$$\begin{bmatrix} \sum_{i=1}^{N}\sum_{j=1}^{N}(M_{ij})^2 & \sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}K_{ij} \\ \sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}K_{ij} & \sum_{i=1}^{N}\sum_{j=1}^{N}(K_{ij})^2 \end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}M_{ij} \\ \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}K_{ij} \end{bmatrix}, \tag{22}$$

which can be readily solved for the parameters α and β that minimize Eq. (21).

A first-order perturbation approach will be used to obtain the approximate eigensolutions of an arbitrarily damped but weakly non-proportional system. The proportionally damped system will be considered as the unperturbed configuration, and the first-order damping matrix $[\delta C]$ will be considered as the perturbation. The eigensolutions of the proportionally damped or the unperturbed system depend only on the eigensolutions of the undamped configuration.

Because the damping matrix $[C]$ is slightly perturbed from the proportionally damped matrix $[C_p]$, the jth eigensolution of the system will be a perturbation of the jth eigensolution of the proportionally damped system

$$\lambda_j = \lambda_{0j} + \delta\lambda_j \quad \text{and} \quad \underline{u}_j = \underline{u}_{0j} + \delta\underline{u}_j, \tag{23}$$

where λ_{0j} is the jth eigenvalue of the unperturbed or the proportionally damped system, thus $\lambda_{0j} = \lambda_{pj}$, and $\underline{u}_{0j}$ represents the jth eigenvector of the undamped system; $\delta\lambda_j$ and $\delta\underline{u}_j$ denote the first-order eigenvalue and eigenvector perturbations, respectively. Substituting Eqs. (20) and (23) into Eq. (7), expanding, keeping only the first-order terms, and noting that the unperturbed eigensolution satisfies

$$(\lambda_{0j}^2[M] + \lambda_{0j}[C_p] + [K])\underline{u}_{0j} = \underline{0}, \tag{24}$$

one gets

$$2\lambda_{0j}\delta\lambda_j[M]\underline{u}_{0j} + \lambda_{0j}[\delta C]\underline{u}_{0j} + \delta\lambda_j[C_p]\underline{u}_{0j} + (\lambda_{0j}^2[M] + \lambda_{0j}[C_p] + [K])\delta\underline{u}_j = \underline{0}. \tag{25}$$

Premultiplying Eq. (25) by $\underline{u}_{0j}^{\mathrm{T}}$, one obtains

$$2\lambda_{0j}\delta\lambda_j\underline{u}_{0j}^{\mathrm{T}}[M]\underline{u}_{0j} + \lambda_{0j}\underline{u}_{0j}^{\mathrm{T}}[\delta C]\underline{u}_{0j} + \delta\lambda_j\underline{u}_{0j}^{\mathrm{T}}[C_p]\underline{u}_{0j} = 0. \tag{26}$$

Recalling Eq. (14) and the orthogonality conditions of Eq. (3), one finds an expression for the first-order eigenvalue perturbation

$$\delta\lambda_j = -\frac{\lambda_{0j}\underline{u}_{0j}^{\mathrm{T}}[\delta C]\underline{u}_{0j}}{2\lambda_{0j} + \alpha + \beta\omega_j^2}. \tag{27}$$

The first-order eigenvector perturbation $\delta\underline{u}_j$ of Eq. (23) can be obtained by premultiplying Eq. (25) by $\underline{u}_{0i}^{\mathrm{T}}$, where $i \neq j$, to yield

$$2\lambda_{0j}\delta\lambda_j\underline{u}_{0i}^{\mathrm{T}}[M]\underline{u}_{0j} + \lambda_{0j}\underline{u}_{0i}^{\mathrm{T}}[\delta C]\underline{u}_{0j} + \delta\lambda_j\underline{u}_{0i}^{\mathrm{T}}[C_p]\underline{u}_{0j} + \lambda_{0j}^2\underline{u}_{0i}^{\mathrm{T}}[M]\delta\underline{u}_j + \lambda_{0j}\underline{u}_{0i}^{\mathrm{T}}[C_p]\delta\underline{u}_j + \underline{u}_{0i}^{\mathrm{T}}[K]\delta\underline{u}_j = 0. \tag{28}$$

Recall that the natural frequencies of the undamped system are assumed to be distinct, and that the eigenvectors $\underline{u}_{0j}$ are normalized according to Eq. (3). Thus, the $\underline{u}_{0j}$'s form a complete orthonormal set (with respect to $[M]$) in the N-dimensional space, and any vector in that N-dimensional space may be expressed as a linear combination of the $\underline{u}_{0j}$'s. Hence, the jth first-order eigenvector perturbation can be written as

$$\delta\underline{u}_j = \sum_{r=1}^{N} \varepsilon_{rj}\underline{u}_{0r}, \tag{29}$$

where the ε_{rj}'s are small coefficients to be determined. Substituting Eq. (29) into Eq. (28) and applying the orthogonality properties of Eq. (3), one obtains

$$\varepsilon_{ij} = -\frac{\lambda_{0j}\underline{u}_{0i}^{\mathrm{T}}[\delta C]\underline{u}_{0j}}{\lambda_{0j}(\lambda_{0j}+\alpha)+\omega_i^2(\beta\lambda_{0j}+1)}. \tag{30}$$

To determine the coefficients ε_{jj}, the perturbed eigenvectors $\underline{u}_j$'s are assumed to satisfy the following orthogonality condition:

$$\underline{u}_j^{\mathrm{T}}[M]\underline{u}_j = 1. \tag{31}$$

Inserting Eq. (23) into Eq. (31), expanding and keeping only the first-order terms, one reduces Eq. (31) to

$$\underline{u}_{0j}^{\mathrm{T}}[M]\underline{u}_{0j} + \underline{u}_{0j}^{\mathrm{T}}[M]\delta\underline{u}_j + \delta\underline{u}_j^{\mathrm{T}}[M]\underline{u}_{0j} = 1. \tag{32}$$

Upon substituting Eq. (29) into Eq. (32) and noting the orthogonality properties, one finds $\varepsilon_{jj} = 0$. Thus, the jth first-order eigenvector perturbation is

$$\delta\underline{u}_j = -\sum_{i=1,i\neq j}^{N} \frac{\lambda_{0j}\underline{u}_{0i}^{\mathrm{T}}[\delta C]\underline{u}_{0j}}{\lambda_{0j}(\lambda_{0j}+\alpha)+\omega_i^2(\beta\lambda_{0j}+1)}\,\underline{u}_{0i}. \tag{33}$$

In summary, the jth perturbed eigenvalue for an arbitrarily damped nearly proportional system can be approximated as

$$\lambda_j = \lambda_{0j}\left(1 - \frac{\underline{u}_{0j}^{\mathrm{T}}[\delta C]\underline{u}_{0j}}{2\lambda_{0j}+\alpha+\beta\omega_j^2}\right), \tag{34}$$

and the corresponding perturbed eigenvector is given by

$$\underline{u}_j = \underline{u}_{0j} - \sum_{i=1,i\neq j}^{N} \frac{\lambda_{0j}\underline{u}_{0i}^{\mathrm{T}}[\delta C]\underline{u}_{0j}}{\lambda_{0j}(\lambda_{0j}+\alpha)+\omega_i^2(\beta\lambda_{0j}+1)}\,\underline{u}_{0i}, \tag{35}$$

where λ_{0j} is the jth eigenvalue of the proportionally damped system, and ω_j and $\underline{u}_{0j}$ are the jth natural frequency and mode shape of the undamped configuration. The parameters α and β are found using a least squares approach by minimizing Eq. (21).

2.4.2. Transformation approach

For a proportionally damped system, the modal matrix of the undamped system $[U_0]$ can be used to diagonalize the damping matrix $[C_p]$, yielding

$$[U_0]^{\mathrm{T}}[C_p][U_0] = \mathrm{diag}[2\zeta_j\omega_j] \quad \text{for } j = 1,\ldots,N. \tag{36}$$

For a generally damped system, however, diagonalization is seldom possible. Nevertheless, it is always possible to express

$$[U_0]^{\mathrm{T}}[C][U_0] = [D] + [ND], \tag{37}$$

as the sum of a diagonal matrix $[D]$ and a non-diagonal matrix $[ND]$. Using Eq. (4), one can manipulate Eq. (37) and find an expression for the damping matrix

$$[C] = [M][U_0][D][U_0]^{\mathrm{T}}[M] + [M][U_0][ND][U_0]^{\mathrm{T}}[M] = [C_d] + [\delta C'], \tag{38}$$

where

$$[C_d] = [M][U_0][D][U_0]^{\mathrm{T}}[M] \quad \text{and} \quad [\delta C'] = [M][U_0][ND][U_0]^{\mathrm{T}}[M]. \tag{39}$$

Here, $[C_d]$ is an arbitrary damping matrix that can be diagonalized by the same transformation that uncouples the undamped system. It does not have to be a linear combination of the mass and stiffness matrices. Thus in general, $[C_d] \neq [C_p]$ and $[\delta C'] \neq [\delta C]$.

If the system is nearly proportionally damped, $[\delta C']$ will be of first-order relative to $[C_d]$, and the perturbation approach can again be applied to find the approximate eigensolutions of the system. Thus, the perturbation results obtained in Section 2.4.1 can be easily extended to this case. Using this particular approach, the unperturbed system is the damped system with $[C_d]$, and the perturbation is simply $[\delta C']$.

Because $[C_d]$ can be diagonalized by the same transformation that uncouples the undamped system, i.e.,

$$[U_0]^{\mathrm{T}}[C_d][U_0] = [D], \tag{40}$$

the unperturbed system considered here possesses the same normal modes as the undamped configuration [16]. Thus, the damped system with $[C_d]$ can be easily uncoupled in the normal coordinates. After some algebraic manipulation, one obtains the following equations of motion in the normal coordinates:

$$\ddot{\eta}_{dj} + 2\zeta'_j\omega_j\dot{\eta}_{dj} + \omega_j^2\eta_{dj} = 0 \quad \text{for } j = 1, \ldots, N, \tag{41}$$

where the jth damping factor is related to the jth diagonal element of $[D]$

$$\zeta'_j = \frac{D_{jj}}{2\omega_j}, \tag{42}$$

and ω_j denotes the jth undamped natural frequency of the system. Thus, the unperturbed eigenvalues using this approach are given by

$$\lambda_{dj} = \left(-\zeta'_j \pm \sqrt{\zeta_j'^2 - 1}\right)\omega_j \quad \text{for } j = 1, \ldots, N, \tag{43}$$

and the unperturbed eigenvectors are $\underline{u}_{0j}$. Like before, two eigenvalues λ_{dj}^+ and λ_{dj}^- are possible for a given ζ'_j and ω_j.

Following the same procedure that was outlined in Section 2.4.1, one obtains the following approximation for the jth eigenvalue:

$$\lambda_j = \lambda_{dj}\left(1 - \frac{\underline{u}_{0j}^{\mathrm{T}}[\delta C']\underline{u}_{0j}}{2\lambda_{dj} + D_{jj}}\right) = \lambda_{dj}\left(1 - \frac{ND_{jj}}{2\lambda_{dj} + D_{jj}}\right) = \lambda_{dj}, \tag{44}$$

because $ND_{jj} = 0$, implying that the perturbed and unperturbed eigenvalues are exactly the same. Similarly, the corresponding jth eigenvector is found to be

$$\underline{u}_j = \underline{u}_{0j} - \sum_{i=1, i\neq j}^{N} \frac{\lambda_{dj} ND_{ij}}{\lambda_{dj}(\lambda_{dj} + D_{ii}) + \omega_i^2}\, \underline{u}_{0i}, \tag{45}$$

where λ_{dj} is given by Eq. (43), and ω_j and $\underline{u}_{0j}$ are the jth natural frequency and mode shape of the undamped system. While no term of $[ND]$ appears in the expression for the perturbed eigenvalues, elements of $[ND]$ do affect the perturbed eigenvectors.

Using this approach, the perturbed and unperturbed eigenvalues are shown to be identical. They can also be obtained by simply approximating $[U_0]^{\mathrm{T}}[C][U_0] \approx [D]$, thus justifying the common practice of neglecting the off-diagonal terms of $[U_0]^{\mathrm{T}}[C][U_0]$ when they are small compared to the diagonal [13]. While this approximation is well known, to the best knowledge of this author, it has not previously been validated using this approach. Thus, it may appear that nothing is gained by introducing the transformation approach. However, the proposed scheme allows one to find a first-order approximation for the eigenvectors. This, in turn, enables one to compute $[Z_{\mathrm{pert}}]^{\mathrm{T}}[B][Z_{\mathrm{pert}}]$, whose diagonal elements offer yet another approximation to the exact eigenvalues. Finally, note that if $[C_d] = [C_p]$ and $[\delta C'] = [\delta C]$, then $\lambda_{dj} = \lambda_{0j}$, $ND_{ij} = \underline{u}_{0i}^{\mathrm{T}}[\delta C]\underline{u}_{0j}$, $D_{ii} = \alpha + \beta\omega_i^2$, and Eqs. (44) and (45) become Eqs. (34) and (35), respectively.

3. Results

Various examples will be considered to validate the effectiveness of the proposed methods. The system of Fig. 1 will be analyzed, which consists of a discrete system with five dof. When $c_1 = k_1 = 0$, the system possesses proportional damping with $\alpha = 0$ and $\beta = c_0/k_0$. By simply changing c_1 and k_1, the extent of the non-proportionality of the damping matrix can be varied. Multiplying the damping matrix $[C]$ by a parameter σ allows the system damping factors to be modified.

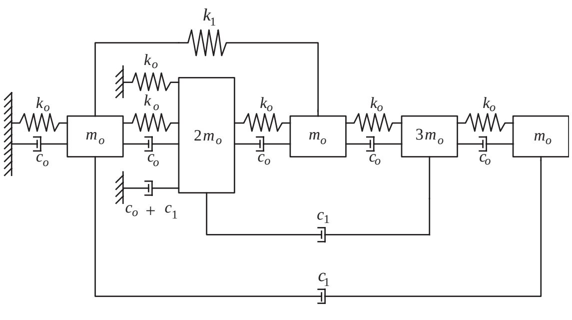

Fig. 1. Discrete model with five degrees of freedom.

The amount of damping in the system can be arbitrary, but the damping matrix is restricted to nearly proportional or weakly non-proportional. To determine quantitatively the degree of non-proportional damping present in the system, a non-proportionality index is introduced, defined as the quotient of the sum of the absolute value of all the terms in the transformed first-order damping matrix and the sum of the absolute value of all the terms in the transformed damping matrix,

$$\delta = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N}|ND'_{ij}|}{\sum_{i=1}^{N}\sum_{j=1}^{N}|C'_{ij}|}, \tag{46}$$

where the transformed damping matrix is $[C'] = [U_0]^{\mathrm{T}}[C][U_0]$, and the transformed first-order damping matrix corresponds to $[ND'] = [U_0]^{\mathrm{T}}[\delta C][U_0]$ for the least squares method, and to $[ND'] = [ND]$ for the transformation approach. Using the latter approach, the non-proportionality index of Eq. (46) coincides with the summation based index that Prater and Singh introduced in Ref. [17] to characterize the extent of non-proportional damping present within a discrete vibratory system. Using either approach, as long as $\delta \ll 1$, the system is nearly proportional damped, and for $\delta = 0$ the system is exactly proportionally damped.

To demonstrate the effectiveness of the current methods, the perturbed eigensolutions are compared with the exact results. The error norm of the eigenvalues, defined as

$$\varepsilon_{\lambda_i} = \frac{\|(\lambda_{\mathrm{exact}})_i - (\lambda_{\mathrm{pert}})_i\|}{\|(\lambda_{\mathrm{exact}})_i\|} \quad \text{for } i = 1, \ldots, 2N, \tag{47}$$

will be used to quantify the accuracy of the perturbed eigenvalues. The exact eigenvalues, λ_{exact}, are obtained by solving the generalized eigenvalue problem of Eq. (11). The perturbed eigenvalues, $(\lambda_{\mathrm{pert}})_i$, correspond to either Eq. (34) or Eq. (44), depending on the approach used. To check for the accuracy of the perturbed eigenvectors, the orthogonality conditions of Eq. (12) will be utilized. A modal matrix $[Z_{\mathrm{pert}}]$ is constructed, whose ith column is given by

$$(\underline{z}_{\mathrm{pert}})_i = \begin{bmatrix} (\lambda_{\mathrm{pert}})_i\, \underline{u}_i \\ \underline{u}_i \end{bmatrix}, \tag{48}$$

where $\underline{u}_i$ is the ith perturbed eigenvector of the system, and it corresponds to either Eq. (35) or Eq. (45). If the modal matrix is normalized such that the diagonal elements of

$$[I'] = [Z_{\mathrm{pert}}]^{\mathrm{T}}[A][Z_{\mathrm{pert}}] \tag{49}$$

are identically one, then the average of the magnitudes of the off-diagonal terms, defined as

$$\varepsilon_{z_{\mathrm{pert}}} = \frac{1}{2N(2N-1)} \sum_{i=1}^{2N} \sum_{j=1, j\neq i}^{2N} |I'_{ij}|, \tag{50}$$

can be used as an error parameter to quantify the correctness of the perturbed eigenvectors. The smaller this error parameter is relative to one, the closer the perturbed modal matrix is to the exact. Interestingly, numerical experiments show that the triple product $[Z_{\mathrm{pert}}]^{\mathrm{T}}[B][Z_{\mathrm{pert}}]$ (where $[Z_{\mathrm{pert}}]$ is properly normalized) returns a matrix whose diagonal elements are consistently closer to the exact eigenvalues than the perturbed eigenvalues obtained using either Eq. (34) or Eq. (44). The diagonal elements of $[Z_{\mathrm{pert}}]^{\mathrm{T}}[B][Z_{\mathrm{pert}}]$ will be referred to as the approximate eigenvalues, and are denoted by λ_j' to distinguish them from the perturbed eigenvalues. Thus, after the perturbed modal matrix is obtain and normalized, $[Z_{\text{pert}}]^{\text{T}}[B][Z_{\text{pert}}]$ is evaluated and its diagonal elements can also be used as approximations to the exact eigenvalues.

For definiteness, let $m_0 = 2\,\text{kg}$, $c_0 = 100\,\text{N s/m}$ and $k_0 = 5000\,\text{N/m}$. When $k_1 = c_1 = 0$, the system is proportionally damped regardless of the parameter σ (recall that σ is the parameter by which the damping matrix is multiplied), and as expected, both the least squares approach and the transformation approach return eigenvalues and eigenvectors that are exact. Consider now a slightly different system by letting $k_1 = 700\,\text{N/m}$, $c_1 = 0$ and $\sigma = 1$. The presence of k_1 renders the damping matrix weakly non-proportional. Using the least squares (ls) approach, $\alpha = 5.7235 \times 10^{-1}$, $\beta = 1.9464 \times 10^{-2}$, and the summation based non-proportionality index of Eq. (46) gives $\delta_{\text{ls}} = 8.6340 \times 10^{-2}$. For the transformation (tf) method, $\delta_{\text{tf}} = 4.7289 \times 10^{-2}$. Table 1(a) shows the exact eigenvalues, the perturbed eigenvalues of Eqs. (34) and (44), and their error norms.

Table 1

(a) The exact and the perturbed eigenvalues (rad/s) for the system of Fig. 1[a]

i	$(\lambda_{\text{exact}})_i$	$(\lambda_{\text{ls}})_i$	$(\lambda_{\text{tf}})_i$
1	−2.0286e + 00 + 1.4629e + 01i	−2.0285e + 00 + 1.4628e + 01i (4.0711e − 05)	−2.0285e + 00 + 1.4623e + 01i (3.8145e − 04)
2	−1.8939e + 01 + 3.9197e + 01i	−1.8939e + 01 + 3.9196e + 01i (2.6524e − 05)	−1.8939e + 01 + 3.9195e + 01i (4.2326e − 05)
3	−3.4152e + 01 + 4.7975e + 01i	−3.4114e + 01 + 4.7818e + 01i (2.7402e − 03)	−3.4114e + 01 + 4.7817e + 01i (2.7564e − 03)
4	−5.3073e + 01 + 5.5673e + 01i	−5.3145e + 01 + 5.6396e + 01i (9.4545e − 03)	−5.3145e + 01 + 5.5877e + 01i (2.8165e − 03)
5	−7.0974e + 01 + 4.5429e + 01i	−7.0941e + 01 + 4.5621e + 01i (2.3167e − 03)	−7.0941e + 01 + 4.5545e + 01i (1.4387e − 03)

(b) The exact and the approximate eigenvalues for the system parameters of (a)[b]

i	$(\lambda_{\text{exact}})_i$	$(\lambda'_{\text{ls}})_i$	$(\lambda'_{\text{tf}})_i$
1	−2.0286e + 00 + 1.4629e + 01i	−2.0286e + 00 + 1.4629e + 01i (7.0131e − 09)	−2.0286e + 00 + 1.4629e + 01i (1.2802e − 07)
2	−1.8939e + 01 + 3.9197e + 01i	−1.8939e + 01 + 3.9197e + 01i (7.5815e − 07)	−1.8939e + 01 + 3.9197e + 01i (7.6413e − 08)
3	−3.4152e + 01 + 4.7975e + 01i	−3.4143e + 01 + 4.7980e + 01i (1.6798e − 04)	−3.4152e + 01 + 4.7975e + 01i (4.5872e − 06)
4	−5.3073e + 01 + 5.5673e + 01i	−5.3034e + 01 + 5.5621e + 01i (8.3975e − 04)	−5.3073e + 01 + 5.5673e + 01i (4.9420e − 06)
5	−7.0974e + 01 + 4.5429e + 01i	−7.1067e + 01 + 4.5491e + 01i (1.3285e − 03)	−7.0975e + 01 + 4.5430e + 01i (1.7092e − 05)

[a]The perturbed eigenvalues of the third and fourth columns, obtained by the least squares and the transformation approaches, are given by Eqs. (34) and (44), respectively. The system parameters are $m_0 = 2\,\text{kg}$, $k_0 = 5000\,\text{N/m}$, $c_0 = 100\,\text{N s/m}$, $c_1 = 0$, $k_1 = 700\,\text{N/m}$ and $\sigma = 1$. The term in the parentheses represents the error norm of the eigenvalue.

[b]The approximate eigenvalues correspond to the diagonal elements of $[Z_{\text{pert}}]^{\text{T}}[B][Z_{\text{pert}}]$.

When an eigenvalue is complex, its conjugate is also an eigenvalue. The eigenvectors corresponding to complex conjugate eigenvalues are also complex conjugates. For the chosen set of system parameters, all of the eigenvalues are complex. Thus, only 5 eigenvalues are presented in Table 1(a). Note how well the perturbed eigenvalues track the exact results. Using the least squares approach, the maximum error norm of the eigenvalues is less than 0.95%, and using the transformation method, it is less than 0.29%. The accuracy of the perturbed modal matrix is given by the error parameter $\varepsilon_{z_{\mathrm{pert}}}$. Using the least squares approach, $(\varepsilon_{z_{\mathrm{pert}}})_{\mathrm{ls}} = 3.4691 \times 10^{-3}$. Using the transformation approach, $(\varepsilon_{z_{\mathrm{pert}}})_{\mathrm{tf}} = 1.4192 \times 10^{-3}$. Because both error parameters are small relative to one, they indicate that the perturbed eigenvectors are close to the exact. The accuracy of the perturbed eigenvectors can also be inferred from the diagonal elements of $[Z_{\mathrm{pert}}]^{\mathrm{T}}[B][Z_{\mathrm{pert}}]$. Table 1(b) shows the exact eigenvalues and the approximate eigenvalues obtained by expanding the triple product $[Z_{\mathrm{pert}}]^{\mathrm{T}}[B][Z_{\mathrm{pert}}]$. Note that the diagonal elements of $[Z_{\mathrm{pert}}]^{\mathrm{T}}[B][Z_{\mathrm{pert}}]$ return approximate eigenvalues that are very accurate, as clearly indicated by their error norms, implying that the perturbed modal matrices are nearly exact. Incidentally, the error norms are never zero even though some of the exact and approximate eigenvalues appear to be identical. The eigenvalues are presented with a format of 5 digits plus 2-digit exponent to make the results more readable. In calculating the error norms, all 16 digits plus 2-digit exponent are used.

Consider a new system where $k_1 = 0$, $c_1 = 20\,\mathrm{N\,s/m}$ and $\sigma = 1$. Now the non-proportionality is attributed to c_1. The parameters m_0, c_0 and k_0 remain unchanged. For this set of parameters, $\alpha = 4.3455 \times 10^{0}$ and $\beta = 2.0482 \times 10^{-2}$ for the least squares approach. The non-proportionality indices for the least squares approach and the transformation method are $\delta_{\mathrm{ls}} = 1.6837 \times 10^{-1}$ and $\delta_{\mathrm{tf}} = 1.3774 \times 10^{-1}$, respectively. Compared to the system of Table 1(a), note that the degree of non-proportionality has increased. Table 2(a) depicts the exact eigenvalues and the perturbed eigenvalues obtained by using the current methods. For the least squares approach, the maximum error norm of the eigenvalues is less than 3.2%, and for the transformation approach, it is less than 3.0%. The error parameter $\varepsilon_{z_{\mathrm{pert}}}$ is used to quantify the accuracy of the perturbed modal matrix. Using the current methods, $(\varepsilon_{z_{\mathrm{pert}}})_{\mathrm{ls}} = 1.5317 \times 10^{-2}$ and $(\varepsilon_{z_{\mathrm{pert}}})_{\mathrm{tf}} = 1.4931 \times 10^{-2}$, both of which are much less than one. Thus, the perturbed modal matrices obtained by using the least squares approach and the transformation method agree well with the exact modal matrix. Table 2(b) shows the exact eigenvalues and the approximate eigenvalues corresponding to the diagonal elements of $[Z_{\mathrm{pert}}]^{\mathrm{T}}[B][Z_{\mathrm{pert}}]$. Note the improvement in accuracy compared to the perturbed eigenvalues, as evidenced by the decrease in all of the error norms. The results of Table 2(b) also reflect the accuracy of the perturbed eigenvectors.

Consider a system with the same m_0, c_0 and k_0, but now with $k_1 = 550\,\mathrm{N/m}$, $c_1 = 15\,\mathrm{N\,s/m}$ and $\sigma = 1.45$. The non-proportionality in the system is caused by k_1 and c_1, and σ affects the damping factors of the system. This set of parameters yields $\alpha = 5.3599 \times 10^{0}$ and $\beta = 2.8911 \times 10^{-2}$ for the least squares approach. The non-proportionality indices for the damping matrix are $\delta_{\mathrm{ls}} = 1.5320 \times 10^{-1}$ and $\delta_{\mathrm{tf}} = 1.1866 \times 10^{-1}$. Table 3(a) shows the exact and the perturbed eigenvalues. For the chosen set of system parameters, there are 4 real, negative and distinct eigenvalues in addition to 3 pairs of complex conjugate eigenvalues. Thus, a total of 7 eigenvalues are shown. Using the least squares approach, the error norms are all less than 6.5% and $(\varepsilon_{z_{\mathrm{pert}}})_{\mathrm{ls}} = 2.8450 \times 10^{-2}$, while using the transformation method, the error norms are less than 7.0% and $(\varepsilon_{z_{\mathrm{pert}}})_{\mathrm{tf}} = 2.9392 \times 10^{-2}$. Table 3(b) illustrates the exact and approximate eigenvalues. Note again

Table 2

(a) The exact and the perturbed eigenvalues (rad/s) for the system of Fig. 1[a]

i	$(\lambda_{\text{exact}})_i$	$(\lambda_{\text{ls}})_i$	$(\lambda_{\text{tf}})_i$
1	$-3.5390e+00+1.3830e+01i$	$-3.5256e+00+1.3741e+01i$ (6.3142e − 03)	$-3.5256e+00+1.3721e+01i$ (7.6839e − 03)
2	$-2.5165e+01+3.5421e+01i$	$-2.5087e+01+3.5787e+01i$ (8.6301e − 03)	$-2.5087e+01+3.5557e+01i$ (3.6147e − 03)
3	$-3.6455e+01+4.6273e+01i$	$-3.6534e+01+4.5435e+01i$ (1.4292e − 02)	$-3.6534e+01+4.5431e+01i$ (1.4354e − 02)
4	$-5.7849e+01+4.4732e+01i$	$-5.7405e+01+4.5064e+01i$ (7.5948e − 03)	$-5.7405e+01+4.5051e+01i$ (7.4835e − 03)
5	$-7.2825e+01+3.9185e+01i$	$-7.3282e+01+4.1719e+01i$ (3.1139e − 02)	$-7.3282e+01+4.1575e+01i$ (2.9429e − 02)

(b) The exact and the approximate eigenvalues for the system parameters of (a)[b]

i	$(\lambda_{\text{exact}})_i$	$(\lambda'_{\text{ls}})_i$	$(\lambda'_{\text{tf}})_i$
1	$-3.5390e+00+1.3830e+01i$	$-3.5392e+00+1.3833e+01i$ (1.6840e − 04)	$-3.5392e+00+1.3831e+01i$ (3.9172e − 05)
2	$-2.5165e+01+3.5421e+01i$	$-2.5158e+01+3.5415e+01i$ (2.0334e − 04)	$-2.5165e+01+3.5422e+01i$ (3.7441e − 05)
3	$-3.6455e+01+4.6273e+01i$	$-3.6439e+01+4.6289e+01i$ (3.7814e − 04)	$-3.6455e+01+4.6282e+01i$ (1.5917e − 04)
4	$-5.7849e+01+4.4732e+01i$	$-5.7762e+01+4.4688e+01i$ (1.3401e − 03)	$-5.7848e+01+4.4706e+01i$ (3.4702e − 04)
5	$-7.2825e+01+3.9185e+01i$	$-7.3073e+01+3.9397e+01i$ (3.9421e − 03)	$-7.2977e+01+3.9329e+01i$ (2.5279e − 03)

[a]The perturbed eigenvalues of the third and fourth columns are given by Eqs. (34) and (44), respectively. The system parameters are $m_0 = 2\,\text{kg}$, $k_0 = 5000\,\text{N/m}$, $c_0 = 100\,\text{N s/m}$, $c_1 = 20\,\text{N s/m}$, $k_1 = 0$ and $\sigma = 1$.

[b]The approximate eigenvalues correspond to the diagonal elements of $[Z_{\text{pert}}]^{\text{T}}[B][Z_{\text{pert}}]$.

the improvement in accuracy for all of the approximate eigenvalues compared to the perturbed eigenvalues. The results of Table 3(b) imply that the perturbed modal matrices obtained by using the current methods track the exact modal matrix fairly accurately.

A least squares approach and a transformation method are developed that can be used to obtain the eigenvalues and eigenvectors of an arbitrarily damped nearly proportional or weakly non-proportional system. The proposed schemes require only the eigensolutions of the undamped configuration, and are computationally efficient because they only involve simple algebraic operations. Numerical experiments showed that both methods return perturbed eigensolutions that agree well with the exact even for non-proportionality indices as large as 15%, and that approximations to the exact eigenvalues can be consistently improved by using the diagonal elements of $[Z_{\text{pert}}]^{\text{T}}[B][Z_{\text{pert}}]$.

The proposed approaches can be exploited in other applications. Specifically, the effects of small changes made to a weakly non-proportionally damped system can be easily analyzed. Using the methods developed in this paper, one can first determine the approximate eigensolutions for

Table 3

(a) The exact and the perturbed eigenvalues (rad/s) for the system of Fig. 1[a]

i	$(\lambda_{\text{exact}})_i$	$(\lambda_{\text{ls}})_i$	$(\lambda_{\text{tf}})_i$
1	−4.5441e + 00 + 1.4050e + 01i	−4.5332e + 00 + 1.3997e + 01i	−4.5332e + 00 + 1.3931e + 01i
		(3.6372e − 03)	(8.0780e − 03)
2	−4.0170e + 01	−4.2231e + 01	−4.2231e + 01
		(5.1299e − 02)	(5.1306e − 02)
3	−3.4252e + 01 + 2.6805e + 01i	−3.4133e + 01 + 2.7590e + 01i	−3.4133e + 01 + 2.7012e + 01i
		(1.8263e − 02)	(5.4785e − 03)
4	−4.9372e + 01	−5.1207e + 01	−5.2813e + 01
		(3.7180e − 02)	(6.9697e − 02)
5	−5.2737e + 01 + 2.9870e + 01i	−5.2594e + 01 + 2.5989e + 01i	−5.2594e + 01 + 2.5986e + 01i
		(6.4082e − 02)	(6.4126e − 02)
6	−1.1286e + 02	−1.1164e + 02	−1.1004e + 02
		(1.0808e − 02)	(2.5033e − 02)
7	−1.7036e + 02	−1.6823e + 02	−1.6823e + 02
		(1.2512e − 02)	(1.2514e − 02)

(b) The exact and the approximate eigenvalues for the system parameters of (a)[b]

i	$(\lambda_{\text{exact}})_i$	$(\lambda'_{\text{ls}})_i$	$(\lambda'_{\text{tf}})_i$
1	−4.5441e + 00 + 1.4050e + 01i	−4.5430e + 00 + 1.4055e + 01i	−4.5442e + 00 + 1.4051e + 01i
		(3.4345e − 04)	(4.6416e − 05)
2	−4.0170e + 01	−4.1828e + 01	−4.0595e + 01
		(4.1271e − 02)	(1.0575e − 02)
3	−3.4252e + 01 + 2.6805e + 01i	−3.4235e + 01 + 2.6787e + 01i	−3.4251e + 01 + 2.6816e + 01i
		(5.6199e − 04)	(2.4876e − 04)
4	−4.9372e + 01	−4.9175e + 01	−4.8655e + 01
		(3.9901e − 03)	(1.4522e − 02)
5	−5.2737e + 01 + 2.9870e + 01i	−5.2643e + 01 + 3.0044e + 01i	−5.2554e + 01 + 3.0057e + 01i
		(3.2708e − 03)	(4.3086e − 03)
6	−1.1286e + 02	−1.1274e + 02	−1.1300e + 02
		(1.1061e − 03)	(1.2056e − 03)
7	−1.7036e + 02	−1.7019e + 02	−1.7033e + 02
		(1.0392e − 03)	(1.6371e − 04)

[a]The perturbed eigenvalues of the third and fourth columns are given by Eqs. (34) and (44), respectively. The system parameters are $m_0 = 2$ kg, $k_0 = 5000$ N/m, $c_0 = 100$ N s/m, $c_1 = 15$ N s/m, $k_1 = 550$ N/m and $\sigma = 1.45$.

[b]The approximate eigenvalues correspond to the diagonal elements of $[Z_{\text{pert}}]^{\text{T}}[B][Z_{\text{pert}}]$.

the arbitrarily damped but nearly proportional system. Then applying the classical first-order eigensolution perturbation techniques, one can find the changes in the eigendata due to small modifications that are subsequently introduced. The unperturbed system corresponds to the weakly non-proportionally damped structure, and the perturbation consists of the small changes that are introduced. In addition, because closed-form expressions for the perturbed eigenvalues and eigenvectors are derived, the current methods can be applied to study the sensitivities of the eigensolutions on the various system parameters.

4. Conclusions

In this paper, two distinct approaches are proposed that can be used to obtain the approximate eigensolutions for arbitrarily damped nearly proportional systems without resorting to state form. The approximate eigensolutions are obtained by means of a first-order perturbation analysis, where the unperturbed system consists of a damped configuration whose damping matrix can be diagonalized by the same transformation that uncouples the mass and stiffness matrices of the undamped system, and the perturbation consists of a first-order damping matrix given by the deviation of this diagonalizable damping matrix from the actual damping matrix. Both schemes require only the eigensolutions of the undamped system and are easy to implement. Interestingly, if the first-order perturbed modal matrix, $[Z_{\text{pert}}]$, is properly normalized such that the diagonal elements of $[Z_{\text{pert}}]^{\mathrm{T}}[A][Z_{\text{pert}}]$ are identically one, then the diagonal elements of $[Z_{\text{pert}}]^{\mathrm{T}}[B][Z_{\text{pert}}]$ can also be used to approximate the exact eigenvalues, and numerical case studies show that the resulting diagonal elements are consistently more accurate than the first-order perturbed eigenvalues. Various numerical experiments were performed, and excellent agreement with the exact eigensolutions was demonstrated.

References

[1] J.C. Chen, B.K. Wada, Matrix perturbation for structural dynamic analysis, *AIAA Journal* 15 (8) (1977) 1095–1100.
[2] W.B. Bickford, An improved computational technique for perturbations of the generalized symmetric linear algebraic eigenvalue problem, *International Journal for Numerical Methods in Engineering* 24 (3) (1987) 529–541.
[3] T. Inamura, Eigenvalue reanalysis by improved perturbation, *International Journal for Numerical Methods in Engineering* 26 (1) (1988) 167–181.
[4] M.S. Eldred, P.B. Lerner, W.J. Anderson, Higher order eigenpair perturbations, *AIAA Journal* 30 (7) (1992) 1870–1876.
[5] X. Liu, Perturbation method for reduction of eigenvalue analysis of structures with large stiffnesses and small masses, *International Journal for Numerical Methods in Engineering* 40 (11) (1997) 2019–2035.
[6] P.B. Nair, A.J. Keane, R.S. Langley, Improved first-order approximation of eigenvalues and eigenvectors, *AIAA Journal* 36 (9) (1998) 1721–1727.
[7] T.K. Hasselman, Modal coupling in lightly damped structures, *AIAA Journal* 14 (11) (1976) 1627–1628.
[8] L. Meirovitch, G. Ryland II, Response of slightly damped gyroscopic systems, *Journal of Sound and Vibration* 67 (1) (1979) 1–19.
[9] K.R. Chung, C.W. Lee, Dynamic reanalysis of weakly non-proportionally damped systems, *Journal of Sound and Vibration* 111 (1) (1986) 37–50.
[10] M.K. Kwak, Perturbation method for the eigenvalue problem of lightly damped systems, *Journal of Sound and Vibration* 160 (2) (1993) 351–357.
[11] A. Craggs, The free vibration of systems with proportional or light damping, *Journal of Sound and Vibration* 170 (1) (1994) 130–139.
[12] R. F. Boisvert, S. E. Howe, D. K. Kahaner, J. L. Springmann, *Guide to Available Mathematical Software*, Center for Computing and Applied Mathematics, National Institute of Standards and Technology, 1990.
[13] L. Meirovitch, *Computational Methods in Structural Dynamics*, Sijthoff & Noordhoff, The Netherlands, 1980.
[14] B.T. Smith, J.M. Boyle, J.J. Dongarra, B.S. Garbow, Y. Ikebe, V.C. Klema, C.B. Moler, *Matrix EigenSystem Routines–Eispack Guide*, Springer, Berlin, 1976.
[15] J.W.S. Rayleigh, *The Theory of Sound*, Dover Publications, New York, 1976.
[16] T.K. Caughey, Classical normal modes in damped linear dynamic systems, *Journal of Applied Mechanics* 27 (1960) 269–271.
[17] G. Prater Jr., R. Singh, Quantification of the extent of non-proportional viscous damping in discrete vibratory systems, *Journal of Sound and Vibration* 104 (1) (1986) 109–125.