Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Sound and Vibration 289 (2006) 171–191

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Moment Lyapunov exponents of a two-dimensional system under both harmonic and white noise parametric excitations

Wei-Chau Xie*

Solid Mechanics Division, Faculty of Engineering, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

Received 7 October 2003; received in revised form 18 January 2005; accepted 3 February 2005
Available online 23 March 2005

Abstract

The moment Lyapunov exponents and the Lyapunov exponents of a 2D system under both harmonic and white noise excitations are studied. The moment Lyapunov exponents and the Lyapunov exponents are important characteristics determining the moment and almost-sure stability of a stochastic dynamical system. The eigenvalue problem governing the moment Lyapunov exponent is established. A singular perturbation method is applied to solve the eigenvalue problem to obtain second-order, weak noise expansions of the moment Lyapunov exponents. The influence of the white noise excitation on the parametric resonance due to the harmonic excitation is investigated.

1. Introduction

The study of the dynamic behaviour of a mechanical or structural system under periodic parametric excitation is one of the oldest problems in the theory of vibration. There are numerous engineering systems that are subject to periodic parametric excitations. A 2D system under harmonic parametric excitation is the well-known Mathieu equation. However, in engineering applications, loadings imposed on the mechanical or structural systems are quite often random forces, such as those arising from earthquakes, wind and ocean waves, which can be described satisfactorily only in probabilistic terms. Under the action of such loadings, the parameters that characterize the motions of the systems fluctuate in a stochastic manner. Because of the

*Tel.: +1 519 888 4567x3988; fax: +1 519 888 6197.
E-mail address: xie@uwaterloo.ca.

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.02.001

coexistence of periodic loadings and random forces, the responses of the systems are governed by differential equations of motion with parameters or coefficients containing both periodic functions and stochastic processes. It is therefore of practical importance to investigate the dynamic behaviour, especially dynamic stability, of systems under parametric periodic and stochastic excitations.

The almost-sure stability of dynamical systems under combined harmonic and stochastic excitations was studied by Sri Namachchivaya [1] by evaluating the largest Lyapunov exponents. Baxendale [2] determined weak noise expansions of the moment Lyapunov exponents and Lyapunov exponents of a similar system using a different approach as presented in this paper. An example of such a system is the uncoupled flapping motion of rotor blades in forward flight under the effect of atmospheric turbulence.

The sample or almost-sure stability of the trivial solution of a stochastic dynamical system is determined by the Lyapunov exponent, which characterizes the average exponential rate of growth of the solutions of system for time parameter t large, defined as

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \|\mathbf{x}(t)\|, \tag{1}$$

where $\mathbf{x}(t)$ is the vector of states of the system and $\|\cdot\|$ denotes the Euclidean vector norm. Depending on the initial conditions $\mathbf{x}(0)$, there are n Lyapunov exponents for the system, where n is the dimension of the system. The trivial solution of the dynamical system is stable with probability one (w.p.1) if the top Lyapunov exponent is negative, whereas it is unstable w.p.1 if the top Lyapunov exponent is positive.

On the other hand, the stability of the pth moment, $E[\|\mathbf{x}(t)\|^p]$, of the trivial solution of the dynamical system is determined by the moment Lyapunov exponent defined as

$$\Lambda(p) = \lim_{t\to\infty} \frac{1}{t} \log E[\|\mathbf{x}(t)\|^p], \tag{2}$$

where $E[\cdot]$ denotes expected value. If $\Lambda(p)<0$, then $E[\|\mathbf{x}(t)\|^p] \to 0$ as $t \to \infty$. The pth moment Lyapunov exponent $\Lambda(p)$ is a convex analytic function in p with $\Lambda(0) = 0$ and $\Lambda'(0)$ is equal to the top Lyapunov exponent λ. The non-trivial zero δ of $\Lambda(p)$, i.e. $\Lambda(\delta) = 0$, is called the stability index.

However, suppose the top Lyapunov exponent λ is negative, implying that the dynamical system is stable w.p.1, the pth moment grows exponentially when $p>\delta$, where δ is the stability index, implying that the pth moment of the dynamical system is unstable when $p>\delta$. This can be explained by the theory of large deviation as follows. Although the solution of the system $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$ w.p.1 at an exponential rate λ, there is a small probability that $\|\mathbf{x}(t)\|$ is large, which makes the expected value $E[\|\mathbf{x}(t)\|^p]$ of this rare event large for large enough values of p, leading to pth moment instability.

To have a complete picture of the dynamic stability of a dynamical system, it is important to study both the sample and moment stability and to determine both the top Lyapunov exponent and the pth moment Lyapunov exponent. A systematic study of moment Lyapunov exponents is presented in Ref. [3] for linear Itô systems and Ref. [4] for linear stochastic systems under real noise excitations. A systematic presentation of the theory of random dynamical systems and a comprehensive list of references can be found in Arnold [5].

Although the moment Lyapunov exponents are important in the study of dynamic stability of stochastic systems, the actual evaluations of the moment Lyapunov exponents are very difficult. Using the analytic property of the moment Lyapunov exponents, Arnold et al. [6] obtained weak noise expansions of the moment Lyapunov exponents of a 2D system in terms of εp, where ε is a small parameter, under both white noise and real noise excitations. Khasminskii and Moshchuk [7] obtained an asymptotic expansion of the moment Lyapunov exponent of a 2D system under white noise parametric excitation in terms of the small fluctuation parameter ε, from which the stability index was obtained. Sri Namachchivaya and Vedula [8] obtained general asymptotic approximation for the moment Lyapunov exponent and the Lyapunov exponent for a 4D system with one critical mode and another asymptotically stable mode driven by a small intensity stochastic process. Sri Namachchivaya and Van Roessel [9] studied the moment Lyapunov exponents of two coupled oscillators driven by real noise. Xie [10] obtained weak noise expansions of the moment Lyapunov exponent, the Lyapunov exponent, and the stability index of a 2D system exhibiting pitch-fork bifurcation under real noise excitation in terms of the small fluctuation parameter. Xie [11] determined small noise expansions of the moment Lyapunov exponent of a 2D viscoelastic system under bounded noise excitation.

In this paper, a 2D system under both harmonic and white noise parametric excitations is studied. Weak noise expansions of the moment Lyapunov exponents and Lyapunov exponents are obtained. The effect of the white noise excitation on the primary and secondary parametric resonance due to the harmonic excitation is investigated.

2. Formulation

Consider the dynamic stability of the following 2D system under both harmonic and white noise excitations:

$$\frac{d^2 q(\tau)}{d\tau^2} + 2\varepsilon\beta \frac{dq(\tau)}{d\tau} + [\Omega_0^2 + \varepsilon\mu_0 \sin \hat{\nu}_0\tau + \varepsilon^{n/2}\sigma_0\xi(\tau)]q(\tau) = 0, \tag{3}$$

where $\xi(\tau)$ is a unit Gaussian white noise process in time τ, $\varepsilon > 0$ is a small parameter. It is well-known that, in the absence of the white noise excitation, i.e. $\sigma_0 = 0$, the primary and secondary parametric resonance occurs when $\hat{\nu}_0/(2\Omega_0)$ is in the vicinity of 1 and $\frac{1}{2}$, respectively. In Eq. (3), $n = 1$ and 2 corresponds to the primary and secondary parametric resonance in the absence of the white noise excitation.

The damping term in Eq. (3) can be removed by applying the transformation $q(\tau) = x(\tau)e^{-\varepsilon\beta\tau}$ to yield

$$\frac{d^2 x(\tau)}{d\tau^2} + [\Omega^2 + \varepsilon\mu_0 \sin \hat{\nu}_0\tau + \varepsilon^{n/2}\sigma_0\xi(\tau)]x(\tau) = 0, \tag{4}$$

where $\Omega^2 = \Omega_0^2 - \varepsilon^2\beta^2$. Eq. (4) can be further simplified by applying the time scaling $t = \Omega\tau$ to result in

$$\frac{d^2 x(t)}{dt^2} + [1 + \varepsilon\mu \sin \nu t + \varepsilon^{n/2}\sigma\zeta(t)]x(t) = 0, \tag{5}$$

where $\mu = \mu_0/\Omega$, $\nu = \hat{\nu}_0/\Omega$, $\sigma = \sigma_0/\Omega^{3/2}$, and $\zeta(t)$ is a unit Gaussian white noise process in time t. In the absence of the white noise excitation, the primary and secondary parametric resonance occurs when $\nu/2$ is in the vicinity of 1 and $\frac{1}{2}$, respectively.

The Lyapunov exponents and the moment Lyapunov exponents of systems (3)–(5) are related by

$$\begin{aligned} \lambda_{q(\tau)} &= -\varepsilon\beta + \lambda_{x(\tau)} = -\varepsilon\beta + \Omega\lambda_{x(t)}, \\ \Lambda_{q(\tau)}(p) &= -\varepsilon p\beta + \Lambda_{x(\tau)}(p) = -\varepsilon p\beta + \Omega\Lambda_{x(t)}(p). \end{aligned} \tag{6}$$

The eigenvalue problem satisfied by the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ of system (5) may be established following a procedure proposed by Baxendale [12].

Denoting $\theta = \nu t$, θ may be considered as a random process with generator $G = \nu\partial/\partial\theta$. Letting $x_1 = x$, $x_2 = \dot{x}$, Eq. (5) may be written in the form of state equations

$$\mathrm{d}\mathbf{x} = \mathbf{B}_0\mathbf{x}\,\mathrm{d}t + \mathbf{B}_1\mathbf{x}\,\mathrm{d}W, \tag{7}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} 0 & 1 \\ -(1+\varepsilon\mu\sin\theta) & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 0 \\ -\varepsilon^{n/2}\sigma & 0 \end{bmatrix},$$

and $W(t)$ is the standard Wiener process. Applying the Khasminskii transformation [13]:

$$s_1 = \frac{x_1}{a} = \cos\varphi, \quad s_2 = \frac{x_2}{a} = \sin\varphi, \quad \mathbf{s} = \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix}, \quad \hat{\mathbf{s}} = \begin{Bmatrix} \sin\varphi \\ -\cos\varphi \end{Bmatrix}, \tag{8}$$

where $a = \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ is the Euclidean norm of vector $\mathbf{x}$, the random process $\mathbf{x}$ is projected onto the unit sphere $\mathbf{s}$. From the general theory of moment Lyapunov exponents [12], it is well-known that the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ of system (5) is the principal simple eigenvalue of the infinitesimal differential operator $\mathscr{L}(p)$ given by

$$\mathscr{L}(p)T(\varphi,\theta) = \Lambda_{x(t)}(p)T(\varphi,\theta), \tag{9}$$

where $\mathscr{L}(p) = L + pX + pQ + \frac{1}{2}p^2R + G$.

To evaluate L, X, Q, and R, it is necessary to determine β_i, h_i, and q_i, for $i = 0$ and 1, as

$$\begin{aligned} &\beta_i = \hat{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_i\mathbf{s}, \quad \beta_0 = 1 + \varepsilon\mu\sin\theta\cos^2\varphi, \quad \beta_1 = \varepsilon^{n/2}\sigma\cos^2\varphi, \\ &h_i = -\beta_i\frac{\partial}{\partial\varphi}, \quad h_0 = -(1+\varepsilon\mu\sin\theta\cos^2\varphi)\frac{\partial}{\partial\varphi}, \quad h_1 = -\varepsilon^{n/2}\sigma\cos^2\varphi\frac{\partial}{\partial\varphi}, \\ &q_i = \mathbf{s}^{\mathrm{T}}\mathbf{B}_i\mathbf{s}, \quad q_0 = -\varepsilon\mu\sin\theta\cos\varphi\sin\varphi, \quad q_1 = -\varepsilon^{n/2}\sigma\cos\varphi\sin\varphi. \end{aligned}$$

Hence

$$\begin{aligned} L &= h_0 + \frac{1}{2}h_1^2 = -(1+\varepsilon\mu\sin\theta\cos^2\varphi)\frac{\partial}{\partial\varphi} - \varepsilon^n\sigma^2\cos^3\varphi\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{2}\varepsilon^n\sigma^2\cos^4\varphi\frac{\partial^2}{\partial\varphi^2}, \\ X &= q_1h_1 = \varepsilon^n\sigma^2\cos^3\varphi\sin\varphi\frac{\partial}{\partial\varphi}, \\ Q &= q_0 - q_1^2 + \tfrac{1}{2}\mathbf{s}^{\mathrm{T}}(\mathbf{B}_1 + \mathbf{B}_1^{\mathrm{T}})\mathbf{B}_1\mathbf{s} = -\varepsilon\mu\sin\theta\cos\varphi\sin\varphi - \varepsilon^n\sigma^2\cos^2\varphi\sin^2\varphi + \tfrac{1}{2}\varepsilon^n\sigma^2\cos^2\varphi, \\ R &= q_1^2 = \varepsilon^n\sigma^2\cos^2\varphi\sin^2\varphi. \end{aligned}$$

Hence, the infinitesimal differential operator $L(p)$ is

$$\mathscr{L}(p) = \frac{1}{2}\varepsilon^n\sigma^2\cos^4\varphi\frac{\partial^2}{\partial\varphi^2} + [-1 - \varepsilon\mu\sin\theta\cos^2\varphi + \varepsilon^n(p-1)\sigma^2\cos^3\varphi\sin\varphi]\frac{\partial}{\partial\varphi} + v\frac{\partial}{\partial\theta} + p\left\{-\varepsilon\mu\sin\theta\cos\varphi\sin\varphi + \frac{1}{2}\varepsilon^n\sigma^2\cos^2\varphi[(p-2)\sin^2\varphi + 1]\right\}. \tag{10}$$

The eigenvalue problem (9) for the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ can also be derived using a more straightforward approach, which was first employed by Wedig [14] to derive the eigenvalue problem for the moment Lyapunov exponent of a 2D linear Itô stochastic system.

For system (5) or (7) in the form of state equations, apply the Khasminskii transformation (8) and define the pth norm $P = a^p$. The Itô equations for P and φ can be obtained using Itô's Lemma:

$$\begin{aligned} \mathrm{d}P &= pP\{-\varepsilon\mu\sin\theta\cos\varphi\sin\varphi + \tfrac{1}{2}\varepsilon^n\sigma^2\cos^2\varphi[(p-2)\sin^2\varphi + 1]\}\,\mathrm{d}t \\ &\quad + \varepsilon^{n/2}\sigma pP\cos\varphi\sin\varphi\,\mathrm{d}W, \\ \mathrm{d}\varphi &= (-1 - \varepsilon\mu\sin\theta\cos^2\varphi - \varepsilon^n\sigma^2\cos^3\varphi\sin\varphi)\,\mathrm{d}t + \varepsilon^{n/2}\sigma\cos^2\varphi\,\mathrm{d}W. \end{aligned} \tag{11}$$

Applying a linear transformation

$$S = T(\varphi,\theta)P, \quad P = T^{-1}(\varphi,\theta)S, \quad -\tfrac{1}{2}\pi \leqslant \varphi < \tfrac{1}{2}\pi, \quad 0 \leqslant \theta < 2\pi,$$

the Itô equation for the new pth norm process S can also be obtained from Itô's Lemma:

$$\begin{aligned} \mathrm{d}S = &\{\tfrac{1}{2}\varepsilon^n\sigma^2\cos^4\varphi T_{\varphi\varphi} + [-1 - \varepsilon\mu\sin\theta\cos^2\varphi + \varepsilon^n(p-1)\sigma^2\cos^3\varphi\sin\varphi]T_\varphi + vT_\theta \\ &+ p\{-\varepsilon\mu\sin\theta\cos\varphi\sin\varphi + \tfrac{1}{2}\varepsilon^n\sigma^2\cos^2\varphi[(p-2)\sin^2\varphi + 1]\}T\}P\,\mathrm{d}t \\ &+ \varepsilon^{n/2}\sigma\cos\varphi(p\sin\varphi T + \cos\varphi T_\varphi)P\,\mathrm{d}W. \end{aligned} \tag{12}$$

For bounded and non-singular transformation $T(\varphi,\theta)$, both processes P and S are expected to have the same stability behaviour. Therefore, $T(\varphi,\theta)$ is chosen so that the drift term of the Itô differential (12) is independent of the processes φ and θ, i.e.

$$\mathrm{d}S = \Lambda S\,\mathrm{d}t + \varepsilon^{n/2}\sigma\cos\varphi(p\sin\varphi + \cos\varphi T_\varphi T^{-1})S\,\mathrm{d}W. \tag{13}$$

Comparing Eqs. (12) and (13), it is seen that such a transformation $T(\varphi,\theta)$ satisfies the differential equation

$$\begin{aligned} &\tfrac{1}{2}\varepsilon^n\sigma^2\cos^4\varphi T_{\varphi\varphi} + [-1 - \varepsilon\mu\sin\theta\cos^2\varphi + \varepsilon^n(p-1)\sigma^2\cos^3\varphi\sin\varphi]T_\varphi + vT_\theta \\ &\quad + p\{-\varepsilon\mu\sin\theta\cos\varphi\sin\varphi + \tfrac{1}{2}\varepsilon^n\sigma^2\cos^2\varphi[(p-2)\sin^2\varphi + 1]\}T = \Lambda T. \end{aligned} \tag{14}$$

Eq. (14) is an eigenvalue problem with Λ being the eigenvalue and $T(\varphi,\theta)$ the associated eigenvector. From Eq. (13), it is seen that Λ is the Lyapunov exponent of the transformed pth moment process S, implying that $\Lambda = \Lambda_{x(t)}(p)$. Hence, the eigenvalue problem (9), with the infinitesimal differential operator given by Eq. (10), established using the general theory of moment Lyapunov exponent [12] is the same as Eq. (14) derived using a more straightforward approach originally employed by Wedig [14].

3. Moment Lyapunov exponents

In this section, the eigenvalue problem (14) is solved to obtain the moment Lyapunov exponent for system (5). Since the small parameter ε appears in the coefficient of the second-order partial derivative term $T_{\varphi\varphi}$, a method of singular perturbation is applied to Eq. (14) to obtain a small noise expansion of the moment Lyapunov exponent $\Lambda_{x(t)}(p)$.

Denote the frequency $v = v_0 + \varepsilon^n \Delta$, $n = 1, 2$, where $v_0 = 2/n$ is the harmonic excitation frequency corresponding to the primary and secondary parametric resonance in the absence of the white noise excitation, and Δ is the mistune parameter.

3.1. Primary parametric resonance, $n = 1$ and $v_0 = 2$

In the absence of the white noise excitation, primary parametric resonance occurs when $v_0 = 2$. Letting $v = 2 + \varepsilon\Delta$, applying the transformation $\varphi = z - \frac{1}{2}\theta$, $z = \varepsilon^{1/2}\psi$, Eq. (14) becomes

$$\left\{k_1(z,\theta)\frac{\partial^2}{\partial\psi^2} + 2\frac{\partial}{\partial\theta} + \varepsilon^{1/2}k_2(z,\theta)\frac{\partial}{\partial\psi} + \varepsilon\left[\Delta\frac{\partial}{\partial\theta} + k_3(z,\theta)\right]\right\}T(\psi,\theta) = \Lambda_{x(t)}(p)T(\psi,\theta), \tag{15}$$

where

$$\begin{aligned}
k_1(z,\theta) &= \tfrac{1}{2}\sigma^2\cos^4(z - \tfrac{1}{2}\theta),\\
k_2(z,\theta) &= \tfrac{1}{2}\Delta - \mu\sin\theta\cos^2(z - \tfrac{1}{2}\theta) + (p-1)\sigma^2\cos^3(z - \tfrac{1}{2}\theta)\sin(z - \tfrac{1}{2}\theta),\\
k_3(z,\theta) &= p\{-\tfrac{1}{2}\mu\sin\theta\sin(2z-\theta) + \tfrac{1}{2}\sigma^2\cos^2(z - \tfrac{1}{2}\theta)[(p-2)\sin^2(z - \tfrac{1}{2}\theta) + 1]\}.
\end{aligned}$$

Expand the eigenvalue $\Lambda_{x(t)}(p)$ and the eigenfunction $T(\psi,\theta)$ as

$$\Lambda_{x(t)}(p) = \sum_{i=0}^{\infty}\varepsilon^{i/2}\Lambda_i, \quad T(\psi,\theta) = \sum_{i=0}^{\infty}\varepsilon^{i/2}T_i(z,\psi,\theta). \tag{16}$$

Substituting Eqs. (16) into Eq. (15), noting that

$$\begin{aligned}
\frac{\partial T}{\partial\psi} &= \frac{\partial T_0}{\partial\psi} + \varepsilon^{1/2}\left(\frac{\partial T_0}{\partial z} + \frac{\partial T_1}{\partial\psi}\right) + \varepsilon\left(\frac{\partial T_1}{\partial z} + \frac{\partial T_2}{\partial\psi}\right) + \cdots,\\
\frac{\partial^2 T}{\partial\psi^2} &= \frac{\partial^2 T_0}{\partial\psi^2} + \varepsilon^{1/2}\left(2\frac{\partial^2 T_0}{\partial z\partial\psi} + \frac{\partial^2 T_1}{\partial\psi^2}\right) + \varepsilon\left(\frac{\partial^2 T_0}{\partial z^2} + 2\frac{\partial^2 T_1}{\partial z\partial\psi} + \frac{\partial^2 T_2}{\partial\psi^2}\right) + \cdots,
\end{aligned}$$

expanding, and equating terms of equal power of ε results in the perturbation equations,

$$\begin{aligned}
&O(1): && L_0T_0 = \Lambda_0T_0,\\
&O(\varepsilon^{1/2}): && L_0T_1 + L_1T_0 = \Lambda_1T_0 + \Lambda_0T_1,\\
&O(\varepsilon): && L_0T_2 + L_1T_1 + L_2T_0 = \Lambda_2T_0 + \Lambda_1T_1 + \Lambda_0T_2,\\
&O(\varepsilon^{3/2}): && L_0T_3 + L_1T_2 + L_2T_1 = \Lambda_3T_0 + \Lambda_2T_1 + \Lambda_1T_2 + \Lambda_0T_3,\\
&\vdots && \vdots
\end{aligned} \tag{17}$$

where

$$L_0 = k_1(z,\theta)\frac{\partial^2}{\partial\psi^2} + 2\frac{\partial}{\partial\theta}, \quad L_1 = k_2(z,\theta)\frac{\partial}{\partial\psi} + 2k_1(z,\theta)\frac{\partial^2}{\partial z\,\partial\psi},$$

$$L_2 = k_1(z,\theta)\frac{\partial^2}{\partial z^2} + k_2(z,\theta)\frac{\partial}{\partial z} + \Delta\frac{\partial}{\partial\theta} + k_3(z,\theta).$$

3.1.1. Zeroth-order perturbation

The zeroth-order perturbation equation is $L_0 T_0 = \Lambda_0 T_0$, or

$$k_1(z,\theta)\frac{\partial^2 T_0}{\partial\psi^2} + 2\frac{\partial T_0}{\partial\theta} = \Lambda_0 T_0. \tag{18}$$

From the property of moment Lyapunov exponent, it is well-known that $\Lambda_0(0) = 0$. Since Eq. (18) does not contain the parameter p explicitly, $\Lambda_0(0) = 0$ implies that $\Lambda_0(p) = 0$ for all values of p. A solution of Eq. (18) may be taken as $T_0(z,\psi,\theta) = Z_0(z)$, where $Z_0(z)$ is a periodic function of period π.

The adjoint equation of Eq. (18) is

$$k_1(z,\theta)\frac{\partial^2 T_0^*}{\partial\psi^2} - 2\frac{\partial T_0^*}{\partial\theta} = 0. \tag{19}$$

A solution of Eq. (19) may be taken as $T_0^*(z,\psi,\theta) = Z_0^*(z)\Psi_0^*(\psi)\Theta_0^*(\theta)$. Because the coefficients of Eq. (19) are periodic functions in θ of period 2π and in z of period of π, one may take $\Theta_0^*(\theta) = 1/2\pi$, $0 \leqslant \theta < 2\pi$. If $\Psi_0^*(\psi)$ is also taken as a constant, the solution of Eq. (19) may be written as $T_0^*(z,\psi,\theta) = Z_0^*(z)$, $0 \leqslant z < \pi$, where $Z_0^*(z)$ is a periodic function of period π.

3.1.2. First-order perturbation

The first-order perturbation equation is

$$L_0 T_1 = \Lambda_1 T_0 - L_1 T_0. \tag{20}$$

Since $T_0 = Z_0(z)$, it is obvious that $L_1 T_0 = 0$. The solvability condition of Eq. (20) is given by, from the Fredholm Alternative,

$$(\Lambda_1 T_0, T_0^*) = 0, \tag{21}$$

which leads to $\Lambda_1(p) = 0$, in which (f, g) denotes the inner product of functions $f(z,\psi,\theta)$ and $g(z,\psi,\theta)$ defined as

$$(f, g) = \int_{z=0}^{\pi}\int_{\psi\in M_\psi}\int_{\theta=0}^{2\pi} f(z,\psi,\theta)g(z,\psi,\theta)\,\mathrm{d}\theta\,\mathrm{d}\psi\,\mathrm{d}z,$$

where M_ψ is the range of variable ψ.

Eq. (20) then becomes $L_0 T_1 = 0$, whose solution may be taken as $T_1(z,\psi,\theta) = Z_1(z)$, where $Z_1(z)$ is a periodic function of period π.

3.1.3. Second-order perturbation

The second-order perturbation equation given in Eqs. (17) becomes

$$L_0 T_2 = \Lambda_2 T_0 - L_1 T_1 - L_2 T_0. \tag{22}$$

It is easy to show that $L_1 T_1 = 0$ and

$$\begin{aligned} L_2 T_0 &= k_1(z,\theta)\frac{\partial^2 T_0}{\partial z^2} + k_2(z,\theta)\frac{\partial T_0}{\partial z} + \Delta\frac{\partial T_0}{\partial \theta} + k_3(z,\theta)T_0 \\ &= k_1(z,\theta)\ddot{Z}_0(z) + k_2(z,\theta)\dot{Z}_0(z) + k_3(z,\theta)Z_0(z). \end{aligned}$$

The solvability condition of Eq. (22) is, from the Fredholm Alternative,

$$(\Lambda_2 T_0 - L_2 T_0, T_0^*) = 0,$$

i.e.

$$\int_{z=0}^{\pi}\int_{\theta=0}^{2\pi}\{k_1(z,\theta)\ddot{Z}_0(z) + k_2(z,\theta)\dot{Z}_0(z) + [-\Lambda_2 + k_3(z,\theta)]Z_0(z)\}Z_0^*(z)\,\mathrm{d}\theta\,\mathrm{d}z = 0. \tag{23}$$

Since Eq. (23) is valid for any periodic function $Z_0^*(z)$, one must have

$$\int_0^{2\pi}\{k_1(z,\theta)\ddot{Z}_0(z) + k_2(z,\theta)\dot{Z}_0(z) + [-\Lambda_2 + k_3(z,\theta)]Z_0(z)\}\,\mathrm{d}\theta = 0,$$

which yields, after integration,

$$3\sigma^2\ddot{Z}_0(z) + (8\Delta - 4\mu\sin 2z)\dot{Z}_0(z) + [-16\Lambda_2 + \sigma^2 p(p+2) + 4\mu p\cos 2z]Z_0(z) = 0. \tag{24}$$

Hence the second-order perturbation $\Lambda_2(p)$ of the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ is the eigenvalue of the eigenvalue problem (24) with a second-order ordinary differential operator.

3.1.4. Determination of Λ_2

The second-order perturbation of the moment Lyapunov exponent Λ_2 can be obtained by solving the eigenvalue problem (24) with a second-order ordinary differential operator.

Since the coefficients of Eq. (24) are periodic functions of z with period π, a series expansion of the function $Z(z)$ may be taken in the form

$$Z(z) = C_0 + \sum_{k=1}^{N}(C_{2k}\cos 2kz + S_{2k}\sin 2kz). \tag{25}$$

Substituting Eq. (25) into Eq. (24), multiplying the resulting equation by $\cos 2nz, \sin 2nz, n = 0, 1, 2, \ldots$, and integrating with respect to z from 0 to π leads to a system of $2N+1$ homogeneous linear equations for the unknown coefficients $C_0, C_2, S_2, \ldots, C_{2N}, S_{2N}$. The existence of a non-trivial solution requires that the determinant of the coefficient matrix $\Delta^{(N)}$ be equal to zero. The determinantal equation $\Delta^{(N)}$ results in a polynomial equation in $\Lambda_2^{(N)}$ of degree $2N+1$ of the form

$$[\Lambda_2^{(N)}]^{2N+1} + d_{2N}[\Lambda_2^{(N)}]^{2N} + \cdots + d_1[\Lambda_2^{(N)}] + d_0 = 0, \tag{26}$$

from which the eigenvalue $\Lambda_2^{(N)}$ can be obtained, where the superscript (N) denotes that N sinusoidal terms are taken in Eq. (25).

Note that when the number of terms N in Eq. (25) approaches infinity, the coefficient matrix $\Delta^{(N)}$ is of infinite dimension. The solution $\Lambda_2^{(N)}$ of Eq. (26) approaches the exact result Λ_2.

When $N=1$, $\Delta^{(1)}=0$ is a cubic equation and an analytical solution for $\Lambda_2^{(1)}$ may be obtained. In this case, the coefficients of $\Delta^{(1)}=0$ are

$$\begin{aligned}
d_2 &= -\tfrac{3}{16}\sigma^2(p+4)(p-2),\\
d_1 &= -\tfrac{1}{256}p(p+2)(-3\sigma^4p^2-6\sigma^4p+48\sigma^4+8\mu^2)+\Delta^2+\tfrac{9}{16}\sigma^4,\\
d_0 &= \tfrac{1}{4096}\sigma^2p(p+2)[-\sigma^4p^4-4\sigma^4p^3+4(2\mu^2+5\sigma^4)p^2+16(\mu^2+3\sigma^4)p\\
&\quad -16(9\sigma^4+6\mu^2+16\Delta^2)].
\end{aligned}$$

The solution of Eq. (26) with $N=1$ is given by

$$\Lambda_2^{(1)}=\frac{1}{6}(A_2-2d_2)-\frac{2}{3}\,\frac{3d_1-d_2^2}{A_2}, \tag{27}$$

where

$$\begin{aligned}
A_2 &= (-108d_0+36d_1d_2-8d_2^3+12A_1)^{1/3},\\
A_1 &= (81d_0^2-54d_0d_1d_2+12d_1^3+12d_0d_2^3-3d_1^2d_2^2)^{1/2}.
\end{aligned}$$

When $N>1$, no analytical solution exists for $\Lambda_2^{(N)}$ from the polynomial Eq. (26). A numerical approach has to be applied to obtain $\Lambda_2^{(N)}$. Typical results of $\Lambda_2^{(N)}$ are shown in Fig. 1. It is seen that, for $\mu=1$, $\Delta=1$, $\Lambda_2^{(1)}$ agrees with $\Lambda_2^{(7)}$ extremely well for all values of σ. For $\sigma=1$, $\Delta=1$, $\Lambda_2^{(1)}$ agrees with $\Lambda_2^{(7)}$ well for smaller values of μ (up to 2); when the value of μ is increased, some discrepancy exists between $\Lambda_2^{(1)}$ and $\Lambda_2^{(7)}$, especially for $-2<p<0$.

The second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(4)}$ is plotted in Fig. 2 for $\sigma=0.5$, 1.0, and 1.5 to illustrate the influence of the white noise excitation on the primary parametric resonance. For small values of σ, the impact of the white noise excitation is small, and the effect of the primary parametric resonance due to the harmonic excitation is very prominent. When the values of σ are increased, the influence of the white noise excitation is increased, and the effect of the primary parametric resonance due to the harmonic excitation is diminished.

Having obtained an approximate result $\Lambda_2^{(N)}$ for Λ_2, a second-order approximation of the moment Lyapunov exponent is given by $\Lambda_{x(t)}(p)\approx\varepsilon\Lambda_2^{(N)}+o(\varepsilon)$.

3.1.5. Determination of second-order perturbation of Lyapunov exponent λ_2

Using the property of the moment Lyapunov exponent,

$$\lambda_{x(t)}=\lim_{p\to 0}\frac{\Lambda_{x(t)}(p)}{p}, \tag{28}$$

the Lyapunov exponent can be obtained easily. A second-order approximation of the Lyapunov exponent is given by

$$\lambda_{x(t)}=\varepsilon\lambda_2+o(\varepsilon), \tag{29}$$

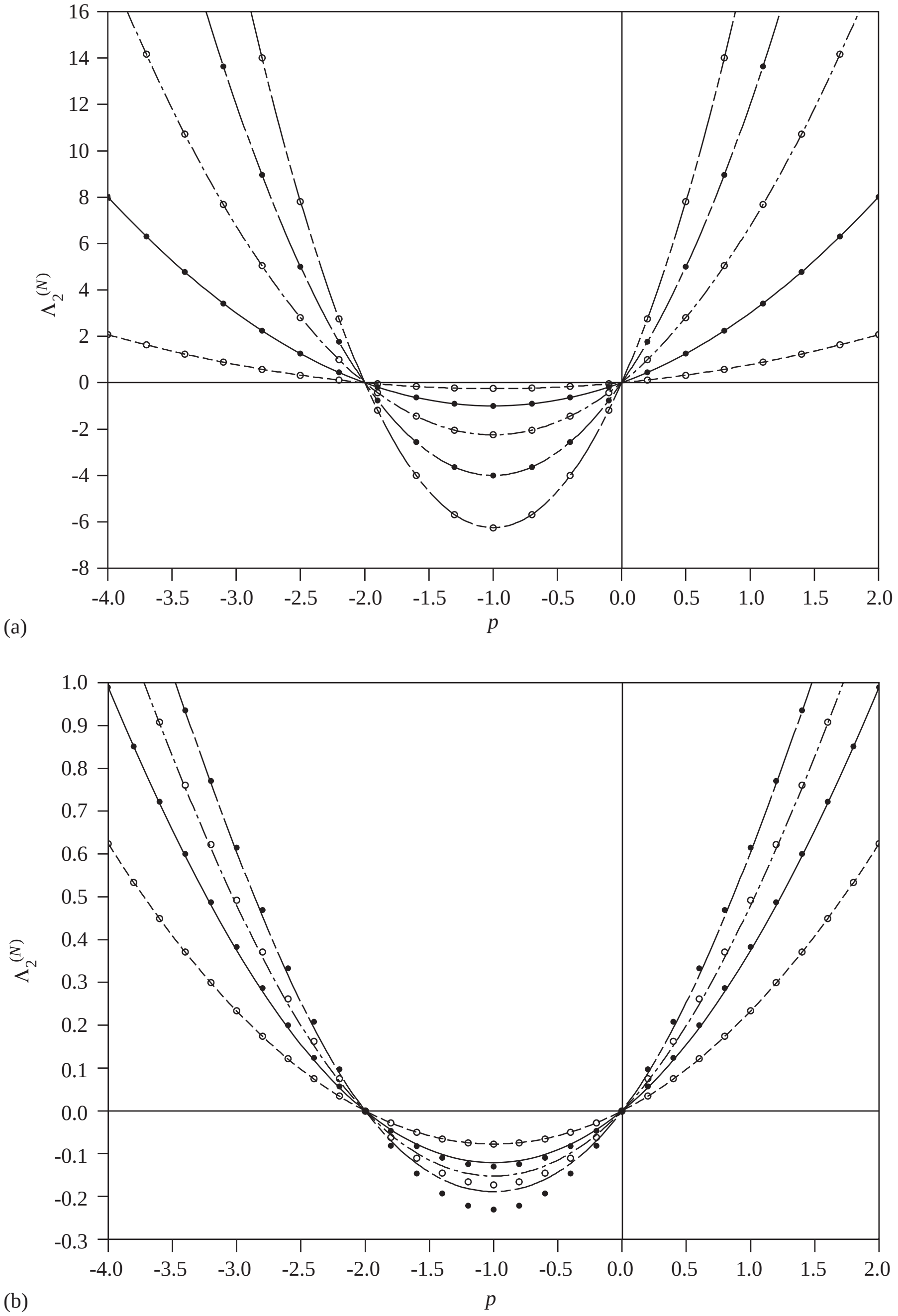

Fig. 1. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(N)}$, primary resonance, $\Delta = 1.0$. Lines, $N = 1$, analytical results; dots (• or ○), $N = 7$, numerical results: (a) $\mu = 1.0$; - - - , $\sigma = 2.0$; —, $\sigma = 4.0$; – – – –, $\sigma = 6.0$; - — -, $\sigma = 8.0$; - - — - - , $\sigma = 10.0$; (b) $\sigma = 1.0$; - - - , $\mu = 1.0$; —, $\mu = 2.0$; – – – –, $\mu = 2.5$; - — -, $\mu = 3.0$.

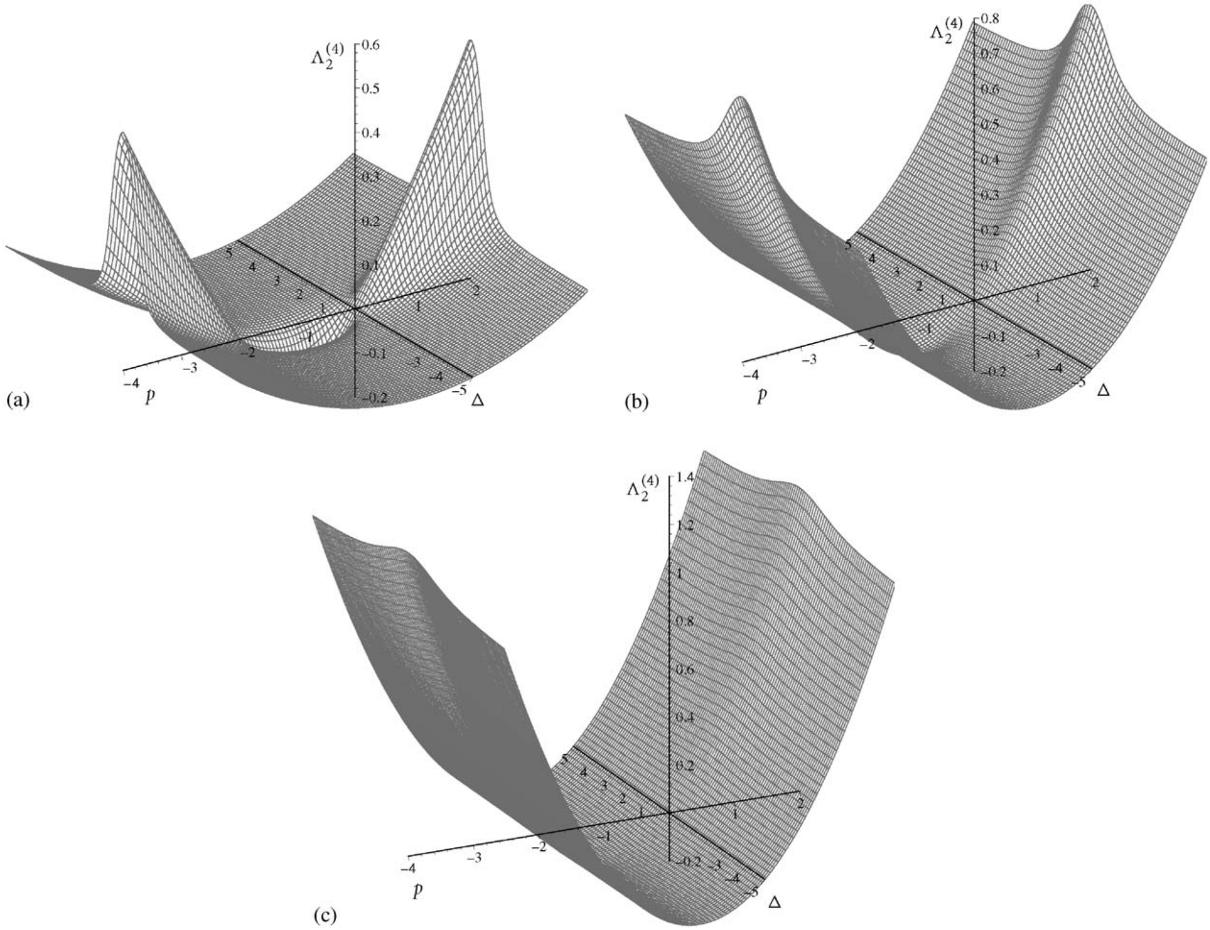

Fig. 2. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(4)}$, primary resonance, $\mu = 1.0$. (a) $\sigma = 0.5$; (b) $\sigma = 1.0$; (c) $\sigma = 1.5$.

where

$$\lambda_2 = \lim_{p \to 0} \frac{\Lambda_2}{p} \quad \text{or} \quad \lambda_2^{(N)} = \lim_{p \to 0} \frac{\Lambda_2^{(N)}}{p}.$$

Since $\Lambda_2 = O(p)$ when $p \to 0$, one has $\Lambda_2^k = o(p)$, $k = 2, 3, \ldots$, and $\lambda_2^{(N)}$ can be easily obtained from Eq. (26) as

$$\lambda_2^{(N)} = \lim_{p \to 0} \frac{d_0}{d_1}. \tag{30}$$

When $N = 4$, Eq. (30) leads to

$$\lambda_2^{(4)} = \frac{\sigma^2}{8} \frac{N^{(4)}}{D^{(4)}},$$

where

$$N^{(4)} = 3779136\sigma^{16} + (3149280\mu^2 + 9564480\Delta^2)\sigma^{12} + (310068\mu^4 + 2021760\mu^2\Delta^2 + 5660928\Delta^4)\sigma^8 + (7452\mu^6 + 46512\mu^4\Delta^2 + 276480\mu^2\Delta^4 + 1105920\Delta^6)\sigma^4 + (37\mu^8 + 192\mu^6\Delta^2 - 256\mu^4\Delta^4 + 65536\Delta^8),$$

$$D^{(4)} = 3779136\sigma^{16} + (629856\mu^2 + 9564480\Delta^2)\sigma^{12} + (30132\mu^4 + 124416\mu^2\Delta^2 + 5660928\Delta^4)\sigma^8 + (324\mu^6 + 6768\mu^4\Delta^2 - 124416\mu^2\Delta^4 + 1105920\Delta^6)\sigma^4 + (\mu^8 - 96\mu^6\Delta^2 + 2816\mu^4\Delta^4 - 24576\mu^2\Delta^6 + 65536\Delta^8).$$

The expressions of $\lambda_2^{(N)}$ for larger values of N can also be determined easily. However, because of the complexity of the expressions, the results are not presented here. Typical results of $\lambda_2^{(N)}$ as a function of the mistune parameter Δ are shown in Fig. 3 for different values of μ. The influence of the white noise excitation on the parametric excitation can also be easily seen.

To determine the accuracy of $\lambda_2^{(N)}$ for different values of N, the results are plotted in Fig. 4 for $N = 4$, 8, and 12. It is seen that $\lambda_2^{(4)}$ agrees with $\lambda_2^{(8)}$ and $\lambda_2^{(12)}$ extremely well for smaller values of μ (up to 5). When μ is larger, there are some discrepancies between $\lambda_2^{(4)}$ and $\lambda_2^{(12)}$, especially in the vicinity of $\Delta = 0$; whereas $\lambda_2^{(8)}$ agrees with $\lambda_2^{(12)}$ extremely well for all values of μ.

It is observed that

$$\lim_{\mu \to 0} \lambda_2^{(N)} = \lim_{\Delta \to \pm\infty} \lambda_2^{(N)} = \frac{\sigma^2}{8}.$$

When $\mu = 0$, the 2D system is excited by the white noise only; whereas when $\Delta \to \pm\infty$, $\mu \neq 0$, the frequency of the harmonic excitation is away from the region of parametric resonance and the influence of the harmonic excitation can be neglected. The second-order approximation $\lambda = \varepsilon\lambda_2^{(N)} = \varepsilon\sigma^2/8$ is the same as that of a 2D system under white noise excitation (see, e.g., Refs. [15,16]).

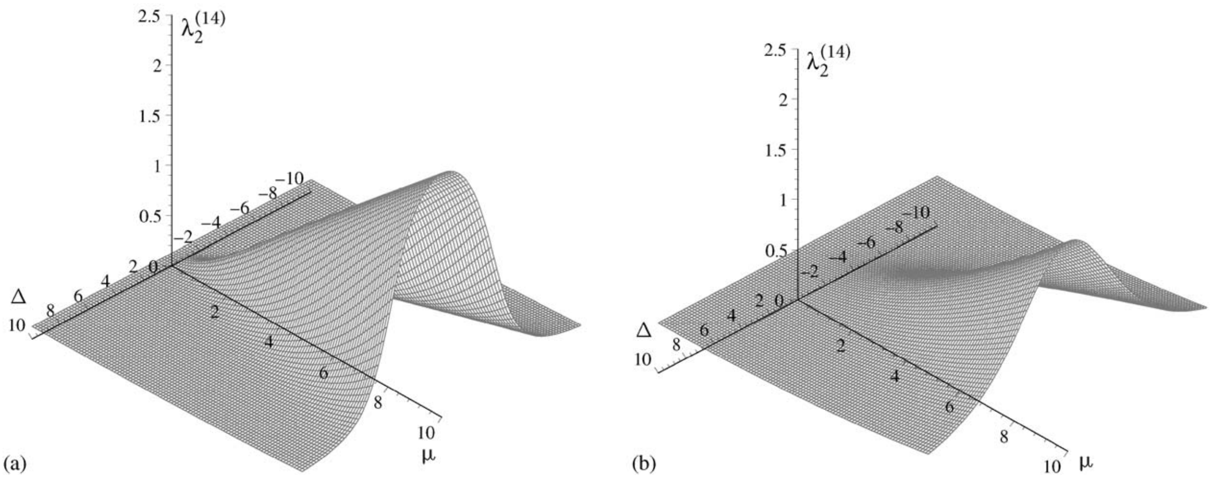

Fig. 3. Second-order perturbation of Lyapunov exponent $\lambda_2^{(14)}$, primary resonance. (a) $\sigma = 1.0$; (b) $\sigma = 2.0$.

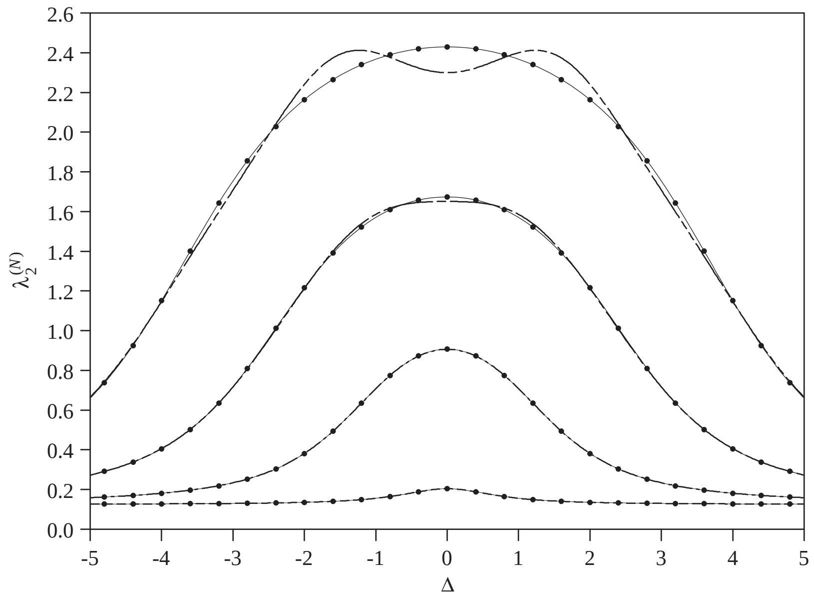

Fig. 4. Second-order perturbation of Lyapunov exponent $\lambda_2^{(N)}$, primary resonance. ●●●, $N=12$; —, $N=8$; - - -, $N=4$, $\mu=1.0$; - – – – -, $N=4$, $\mu=4.0$; - — -, $N=4$, $\mu=7.0$; - - — - -, $N=4$, $\mu=10.0$.

The validity of the second-order perturbation results (29) with λ_2 being approximated by $\lambda_2^{(N)}$ is checked by a digital simulation. Eq. (5) is solved numerically using the Euler scheme as

$$x_1(t+\Delta t) = x_1(t) + x_2(t)\cdot\Delta t,$$
$$x_2(t+\Delta t) = x_2(t) - (1+\varepsilon\mu\sin\nu t)x_1(t)\cdot\Delta t - \varepsilon^{1/2}\sigma x_1(t)\cdot\Delta W.$$

Note that the Milstein scheme for system (5) is the same as the Euler scheme. The numerical algorithm proposed by Wolf et al. [17] is applied to determine the Lyapunov exponent. Typical results are shown in Fig. 5. It is seen that the second-order approximation agrees well with that obtained from digital simulation.

3.2. Secondary parametric resonance, $n=2$ and $\nu_0=1$

In the absence of the white noise excitation, i.e. $\sigma=0$, the secondary parametric resonance occurs when $\nu_0=1$. Let $\nu=1+\varepsilon^2\Delta$, and apply the transformation $\varphi=\varepsilon\psi-\theta$ and $z=\varepsilon\psi$. Eq. (14) becomes

$$\left\{\left[K_1(z,\theta)\frac{\partial^2}{\partial\psi^2}+K_2(z,\theta)\frac{\partial}{\partial\psi}+\frac{\partial}{\partial\theta}\right]+\varepsilon\left\{[\Delta+K_3(z,\theta)]\frac{\partial}{\partial\psi}+K_4(z,\theta)\right\}\right.$$
$$\left.+\varepsilon^2\left[\Delta\frac{\partial}{\partial\theta}+K_5(z,\theta)\right]\right\}T(\psi,\theta)=\Lambda_{x(t)}(p)T(\psi,\theta), \quad (31)$$

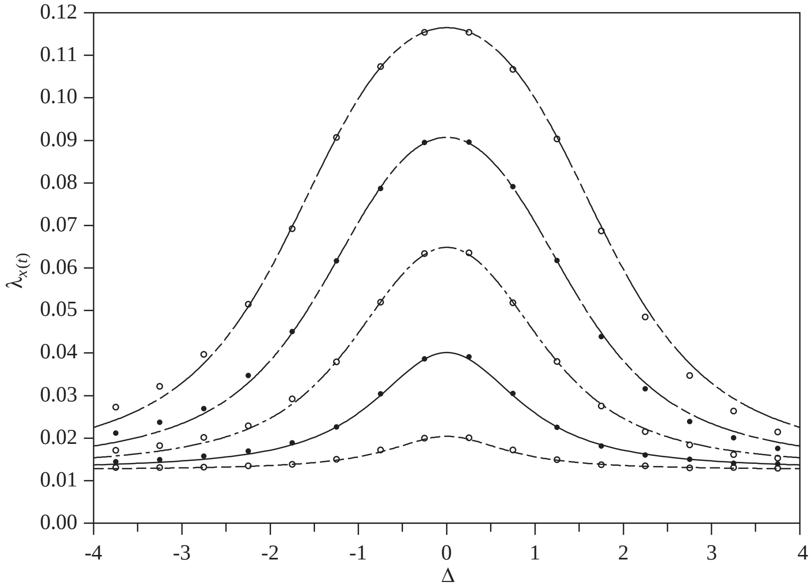

Fig. 5. Lyapunov exponent $\lambda_{x(t)}$, primary resonance. Dots (● or ○), simulation; lines, analytical results, $N = 12$; - - -, $\mu = 1.0$; —, $\mu = 2.0$; - – – – -, $\mu = 3.0$; - — -, $\mu = 4.0$; - - — - -, $\mu = 5.0$.

where

$$\begin{aligned}
K_1(z,\theta) &= \tfrac{1}{2}\sigma^2\cos^4(z-\theta),\\
K_2(z,\theta) &= -\mu\sin\theta\cos^2(z-\theta),\\
K_3(z,\theta) &= (p-1)\sigma^2\cos^3(z-\theta)\sin(z-\theta),\\
K_4(z,\theta) &= -p\mu\sin\theta\,\cos(z-\theta)\sin(z-\theta),\\
K_5(z,\theta) &= \tfrac{1}{2}p\sigma^2\cos^2(z-\theta)[(p-2)\sin^2(z-\theta)+1].
\end{aligned}$$

Expand the eigenvalue $\Lambda_{x(t)}(p)$ and the eigenfunction $T(\psi,\theta)$ as

$$\Lambda_{x(t)}(p) = \sum_{i=0}^{\infty}\varepsilon^i\Lambda_i, \quad T(\psi,\theta) = \sum_{i=0}^{\infty}\varepsilon^i T_i(z,\psi,\theta). \tag{32}$$

Substituting Eqs. (32) into (31), noting that

$$\begin{aligned}
\frac{\partial T}{\partial\psi} &= \frac{\partial T_0}{\partial\psi} + \varepsilon\left(\frac{\partial T_0}{\partial z} + \frac{\partial T_1}{\partial\psi}\right) + \varepsilon^2\left(\frac{\partial T_1}{\partial z} + \frac{\partial T_2}{\partial\psi}\right) + \dots,\\
\frac{\partial^2 T}{\partial\psi^2} &= \frac{\partial^2 T_0}{\partial\psi^2} + \varepsilon\left(2\frac{\partial^2 T_0}{\partial z\partial\psi} + \frac{\partial^2 T_1}{\partial\psi^2}\right) + \varepsilon^2\left(\frac{\partial^2 T_0}{\partial z^2} + 2\frac{\partial^2 T_1}{\partial z\partial\psi} + \frac{\partial^2 T_2}{\partial\psi^2}\right) + \dots,
\end{aligned}$$

expanding, and equating terms of equal power of ε leads to the perturbation equations:

$$\begin{aligned} &O(1): && L_0T_0 = \Lambda_0T_0,\\ &O(\varepsilon): && L_0T_1 + L_1T_0 = \Lambda_1T_0 + \Lambda_0T_1,\\ &O(\varepsilon^2): && L_0T_2 + L_1T_1 + L_2T_0 = \Lambda_2T_0 + \Lambda_1T_1 + \Lambda_0T_2,\\ &\vdots && \vdots \end{aligned} \tag{33}$$

where

$$L_0 = K_1(z,\theta)\frac{\partial^2}{\partial\psi^2} + K_2(z,\theta)\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\theta},$$

$$L_1 = [\Delta + K_3(z,\theta)]\frac{\partial}{\partial\psi} + 2K_1(z,\theta)\frac{\partial^2}{\partial z\partial\psi} + K_2(z,\theta)\frac{\partial}{\partial z} + K_4(z,\theta),$$

$$L_2 = K_1(z,\theta)\frac{\partial^2}{\partial z^2} + [\Delta + K_3(z,\theta)]\frac{\partial}{\partial z} + \Delta\frac{\partial}{\partial\theta} + K_5(z,\theta).$$

3.2.1. Zeroth-order perturbation

The zeroth-order perturbation equation is $L_0T_0 = \Lambda_0T_0$, or

$$K_1(z,\theta)\frac{\partial^2T_0}{\partial\psi^2} + K_2(z,\theta)\frac{\partial T_0}{\partial\psi} + \frac{\partial T_0}{\partial\theta} = \Lambda_0T_0. \tag{34}$$

Following the same procedure as Section 3.1.1, one has $\Lambda_0(p) = 0$ and $T_0(\psi,z,\theta) = Z_0(z)$, where $Z_0(z)$ is a periodic function of period π.

The adjoint equation of Eq. (34) is

$$K_1(z,\theta)\frac{\partial^2T_0^*}{\partial\psi^2} - K_2(z,\theta)\frac{\partial T_0^*}{\partial\psi} - \frac{\partial T_0^*}{\partial\theta} = 0. \tag{35}$$

A solution of Eq. (35) may be taken as $T_0^*(z,\psi,\theta) = Z_0^*(z)\Psi_0^*(\psi)\Theta_0^*(\theta)$, in which $\Psi_0^*(\psi)$ is a constant, $\Theta_0^*(\theta) = 1/2\pi$, $0\leqslant\theta<2\pi$. Hence $T_0^*(z,\psi,\theta) = Z_0^*(z)$, $0\leqslant z<\pi$, where $Z_0^*(z)$ is a periodic function of period π.

3.2.2. First-order perturbation

The first-order perturbation equation is $L_0T_1 = \Lambda_1T_0 - L_1T_0$. Since $T_0 = Z_0(z)$, one has $L_1T_0 = K_2(z,\theta)\dot{Z}_0(z) + K_4(z,\theta)Z_0(z)$. From the Fredholm Alternative, the solvability condition is $(\Lambda_1T_0 - L_1T_0, T_0^*) = 0$, which leads to $\Lambda_1(p) = (L_1T_0, T_0^*)/(T_0, T_0^*) = 0$.

The first-order perturbation equation becomes

$$K_1(z,\theta)\frac{\partial^2T_1}{\partial\psi^2} + K_2(z,\theta)\frac{\partial T_1}{\partial\psi} + \frac{\partial T_1}{\partial\theta} = K_2(z,\theta)\dot{Z}_0(z) + K_4(z,\theta)Z_0(z).$$

Seeking a solution of the form $T_1(z,\psi,\theta) = T_1(z,\theta)$ leads to

$$\begin{aligned} T_1(z,\theta) &= \dot{Z}_0(z)\int K_2(z,\theta)\,\mathrm{d}\theta + Z_0(z)\int K_4(z,\theta)\,\mathrm{d}\theta \\ &= \frac{1}{12}\mu\{[3\cos(2z-\theta)-\cos(2z-3\theta)-6\cos\theta]\dot{Z}_0(z) \\ &\quad + p[3\sin(2z-\theta)-\sin(2z-3\theta)]Z_0(z)\}. \end{aligned}$$

3.2.3. Second-order perturbation

The second-order perturbation equation is reduced to $L_0T_2 = \Lambda_2 T_0 - L_1T_1 - L_2T_0$. From the Fredholm Alternative, the solvability condition is $(\Lambda_2T_0 - L_1T_1 - L_2T_0, T_0^*) = 0$, or

$$\int_{z=0}^{\pi}\int_{\theta=0}^{2\pi}(\Lambda_0T_1 - L_1T_1 - L_2T_0)Z_0^*(z)\,\mathrm{d}\theta\,\mathrm{d}z = 0.$$

Since this condition is satisfied for any periodic function $Z_0^*(z)$, one must have

$$\int_0^{2\pi}(\Lambda_0T_1 - L_1T_1 - L_2T_0)\,\mathrm{d}\theta = 0,$$

which results in, after integration,

$$9\sigma^2\ddot{Z}_0(z) + (4\mu^2 + 48\Delta - 6\mu^2\cos 2z)\dot{Z}_0(z) - [48\Lambda_2 - 3\sigma^2p(p+2) + 6\mu^2p\sin 2z]Z_0(z) = 0. \quad (36)$$

The second-order perturbation Λ_2 of the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ is the eigenvalue of the eigenvalue problem (36).

3.2.4. Determination of Λ_2 and λ_2

The approach applied in Sections 3.1.4 and 3.1.5 can be employed to solve the eigenvalue problem (36) for Λ_2 and then to determine λ_2 using Eq. (28).

When $N = 1$, an analytical solution $\Lambda_2^{(1)}$ is given by Eq. (27) with

$$\begin{aligned} d_2 &= -\tfrac{3}{16}\sigma^2(p+4)(p-2), \\ d_1 &= \tfrac{1}{256}p(p+2)[3\sigma^4p(p+2) - 2\mu^4 - 48\sigma^4] + \tfrac{1}{36}\mu^4 + \tfrac{2}{3}\Delta\mu^2 + \tfrac{9}{16}\sigma^4 + 4\Delta^2, \\ d_0 &= -\tfrac{1}{36864}\sigma^2p(p+2)[9\sigma^4p^3(p+4) - 18(\mu^4+10\sigma^4)p^2 - 36(\mu^4+12\sigma^4)p \\ &\quad + 8(35\mu^4 + 192\Delta\mu^2 + 162\sigma^4 + 1152\Delta^2)]. \end{aligned}$$

Typical results of $\Lambda_2^{(1)}$ are plotted in Fig. 6 along with $\Lambda_2^{(7)}$, which is obtained by solving a polynomial equation of degree 15 in the form of Eq. (26). It can be observed that for $\mu = 1$, $\Delta = 1$, $\Lambda_2^{(1)}$ agrees with $\Lambda_2^{(7)}$ extremely well for all values of σ. However, for $\sigma = 1$, $\Delta = 1$, $\Lambda_2^{(1)}$ agrees with $\Lambda_2^{(7)}$ well for smaller values of μ (up to 2). For larger values of μ, discrepancies exist between $\Lambda_2^{(1)}$ and $\Lambda_2^{(7)}$, especially for $-2<p<0$.

Results of $\Lambda_2^{(4)}$ are plotted in Fig. 7 for $\sigma = 0.5$, 1.0, and 1.5 to illustrate the influence of the white noise excitation. For small values of σ, e.g. $\sigma = 0.5$, the influence of the white noise

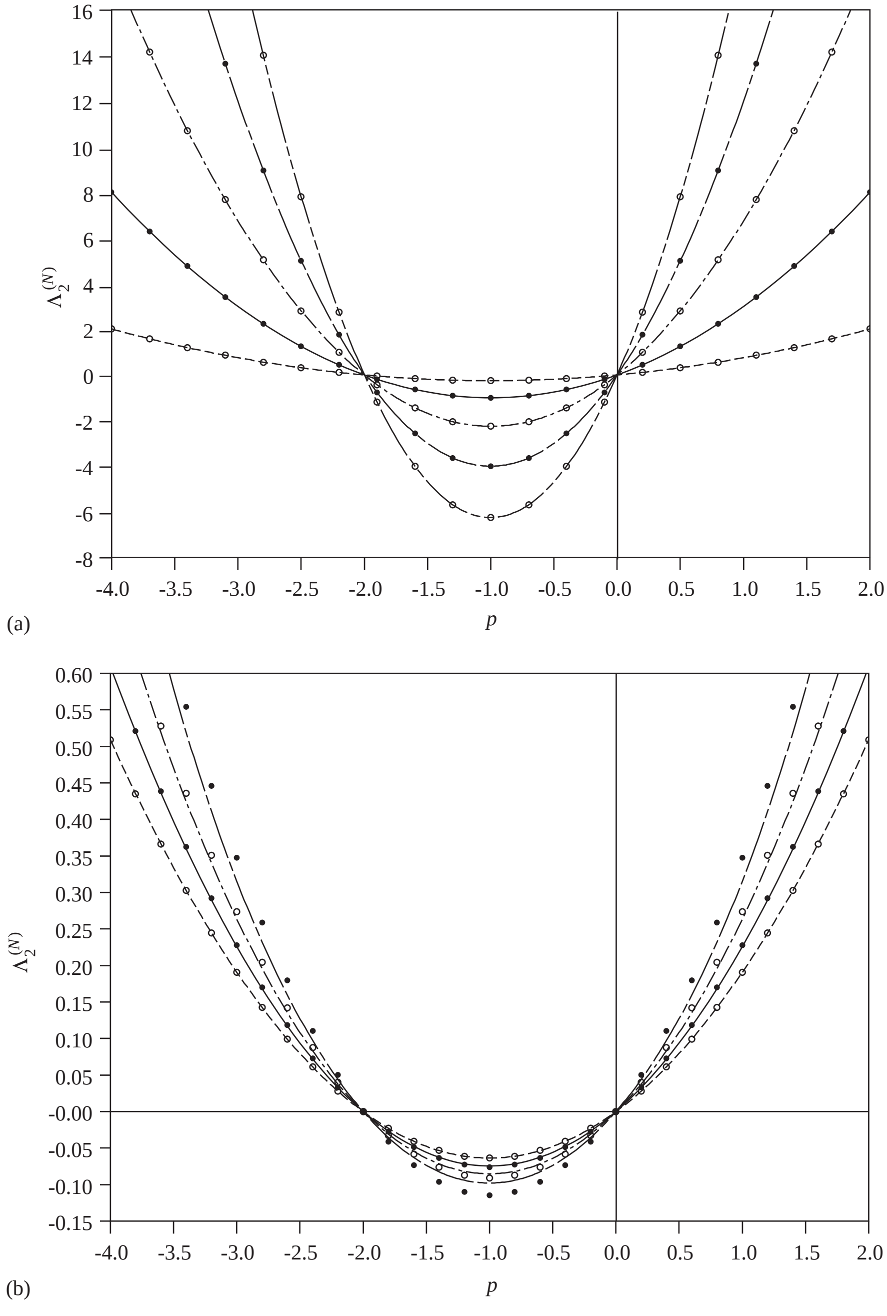

Fig. 6. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(N)}$, secondary resonance, $\Delta = 1.0$. Lines, $N = 1$, analytical results; dots (• or ○), $N = 7$, numerical results: (a) $\mu = 1.0$; - - -, $\sigma = 2.0$; —, $\sigma = 4.0$; – – – –, $\sigma = 6.0$; - — -, $\sigma = 8.0$; - - — - -, $\sigma = 10.0$; (b) $\sigma = 1.0$; - - -, $\mu = 1.0$; —, $\mu = 2.0$; – – – –, $\mu = 2.5$; - — -, $\mu = 3.0$.

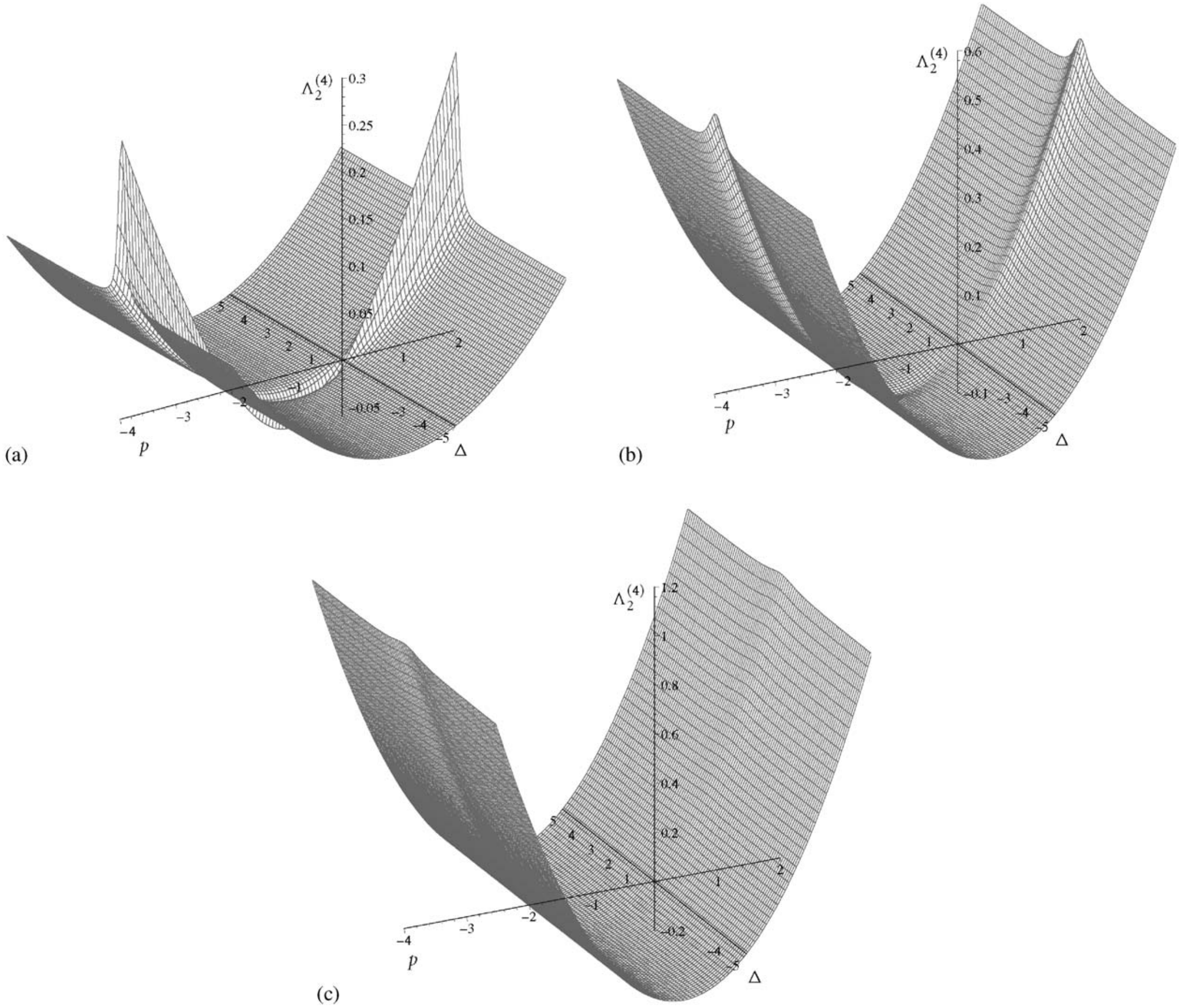

Fig. 7. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(4)}$, secondary resonance, $\mu = 1.0$. (a) $\sigma = 0.5$; (b) $\sigma = 1.0$; (c) $\sigma = 1.5$.

excitation is small and the parametric resonance due to the harmonic excitation is significant. When σ is increased, the impact of the white noise excitation is increased and the prominence of parametric resonance due to the harmonic excitation is reduced.

Having obtained an approximation of the second-order perturbation $\Lambda_2^{(N)}$, the moment Lyapunov exponent is given by $\Lambda_{x(t)}(p) \approx \varepsilon^2 \Lambda_2^{(N)} + o(\varepsilon^2)$.

Second-order perturbations of the Lyapunov exponents are plotted in Fig. 8. The impact of the white noise excitation on the parametric resonance can also be clearly seen. To study the accuracy of $\lambda_2^{(N)}$, the results are shown in Fig. 9 for different values of N. It is seen that $\lambda_2^{(4)}$ gives very good results up to $\mu = 3$; whereas $\lambda_2^{(8)}$ and $\lambda_2^{(12)}$ both give very good results up to $\mu = 6$. It is noted that even $\lambda_2^{(14)}$ does not yield satisfactory results for $\mu \geqslant 7$. The Lyapunov exponent is given by $\lambda_{x(t)} \approx \varepsilon^2 \lambda_2^{(N)} + o(\varepsilon^2)$.

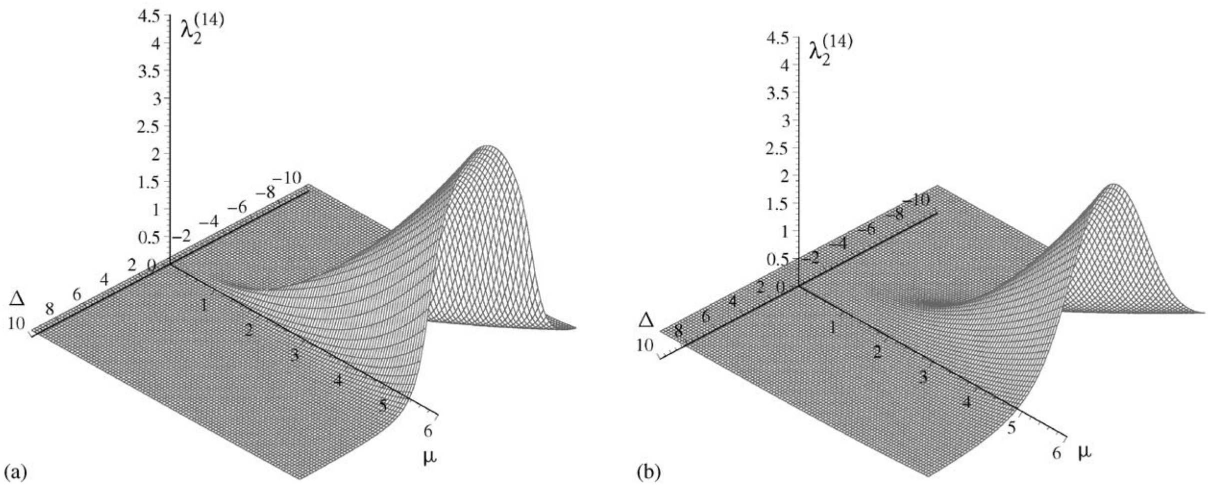

Fig. 8. Second-order perturbation of Lyapunov exponent $\lambda_2^{(14)}$, secondary resonance. (a) $\sigma = 1.0$; (b) $\sigma = 2.0$.

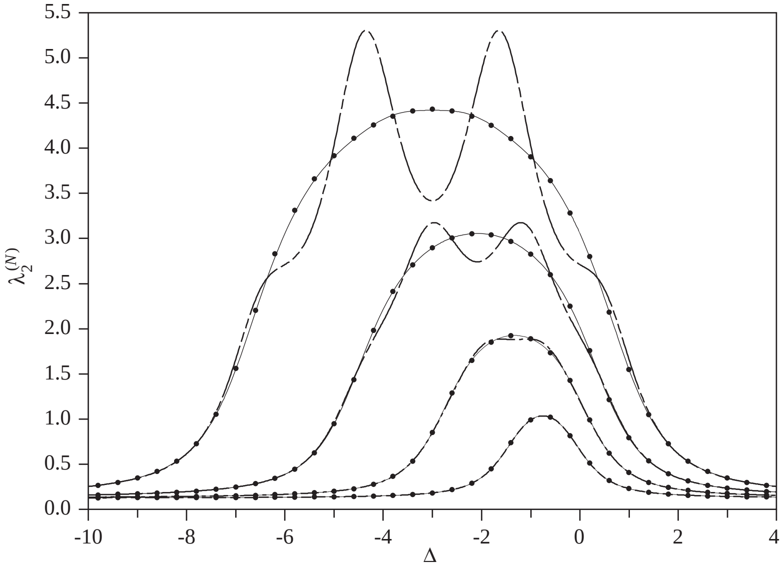

Fig. 9. Second-order perturbation of Lyapunov exponent $\lambda_2^{(N)}$, secondary resonance. ● ● ●, $N = 12$; —, $N = 8$; - - -, $N = 4$, $\mu = 1.0$; - – – – -, $N = 4$, $\mu = 4.0$; - — -, $N = 4$, $\mu = 7.0$; - - — - -, $N = 4$, $\mu = 10.0$.

For checking the correctness of the perturbation results, the approximate results of Lyapunov exponent $\lambda_{x(t)} \approx \varepsilon^2 \lambda_2^{(12)}$ are compared with values obtained using numerical simulation in Fig. 10. It can be seen that both results agree quite well, especially near the top of the hump.

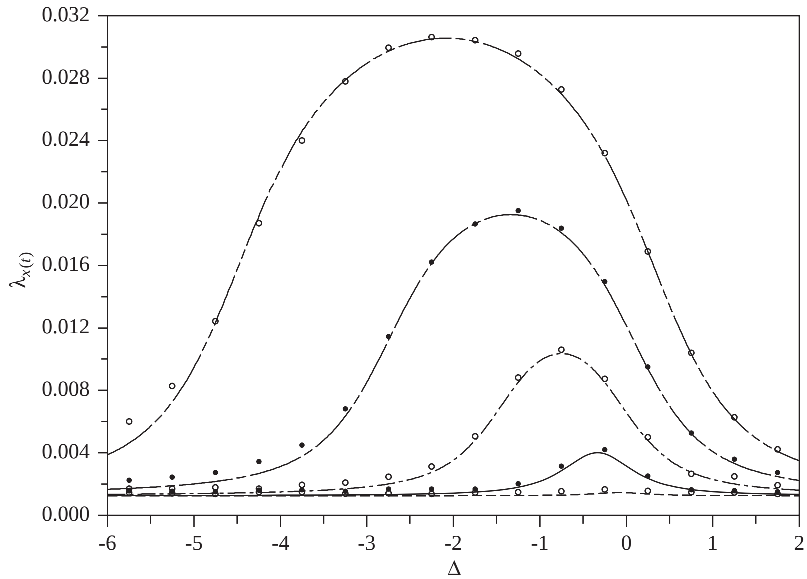

Fig. 10. Lyapunov exponent $\lambda_{x(t)}$, secondary resonance. Dots (• or ∘), simulation; lines, analytical results, $N = 12$; - - -, $\mu = 1.0$; —, $\mu = 2.0$; - — — -, $\mu = 3.0$; - — -, $\mu = 4.0$; - - — - -, $\mu = 5.0$.

It is noted that the Lyapunov exponent is almost symmetric about $\varDelta = 0$ in the vicinity of primary resonance $v_0 = 2$ (Figs. 4 and 5); whereas it is skewed towards $-\varDelta$ in the region of secondary parametric resonance $v_0 = 1$ (Figs. 9 and 10).

4. Conclusions

In this paper, the dynamic stability of a 2D system under both harmonic and white noise excitations is studied through the determination of the moment Lyapunov exponents and the Lyapunov exponents. An eigenvalue problem for the moment Lyapunov exponent is established using the theory of stochastic dynamical systems. A singular perturbation method is applied to obtain second-order small noise expansions of the moment Lyapunov exponent in both the primary and secondary parametric resonance regions due to the harmonic excitation. The Lyapunov exponents are determined using the relationship between moment Lyapunov exponents and Lyapunov exponents. The approximate analytical results compared well with those of numerical simulation. When the values of σ are small, the influence of the white noise excitation is small and the parametric resonance due to the harmonic excitation is prominent. When the values of σ are increased, the impact of the white noise excitation is intensified and the prominence of the parametric resonance is reduced.

When the amplitude of the harmonic excitation $\mu = 0 \to 0$ or when the influence of parametric resonance approaches zero with $\varDelta \to \pm\infty$, the resulting Lyapunov exponent approaches that of a 2D system under white noise excitation. However, the results for the Mathieu equation cannot be obtained by setting $\sigma = 0$, because the eigenvalue problem (14) implies that $\sigma \neq 0$.

Acknowledgements

The research for this paper was supported, in part, by the Natural Sciences and Engineering Research Council of Canada through Grant no. OGPO131355.

References

[1] N. Sri Namachchivaya, Almost sure stability of dynamical systems under combined harmonic and stochastic excitations, *Journal of Sound and Vibration* 151 (1) (1991) 77–90.
[2] P. Baxendale, Lyapunov exponents, and resonance for small periodic and random perturbations of a conservative system, *Stochastics and Dynamics* 2 (2002) 49–66.
[3] L. Arnold, E. Oeljeklaus, E. Pardoux, Almost sure and moment stability for linear Itô equations, in: L. Arnold, V. Wihstutz (Eds.), *Lyapunov Exponents*, Lecture Notes in Mathematics, vol. 1186, Springer, Berlin, 1986, pp. 85–125.
[4] L. Arnold, W. Kliemann, E. Oeljeklaus, Lyapunov exponents of linear stochastic systems, in: L. Arnold, V. Wihstutz (Eds.), *Lyapunov Exponents*, Lecture Notes in Mathematics, vol. 1186, Springer, Berlin, 1986, pp. 129–159.
[5] L. Arnold, *Random Dynamical Systems*, Springer, Berlin, 1998.
[6] L. Arnold, M.M. Doyle, N. Sri Namachchivaya, Small noise expansion of moment Lyapunov exponents for two-dimensional systems, *Dynamics and Stability of Systems* 12 (3) (1997) 187–211.
[7] R. Khasminskii, N. Moshchuk, Moment Lyapunov exponent and stability index for linear conservative system with small random perturbation, *SIAM Journal of Applied Mathematics* 58 (1) (1998) 245–256.
[8] N. Sri Namachchivaya, L. Vedula, Stabilization of linear systems by noise: application to flow induced oscillations, *Dynamics and Stability of Systems* 15 (2) (2000) 185–208.
[9] N. Sri Namachchivaya, H.J. Van Roessel, Moment Lyapunov exponent and stochastic stability of two coupled oscillators driven by real noise, *Journal of Applied Mechanics* 68 (6) (2001) 903–914.
[10] W.-C. Xie, Moment Lyapunov exponents of a two-dimensional system under real noise excitation, *Journal of Sound and Vibration* 239 (1) (2001) 139–155.
[11] W.-C. Xie, Moment Lyapunov exponents of a two-dimensional viscoelastic system under bounded noise excitation, *Journal of Applied Mechanics* 69 (3) (2002) 346–357.
[12] P.H. Baxendale, Moment stability and large deviations for linear stochastic differential equations, in: N. Ikeda (Ed.), *Proceedings of the Taniguchi Symposium on Probabilistic Methods in Mathematical Physics*, Kinokuniya, Tokyo, Japan, 1985, pp. 31–54.
[13] R.Z. Khasminskii, Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems, *Theory of Probability and its Applications* 12 (1967) 144–147 (English Translation).
[14] W. Wedig, Lyapunov exponent of stochastic systems and related bifurcation problems, in: S.T. Ariaratnam, G.I. Schuëller, I. Elishakoff (Eds.), *Stochastic Structural Dynamics—Progress in Theory and Applications*, Elsevier Applied Science, 1988, pp. 315–327.
[15] E. Pardoux, V. Wihstutz, Lyapunov exponent and rotation number of two-dimensional linear stochastic systems with small diffusion, *SIAM Journal of Applied Mathematics* 48 (2) (1988) 442–457.
[16] S.T. Ariaratnam, W.-C. Xie, Lyapunov exponents and stochastic bifurcations, in: W. Schiehlen (Ed.), *Proceedings of IUTAM Symposium on Nonlinear Dynamic Engineering Systems*, Stuttgart, Germany, August 21–25, Springer, Berlin, 1989, pp. 1–8.
[17] A. Wolf, J. Swift, H. Swinney, A. Vastano, Determining Lyapunov exponents from a time series, *Physica D* 16 (1985) 285–317.