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Journal of Sound and Vibration 289 (2006) 210–228
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Vertical accelerations of simple beams due to successive loads traveling at resonant speeds

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Received 1 November 2004; accepted 3 February 2005
Available online 17 May 2005

Abstract

In this paper, the vertical acceleration response of a simple beam traveled by a series of equally spaced moving loads at constant speeds is studied by the superposition method. From the closed-form solution derived, the key parameters dominating the resonant response of the beam are identified, along with the effect of higher modes of vibration on the acceleration response investigated. For the loads moving at resonant speeds, the higher modes can have significant influence on the acceleration amplitude. This is true especially for beams with light damping, for which the maximum acceleration on the beam depends on which vibration mode is excited. As such, the maximum acceleration of the beam need not occur at the mid-point. By considering the resonant speeds associated with the first and second modes, a simplified formula is proposed for checking whether the maximum acceleration may occur at the mid-point of the beam. For the case when the structural damping is taken into account, the contribution of higher modes to the acceleration response tends to be damped out. It is concluded that for a beam properly damped, the maximum acceleration response of the beam is dominated by the fundamental vibration mode.

1. Introduction

With the advancement of related technologies, the operation speed of modern trains has increased from time to time. This has stimulated a huge amount of researches, especially of the

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.02.037

theoretical nature, on the dynamic behaviors of railway bridges under the passage of high-speed trains [1–7]. Particularly, the phenomenon of vehicle-induced resonance has drawn attention from researchers, due to the fact that it may create drastic amplification on the dynamic response of the track structure and cause problems such as ballast destabilization and excessive vibrations of trains at high speeds [1,2,7]. For the purpose of track maintenance, it is recommended that the phenomenon of train-induced resonance be totally avoided from the beginning of design for a railway bridge. To this end, theories that can accurately predict the phenomenon of resonance for the problem of a beam subjected to a series of moving loads at high speeds should be developed.

In the book on vehicle–bridge interaction dynamics recently published by Yang et al. [8], a total of 167 references were cited. No attempt is made herein for a comprehensive review of all the related works. Specifically, this paper will be focused on the effect of higher modes of vibration on the resonant response of a simple beam to a series of moving loads.

In studying the fundamental problem of train-induced vibrations on bridges, a bridge was often modeled as a simply supported beam and a train as a series of moving loads with regular intervals. In the studies by Yang et al. [3,6], the mechanism of resonance for the train-induced vibration of railway bridges have been identified, both by theoretical and experimental means. These studies indicated that whenever the excitation frequency implied by the moving loads coincides with any of the bridge frequencies, the phenomenon of resonance will be developed on the bridge. By taking the effect of damping of the bridge into account, Li and Su [4] dealt with the fundamental characteristics and dominant factors for the resonant response of a girder bridge under high-speed trains. Furthermore, Yau et al. [5] proposed an envelope impact formula for the deflection of a damped beam with elastic bearings to a series of equally spaced moving loads. Savin [2] formulated an analytical expression of the dynamic amplification factor and response spectrum for the beams with various boundary conditions under the action of successive moving loads.

Previously, the resonant response associated with the fundamental bending mode was studied for the mid-point of a simple beam under a series of moving loads [3]. Recently, it was pointed out by Museros [7] by a numerical approach that the second bending mode may also have some influence on the resonant response of some high-speed railway bridges, and for some cases the maximum response may not occur at the mid-point. Unfortunately, no further details were proposed for determining whether the maximum acceleration will occur at the mid-point or not.

In the present study, a general solution for the vertical acceleration of a simple beam subjected to successive moving loads is derived by an analytical approach. It is indicated that for a beam with light damping, higher modes of vibration should be considered in computing the acceleration of the beam, and that the critical position for the maximum acceleration response to occur on the beam relates to the vibration mode that has been excited. In this regard, a simplified formula was derived for identifying whether the maximum acceleration will occur at the mid-point or not. In addition, the effect of damping that tends to reduce the contribution of higher modes to the acceleration of the beam will also be illustrated.

2. Formulation of the theory

With reference to Fig. 1, consider a simply supported beam of length L, subjected to a sequence of wheel loads p with constant interval d moving at speed v. Assume that there are K wheel loads

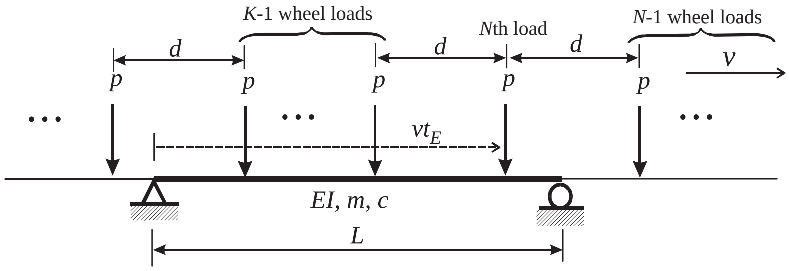

Fig. 1. Simply supported beam subjected to uniform moving loads.

acting simultaneously on the beam, and that the first $(N-1)$ wheel loads have left the beam. The equation of motion for the beam traversed by such moving loads is [5]

$$m\ddot{u} + c\dot{u} + EIu'''' = p \sum_{k=1}^{N+K-1} \delta[x - v(t - t_k)] \times [H(t - t_k) - H(t - t_k - L/v)], \tag{1}$$

where a prime denotes derivative with respect to the coordinate x, an overdot with respect to time t, m is the mass per unit length, $u(x,t)$ the vertical displacement, c the damping coefficient, E the elastic modulus, I the moment of inertia of the beam, δ the Dirac's delta function, $H(t)$ the unit step function, N the Nth moving load on the beam, and $t_k = (k-1)d/v$ the arriving time of the kth load at the beam. In Fig. 1, vt_E denotes the travel distance of the Nth load along the beam axis. Accordingly, the boundary conditions of the beam are

$$u(0,t) = u(L,t) = 0,$$

$$EIu''(0,t) = EIu''(L,t) = 0. \tag{2a,b}$$

Let us assume that the beam starts to vibrate from rest upon the arrival of the first moving load. The initial conditions are

$$u(x,0) = \dot{u}(x,0) = 0. \tag{3}$$

For a simply supported beam, the deflection $u(x,t)$ and acceleration $\ddot{u}(x,t)$ of the beam can be expressed in terms of the vibration mode shapes as follows:

$$u(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin\frac{n\pi x}{L}, \quad \ddot{u}(x,t) = \sum_{n=1}^{\infty} \ddot{q}_n(t) \sin\frac{n\pi x}{L}, \tag{4a,b}$$

where $q_n(t)$ denotes the generalized coordinate associated with the nth vibration shape of the beam. By substituting the displacement $u(x,t)$ of Eq. (4a) into Eq. (1), multiplying both sides of the equation by the shape function $\sin(n\pi x/L)$, and then integrating with respect to the beam axis x over the length L, one can formulate the nth generalized equation of motion of the beam as

$$\ddot{q}_n + 2\xi_n\omega_n\dot{q}_n + \omega_n^2 q_n = \frac{2p}{mL} \sum_{k=1}^{N+K-1} [f_n(t - t_k, v, L)],$$

$$f_n(t-t_k,v,L) = \sin\Omega_n(t-t_k)\times H(t-t_k) + (-1)^{n+1}\sin\Omega_n(t-t_k-L/v)\times H(t-t_k-L/v), \quad (5a, b)$$

where ξ_n is the modal damping ratio, ω_n is the nth vibration frequency of the beam,

$$\omega_n = \left(\frac{n\pi}{L}\right)^2\sqrt{\frac{EI}{m}}, \quad (6)$$

$\Omega_n(=n\pi v/L)$ is the driving frequency, and $f_n(t-t_k,v,t)$ is the generalized forcing function of the kth moving load acting on the beam.

3. Acceleration response of the beam

Before working on the solution to the generalized coordinate $q_n(t)$ in Eq. (5), let us consider the simplest case when the beam is subjected to the passage of a single moving load. For this case, the equation of motion in Eq. (5) reduces to

$$\ddot{q}_n + 2\xi_n\omega_n\dot{q}_n + \omega_n^2 q_n = \frac{2p}{mL}\sin\frac{n\pi vt}{L}, \quad 0\leqslant vt\leqslant L. \quad (7)$$

By Duhamel's integral, the response of the generalized coordinate $q_n(t)$ can be directly derived from Eq. (7) as [8, p. 76]

$$q_n(t) = \frac{2p/(mL\omega_n^2)}{(1-S_n^2)^2+(2\xi_n S_n)^2}\left\{(1-S_n^2)\sin\Omega_n t - 2\xi_n S_n\cos\omega_n t + S_n e^{-\xi_n\omega_n t}\left[2\xi_n\cos\omega_{dn}t - \left(\frac{1-S_n^2-2\xi_n^2}{\sqrt{1-\xi_n^2}}\right)\sin\omega_{dn}t\right]\right\}, \quad (8a)$$

where $S_n = \Omega_n/\omega_n$ = speed parameter, and $\omega_{dn} = \omega_n\sqrt{1-\xi_n^2}$ = damped frequency. The acceleration response can be obtained by differentiating the generalized coordinate q_n in Eq. (8a) twice with respect to time, that is,

$$\ddot{q}_n(t) = \frac{-2p/(mL)}{(1-S_n^2)^2+(2\xi_n S_n)^2}\left\{S_n^2[(1-S_n^2)\sin\Omega_n t - 2\xi_n S_n\cos\omega_n t] + S_n(1-2\xi_n^2)e^{-\xi_n\omega_n t}\left[2\xi_n\cos\omega_{dn}t - \frac{1-S_n^2-2\xi_n^2}{\sqrt{1-\xi_n^2}}\sin\omega_{dn}t\right] - 2S_n\xi_n\sqrt{1-\xi_n^2}e^{-\xi_n\omega_n t}\left[2\xi_n\sin\omega_{dn}t + \frac{1-S_n^2-2\xi_n^2}{\sqrt{1-\xi_n^2}}\cos\omega_{dn}t\right]\right\}. \quad (8b)$$

In this study, only the beams with light damping ($\xi_n < 0.03$) are considered, which implies that the terms involving ξ_n^2, S_n^3 and $\xi_n S_n$ in Eqs. (8) can be neglected and that the damped frequency ω_{dn} is approximately equal to the natural frequency ω_n [5]. As a result, the dynamic responses in Eqs. (8a) and (8b) can be approximated as

$$q_n(t) \approx \frac{1}{n^4} \times \frac{2pL^3}{\pi^4 EI} \frac{\sin \Omega_n t - S_n e^{-\xi_n \omega_n t} \sin \omega_n t}{1 - S_n^2},$$

$$\begin{aligned} \ddot{q}_n(t) &\approx \frac{-2p}{mL} \frac{S_n^2 \sin \Omega_n t - S_n e^{-\xi_n \omega_n t} \sin \omega_n t}{1 - S_n^2} \\ &= \frac{1}{n} \times \frac{2pv}{\pi \sqrt{mEI}} \frac{-S_n \sin \Omega_n t + e^{-\xi_n \omega_n t} \sin \omega_n t}{1 - S_n^2}. \end{aligned} \quad (9a, b)$$

By comparing Eq. (9a) with Eq. (9b), one observes that the contribution of higher modes should be considered in computing the acceleration of the beam, because the acceleration amplitude decreases less rapidly than the displacement amplitude as the mode number n increases, due to the difference in the power of n in the two equations.

Next, consider the general case when a series of equidistant loads move over the beam at a constant speed v, and assume that there are K wheel loads simultaneously acting on the beam, as shown in Fig. 1. When the Nth (leading) load and its following $(K - 1)$ loads are traversing the beam, the generalized acceleration response $\ddot{q}_n(t)$ induced on the beam can be obtained from Eq. (9b) by the method of superposition as follows:

$$\ddot{q}_n(t) \simeq \frac{2p}{mL} \frac{Q_n(v, t)}{1 - S_n^2}, \quad (10a)$$

where

$$\begin{aligned} Q_n(v, t) = &-\left(\sum_{k=N}^{N+K-1} [A_n(v, t - t_k) H(t - t_k)] \right) - \left(\sum_{k=1}^{N-1} [A_n(v, t - t_k) H(t - t_k) \right. \\ &\left. + (-1)^{n+1} A_n(v, t - t_k - L/v) H(t - t_k - L/v)] \right), \end{aligned}$$

$$A_n(v, t) = [S_n^2 \sin \Omega_n t - S_n e^{-\xi_n \omega_n t} \sin \omega_n t], \quad (10b,c)$$

in which the unit step function $H(t - t_k)$ is used to signify the arrival of the kth moving load on the beam, and the function $H(t - t_k - L/v)$ the departure of the kth moving load from the beam. Here, the first term with the summation from $k = N$ to $N + K - 1$ represents the effect of the K moving loads simultaneously acting on the beam, and the second term with the summation from $k = 1$ to $N - 1$ the effect of the first $(N - 1)$ moving loads that have passed the beam. Substituting Eq. (10) into Eq. (4b) yields the acceleration of the beam as follows:

$$\ddot{u}(x, t) = \frac{2p}{mL} \sum_{n=1} \frac{Q_n(v, t)}{1 - S_n^2} \sin \frac{n\pi x}{L}. \quad (11)$$

This equation represents exactly the general solution for the acceleration of a simple beam due to a series of equally spaced moving loads. To identify the resonant condition for the beam, we need to rearrange the acceleration response in Eq. (11) through some mathematical manipulations. In the following section, the resonant response of the beam will be derived.

4. Phenomenon of resonance

In Refs. [3,5], the authors have derived the resonant conditions for the deflection response of simple beams subjected to a series of moving loads. To analytically derive the dominant condition of resonance from the acceleration response in Eq. (10), let us first neglect the effect of damping of the beam. For this case, the acceleration factor $A_n(v,t)$ in Eq. (10c) reduces to

$$A_n(v,t) = [S_n^2 \sin \Omega_n t - S_n \sin \omega_n t]. \tag{12}$$

As depicted in Fig. 1, when the Nth load and its following $(K-1)$ loads are simultaneously moving over the beam, i.e., when $t_N < t < t_{N+K-1} + L/v$, by substituting Eq. (12) into Eq. (10a), the generalized acceleration $\ddot{q}_n(t)$ can be expressed as follows:

$$\begin{aligned}\ddot{q}_n(t) = \frac{2p}{mL}\frac{S_n}{1-S_n^2}\Bigg\{&\left(\sum_{k=N}^{N+K-1}[-S_n \sin \Omega_n(t-t_k) + \sin \omega_n(t-t_k)]H(t-t_k)\right)\\ &+\left(\sum_{k=1}^{N-1}[(-S_n \sin \Omega_n(t-t_k) + \sin \omega_n(t-t_k))H(t-t_k) + (-1)^{n+1}\right.\\ &\left.\times(-S_n \sin \Omega_n(t-t_k-L/v) + \sin \omega_n(t-t_k-L/v))H(t-t_k-L/v)]\right)\Bigg\}. \end{aligned} \tag{13}$$

As was stated previously, the first term with the summation from $k=N$ to $N+K-1$ represents the effect of the K moving loads simultaneously acting on the beam, and the second term with the summation from $k=1$ to $N-1$ the effect of the first $(N-1)$ moving loads that have passed the beam. By using the following formulas for trigonometric functions:

$$\sin \Omega_n(t-t_k) + (-1)^{n+1} \sin \Omega_n(t-t_k-L/v) = 0,$$

$$\sin \omega_n(t-t_k) + (-1)^{n+1}\sin \omega_n\left(t-t_k-\frac{L}{v}\right) = \begin{cases} 2\cos\left(\dfrac{\omega_n L}{2v}\right)\sin\left[\omega_n\left(t-t_k-\dfrac{L}{2v}\right)\right], & n = \text{odd},\\ 2\sin\left(\dfrac{\omega_n L}{2v}\right)\cos\left[\omega_n\left(t-t_k-\dfrac{L}{2v}\right)\right], & n = \text{even} \end{cases} \tag{14a,b}$$

and

$$\sum_{k=1}^{N-1}\sin \omega_n\left(t-t_k-\frac{L}{2v}\right) = \sin \omega_n\left(t-\frac{L}{2v}\right) + \sin \omega_n\left(t-\frac{t_N+L/v}{2}\right)\frac{\sin\left((N-2)\frac{\omega_n d}{2v}\right)}{\sin(\omega_n d/2v)},$$

$$\sum_{k=1}^{N-1}\cos\omega_n\left(t-t_k-\frac{L}{2v}\right)=\sum_{k=1}^{N-1}\sin\left[\omega_n\left(t-t_k-\frac{L}{2v}\right)+\frac{\pi}{2}\right]$$
$$=\cos\omega_n\left(t-\frac{L}{2v}\right)+\cos\omega_n\left(t-\frac{t_N+L/v}{2}\right)\frac{\sin\left((N-2)\frac{\omega_n d}{2v}\right)}{\sin(\omega_n d/2v)},\quad (14c,d)$$

the generalized acceleration $\ddot{q}_n(t)$ in Eq. (13) can be manipulated into a compact form as

$$\ddot{q}_n(t)=\frac{2p}{mL}\frac{S_n}{1-S_n^2}\left\{\left(\sum_{k=N}^{N+K-1}[-S_n\sin\Omega_n(t-t_k)+\sin\omega_n(t-t_k)]H(t-t_k)\right)\right.$$
$$\left.+2D_n(v,t)\times H(t-t_{N-1}-L/v)\right\},\quad (15a)$$

where

$$D_n(v,t)=\begin{cases}\cos\dfrac{\omega_n L}{2v}\left[\sin\omega_n\left(t-\dfrac{L}{2v}\right)+\sin\omega_n\left(t-\dfrac{L}{2v}-\dfrac{t_N}{2}\right)\dfrac{\sin\left((N-2)\frac{\omega_n d}{2v}\right)}{\sin(\omega_n d/2v)}\right], & n=\text{odd},\\ \sin\dfrac{\omega_n L}{2v}\left[\cos\omega_n\left(t-\dfrac{L}{2v}\right)+\cos\omega_n\left(t-\dfrac{L}{2v}-\dfrac{t_N}{2}\right)\dfrac{\sin\left((N-2)\frac{\omega_n d}{2v}\right)}{\sin(\omega_n d/2v)}\right], & n=\text{even}.\end{cases}\quad (15b)$$

As can be seen from Eq. (15b), for any vibration modes considered, the expression of $\ddot{q}_n(t)$ will reduce to an indeterminate form 0/0 when the denominator $\sin(\omega_n d/2v)$ equals zero, i.e., when $\omega_n d/2v=j\pi,\ j=1,2,3\ldots$, where j represents the number of complete oscillation cycles for the nth mode of the beam to vibrate during the passage of two adjacent loads [7]. This is exactly the condition for the jth resonance of the nth mode to be excited. Let us denote the corresponding resonant speed as $v_{r,n}$, where the subscript r,n means the rth resonance of the nth vibration mode to be excited. As for investigation of the resonant phenomenon, a detailed study will be presented in Section 4.1.

Consequently, introducing Eq. (15) into Eq. (11) leads to the acceleration of the beam:

$$\ddot{u}(x,t)=\frac{2p}{mL}\left\{\sum_{n=1}\sin\frac{n\pi x}{L}\frac{S_n}{1-S_n^2}\left[\left(\sum_{k=N}^{N+K-1}[-S_n\sin\Omega_n(t-t_k)+\sin\omega_n(t-t_k)]H(t-t_k)\right)\right.\right.$$
$$\left.\left.+2D_n(v,t)\times H(t-t_{N-1}-L/v)\right]\right\}.\quad (16)$$

This is a general expression for the acceleration of the beam induced by successively moving loads. It is featured by the fact that the contribution of higher modes of vibration of the beam is duly taken into account, which may be important in computation of the maximum acceleration of the beam.

4.1. The first resonance associated with the fundamental mode

Consider the most popular case when the first vibration mode of the beam is excited, i.e., when $\sin(\omega_1 d/2v) = 0$ or $\omega_1 d/2v = j\pi$. Here, the condition $j = 1$ means that the first characteristic frequency, i.e., v/d, of the successive moving loads coincides with the fundamental frequency $\omega_1/2\pi$ of the beam. Correspondingly, the first resonant speed is $v_{r,1} = \omega_1 d/2\pi$. As can be seen, the shorter the interval d of the moving loads, the smaller the resonant speed will be induced. By substituting $v_{r,1}$ into the denominator in Eq. (15b) and by L'Hospital's rule, the following identity can be established:

$$\sin\frac{\omega_n d}{2v_{r,1}} = 0 \quad \Rightarrow \quad \frac{\omega_n d}{2v_{r,1}} = n^2\pi,$$

$$\lim_{\sin(\frac{\omega_n d}{2v_{r,1}})\to 0} \frac{\sin[(N-2)\omega_n d/2v_{r,1}]}{\sin(\omega_n d/2v_{r,1})} = N-2. \tag{17a,b}$$

Thus, the first resonant acceleration of the first mode of the beam can be expressed as

$$\begin{aligned}\ddot{u}_{r,1}(x,t) = \frac{2p}{mL}\Bigg\{\sum_{n=1}\Bigg[\sin\left(\frac{n\pi x}{L}\right)\frac{S_{n,r1}}{1-S_{n,r1}^2} \\ \times\left(\sum_{k=N}^{N+K-1}\left[-S_{n,r1}\sin\Omega_n(t-t_k)+\sin\omega_n(t-t_k)\right]H(t-t_k)\right) \\ + 2D_{n,r1}(v_{r,1},t)\times H(t-t_{N-1}-L/v_{r,1})\Bigg]\Bigg\},\end{aligned} \tag{18a}$$

where

$$S_{n,r1} = \frac{n\pi v_{r,1}}{\omega_n L} = \frac{d}{2nL},$$

$$D_{n,r1}(v_{r,1},t) = \begin{cases} (N-1)\cos\dfrac{\omega_n L}{2v_{r,1}}\sin\omega_n\left(t-\dfrac{L}{2v_{r,1}}\right), & n = \text{odd},\\ (N-1)\sin\dfrac{\omega_n L}{2v_{r,1}}\cos\omega_n\left(t-\dfrac{L}{2v_{r,1}}\right), & n = \text{even}.\end{cases} \tag{18b,c}$$

The preceding equations indicate that once the first mode of the beam is excited, the other higher modes will be developed as well on the beam, as implied by $(N-1)$ in Eq. (18c). As indicated by Eq. (18a), higher mode shapes can influence the maximum acceleration of the beam through the function $\sin(n\pi x/L)$ with $n \geqslant 2$, resulting in some resonant peaks on the response along the beam axis. This explains the reason why the critical position of the maximum acceleration of an undamped beam subjected to successive moving loads may occur at position other than the mid-point. On the other hand, the contribution of higher modes to the acceleration of the beam may gradually decay because the resonant speed parameter $S_{n,r1} = d/2nL$ in the numerator becomes smaller for larger mode number.

4.2. The second resonance associated with the second mode

Next, let us consider a relatively rare case when the second mode of the beam is excited, or when $\omega_2 d/2v_{r,2} = j\pi$ as implied by the condition $\sin(\omega_2 d/2v) = 0$. For this case, one can find the following relations:

$$\frac{\omega_n d}{2v_{r,2}} = \frac{n^2\omega_2 d}{4 \times 2v_{r,2}} = \frac{n^2 j\pi}{4},$$

$$\left.\frac{\sin\left[(N-2)\frac{\omega_n d}{2v_{r,2}}\right]}{\sin\frac{\omega_n d}{2v_{r,2}}}\right|_{\frac{\omega_n d}{2v_{r,2}}=\frac{n^2 j\pi}{4}} = \frac{\sin[(N-2)jn^2\pi/4]}{\sin(jn^2\pi/4)} = \begin{cases} \dfrac{\sin[(N-2)j\pi/4]}{\sin(j\pi/4)}, & n = \text{odd}, \\ N-2, & n = \text{even}. \end{cases} \tag{19a,b}$$

For most commercial trains in operation, the first resonant speed of the second mode, as implied by $v_{r,2} = \omega_2 d/2j\pi$ with $j = 1$, can hardly be reached, as it is too high. Therefore, only the second resonant speed of the second mode, i.e., $v_{r,2} = \omega_2 d/2j\pi$ with $j = 2$, will be considered herein. Consequently, Eq. (19b) becomes

$$\left(\frac{\sin[(N-2)\omega_n d/2v_{r,2}]}{\sin\frac{\omega_n d}{2v_{r,2}}}\right)_{\frac{\omega_n d}{2v_{r,2}}=\frac{n^2 j\pi}{4},\, j=2} = \begin{cases} -\sin(N\pi/2), & n = \text{odd}, \\ N-2, & n = \text{even}, \end{cases} \tag{20a}$$

where

$$v_{r,2} = \frac{\omega_2 d}{4\pi}. \tag{20b}$$

Using Eqs. (20a) and (20b), one can establish the following relation:

$$\sin\omega_n\left(t - \frac{L}{2v_{r,2}}\right) + \sin\omega_n\left(t - \frac{L}{2v_{r,2}} - \frac{t_N}{2}\right)\frac{\sin((N-2)\omega_n d/2v_{r,2})}{\sin(\omega_n d/2v_{r,2})}$$
$$= \begin{cases} (1-\sin^2(N\pi/2))\sin\omega_n(t - L/2v_{r,2}), & n = \text{odd}, \\ (N-1)\sin\omega_n(t - L/2v_{r,2}), & n = \text{even}, \end{cases} \tag{20c}$$

where $1 - \sin^2(N\pi/2)$ is equal to $[1 + (-1)^N]/2$. Thus, the acceleration of the beam associated with the second resonance of the second mode can be given as

$$\ddot{u}_{r,2}(x,t) = \frac{2p}{mL}\left\{\sum_{n=1}\sin\left(\frac{n\pi x}{L}\right)\frac{S_{n,r2}}{1 - S_{n,r2}^2}\left[\sum_{k=N}^{N+K-1}(-S_{n,r2}\sin\Omega_n(t - t_k) + \sin\omega_n(t - t_k))\right.\right.$$
$$\left.\left.\times H(t - t_k) + 2D_{n,r2}(v_{r,2}, t) \times H(t - t_{N-1} - L/v_{r,2})\right]\right\}, \tag{21a}$$

where

$$S_{n,r2} = \frac{n\pi v_{r,2}}{\omega_n L} = \frac{d}{nL},$$

$$D_{n,r2}(v_{r,2},t)=\begin{cases}\dfrac{[1+(-1)^{N+1}]}{2}\cos\dfrac{\omega_n L}{2v_{r,2}}\sin\omega_n\left(t-\dfrac{L}{2v_{r,2}}\right), & n=\text{odd},\\ (N-1)\sin\dfrac{\omega_n L}{2v_{r,2}}\cos\omega_n\left(t-\dfrac{L}{2v_{r,2}}\right), & n=\text{even}.\end{cases} \quad (21\text{b,c})$$

As indicated in Eq. (21c), whenever the running speed of the successive moving loads equals the resonant speed $v_{r,2}$ given in Eq. (20b), the contribution to the acceleration comes mainly from the even vibration modes of the beam, which increases as there are more loads passing the beam. Correspondingly, the maximum acceleration of the beam may not occur at the mid-point. As for the *odd* modes, according to Eq. (21c), their contributions to the acceleration do not increase as there are more loads passing the beam.

Similarly, following the same procedure as outlined above for finding the resonant speeds, one can also formulate the resonant acceleration responses of the beam for the case when any of the higher modes is to be excited. However, the effect of damping has not yet been considered in the derivation. The purpose of the following section is to investigate such an effect on the resonant acceleration response of the beam.

5. Effect of damping

Consider the case that when the running speed of moving loads meets with the first resonant speed of the fundamental mode of the beam, i.e., when $v_{r,1}=\omega_1 d/2\pi$ and $\omega_n d/v_{r,1}=2n^2\pi$. As shown in Fig. 1, when the Nth moving load travels over a distance vt_E from the start end of the beam at the time $t=t_E+t_N$ or $t=t_E+(N-1)d/v_{r,1}$, by the equality of $\omega_n(t-t_k)=\omega_n t_E+(N-k)\omega_n d/v_{r,1}=\omega_n t_E+2n^2\pi(N-k)$, one can derive the following relations:

$$\sin\Omega_n(t-t_k)+(-1)^{n+1}\sin\Omega_n(t-t_k-L/v_{r,1})=0,\quad 0\leqslant t\leqslant t_N,$$

$$\sin\omega_n(t-t_k)=\sin\omega_n t_E,$$

$$\sin\omega_n(t-t_k-L/v_{r,1})=\sin\omega_n(t_E-L/v_{r,1}). \quad (22\text{a–c})$$

By inserting the preceding relations with the resonant speed of $v_{r,1}$ into the response factor $Q_n(v,t)$ in Eq. (10b), considering that the beam is lightly damped with constant modal damping ratios, i.e., $\xi_n=\xi$, one can express the resonant response factor $Q_{n,\text{res}}(v_{r,1},t)$ as

$$\begin{aligned}Q_{n,\text{res}}(v_{r,1},t_E)={}&S_{n,r1}\sum_{k=1}^{K}[-S_{n,r1}\sin\Omega_n(t_E+t_k)+\mathrm{e}^{-\xi\omega_n(t_E+t_k)}\sin\omega_n(t_E+t_k)]H(t_E+t_k)\\&+S_{n,r1}\mathrm{e}^{-\xi\omega_n t_N}\left(\sum_{k=1}^{N-1}\mathrm{e}^{\xi\omega_n t_k}\right)\times\mathrm{e}^{-\xi\omega_n t_E}[\sin\omega_n t_E+(-1)^{n+1}\mathrm{e}^{\xi\omega_n L/v_{r,1}}\\&\times\sin\omega_n(t_E-L/v_{r,1})]H(t_E-(L-d)/v_{r,1}),\end{aligned} \quad (23)$$

where the first term with $H(t_E+t_k)$ indicates the response of the beam induced by the Nth moving load and its following $(K-1)$ loads acting on the beam, and the second term with

$H(t_E-(L-d)/v_{r,1})$ represents the residual vibration of the beam due to the first $(N-1)$ moving loads that have left the beam. By using the following series sum [5]:

$$E_n(\xi, v_r) = e^{-\xi\omega_n t_N} \sum_{k=1}^{N-1} e^{\xi\omega_n t_k} = e^{-\xi\omega_n(N-1)d/v_r} \sum_{k=1}^{N-1} e^{\xi\omega_n(k-1)d/v_r} = \frac{1-e^{-(N-1)\xi\omega_n d/v_r}}{e^{\xi\omega_n d/v_r}-1}, \tag{24}$$

the resonant response factor $Q_{n,\text{res}}(v_{r,1}, t)$ in Eq. (23) can be transformed into

$$Q_{n,\text{res}}(v_{r,1}, t) = S_{n,r1}\{D_{1n}(\xi, v_{r,1}) + E_n(\xi, v_{r,1})e^{-\xi\omega_n t_E} \times [\sin\omega_n t_E + (-1)^{n+1} e^{\xi\omega_n L/v_{r,1}} \sin\omega_n(t_E - L/v_{r,1})] \times H(t_E-(L-d)/v_{r,1})\}, \tag{25}$$

where

$$D_{1n}(\xi, v_{r,1}) = \sum_{k=1}^{K} [-S_{n,r1}\sin\Omega_n(t_E+t_k) + e^{-\xi\omega_n(t_E+t_k)}\sin\omega_n(t_E+t_k)]H(t_E+t_k). \tag{26}$$

Substitution of Eq. (25) into Eq. (10a) yields the acceleration of the beam as

$$\ddot{u}_{r,1}(x,t) = \frac{2p}{mL}\sum_{n=1}\sin\left(\frac{n\pi x}{L}\right)\frac{S_{n,r1}}{1-S_{n,r1}^2}\{D_{1n}(\xi, v_{r,1}) + E_n(\xi, v_{r,1})e^{-\xi\omega_n t_E} \times [\sin\omega_n t_E + (-1)^{n+1}e^{\xi\omega_n L/v_{r,1}}\sin\omega_n(t_E-L/v_{r,1})]H(t_E-(L-d)/v_{r,1})\}. \tag{27}$$

This equation represents exactly the resonant acceleration response of a simple beam with light damping due to successive moving loads at the resonant speed $v_{r,1}$. On the other hand, by letting the damping ratio ξ become zero, one gets $E_n(0, v_{r,1}) = (N-1)$ by applying L'Hospital's rule to Eq. (24). By the relations of $E_n(0, v_{r,1})$ and $t_E = t - t_N$, it can be shown that the acceleration response in Eq. (27) reduces to that in Eq. (18).

Next, let us consider the special case when the second resonance of the second vibration mode is excited. This means that the resonant speed $v_{r,2}$ is equal to $\omega_2 d/4\pi$ and that $\omega_n d/v_{r,2} = n^2\pi$ from Eq. (20b). When the Nth load and its following $(K-1)$ moving loads are simultaneously acting on the beam at the time $t = t_E + t_N$ or $t = t_E + (N-1)d/v_{r,2}$, by the equality of $\omega_n(t-t_k) = \omega_n t_E + (N-k)\omega_n d/v_{r,2} = \omega_n t_E + n^2\pi(N-k)$, one can derive the following relations:

$$\sin\Omega_n(t-t_k) + (-1)^{n+1}\sin\Omega_n(t-t_k-L/v_{r,2}) = 0, \quad 0 \leqslant t \leqslant t_N,$$

$$\sin\omega_n(t-t_k) = \sin[\omega_n t_E + n^2\pi(N-k)] = \begin{cases} (-1)^{N-k}\sin\omega_n t_E, & n = \text{odd}, \\ \sin\omega_n t_E, & n = \text{even}, \end{cases}$$

$$\sin\omega_n(t-t_k-L/v_{r,2}) = \begin{cases} (-1)^{N-k}\sin\omega_n(t_E-L/v_{r,2}), & n = \text{odd}, \\ \sin\omega_n(t_E-L/v_{r,2}), & n = \text{even}. \end{cases} \tag{28a–c}$$

The resonant acceleration response factor $Q_{n,\text{res}}(v_{r,2}, t_E)$ can be written as

$$Q_{n,\text{res}}(v_{r,2}, t_E) = S_{n,r2}[D_{1n}(\xi, v_{r,2}) + e^{-\xi\omega_n t_E}D_{2n}(\xi, v_{r,2})H(t_E-(L-d)/v_{r,2})], \tag{29}$$

where

$$D_{2n}(\xi, v_{r,2}) = \begin{cases} [\sin \omega_n t_E + e^{\xi\omega_n L/v_{r,2}} \sin \omega_n(t_E - L/v_{r,2})] \times e^{-\xi\omega t_N} \sum_{k=1}^{N-1}[(-1)^{N-k} e^{\xi\omega_n t_k}], & n = \text{odd}, \\ [\sin \omega_n t_E - e^{\xi\omega_n L/v_{r,2}} \sin \omega_n(t_E - L/v_{r,2})] \times e^{-\xi\omega t_N} \sum_{k=1}^{N-1} e^{\xi\omega_n t_k}, & n = \text{even}. \end{cases} \tag{30}$$

By the expression for $E_n(\xi, v_{r,2})$ in Eq. (24) and the following series sum:

$$\bar{E}_n(\xi, v_r) = e^{-\xi\omega_n t_N} \sum_{k=1}^{N-1}[(-1)^{N-k} e^{\xi\omega_n t_k}] = \frac{1 + (-1)^N \times e^{-(N-1)\xi\omega_n d/v_r}}{-e^{\xi\omega_n d/v_r} - 1}, \tag{31}$$

the factor $D_{2n}(\xi, v_{r,2})$ in Eq. (30) can be expressed as

$$D_{2n}(\xi, v_{r,2}) = \begin{cases} \bar{E}_n(\xi, v_{r,2})\left[\sin \omega_n t_E + e^{\xi\omega_n L/v_{r,2}} \sin \omega_n\left(t_E - L/v_{r,2}\right)\right], & n = \text{odd}, \\ E_n(\xi, v_{r,2})[\sin \omega_n t_E - e^{\xi\omega_n L/v_{r,2}} \sin \omega_n(t_E - L/v_{r,2})], & n = \text{even}. \end{cases} \tag{32}$$

It follows that the resonant acceleration response for the simply supported beam under the action of successive moving loads at the resonant speed $v_{r,2}$ can be expressed as

$$\ddot{u}_{r,2}(x, t) \approx \frac{2p}{mL} \sum_{n=1} \left\{ \sin\left(\frac{n\pi x}{L}\right) \frac{S_n}{1 - S_n^2} [D_{1n}(\xi, v_{r,2}) H(t_E + t_k) + e^{-\xi\omega_n t_E} D_{2n}(\xi, v_{r,2}) H(t_E - (L - d)/v_{r,2})] \right\}. \tag{33}$$

For the special case of zero damping, i.e., for $\xi = 0$, one can get $\bar{E}_n(0, v_{r,2}) = -[1 + (-1)^N]/2$ and $E_n(0, v_{r,2}) = (N - 1)$. Accordingly, the resonant acceleration response of the beam in Eq. (33) can be reduced to that in Eq. (21), where it is noted that $t_E = t - t_N$.

In addition, by considering the effect of damping ratio ξ in computing the acceleration response of the beam, Eqs. (27) and (33) show that the contributions of higher modes to the beam acceleration quickly decrease owing to the presence of $e^{-\xi\omega_n t_E} (= e^{-n^2\xi\omega_n t_E})$, that is, the higher the mode number n, the faster the decay of the exponential factor $e^{-\xi\omega_n t_E}$ is. A detailed illustration of the effect of damping on the acceleration response of the beam will be illustrated in the section to follow.

6. Illustrative examples

As shown in Fig. 1, a simple beam is subjected to a series of equidistant moving loads. The properties of the beam, assumed to be made of steel, have been listed in Table 1. Three sets of moving loads, T-24, T-18, and T-12, each with identical intervals, are assumed, for which the axle interval d, weight of each load p, total number $(N + K - 1)$ of loads, and the maximum number K of the wheel loads allowed to stay on the beam were listed in Table 2. To compute the acceleration

Table 1
Properties of beam

L (m)	m (t/m)	EI (kN m^2)	ω_1[a] (rad/s)
40	30	4×10^8	22.5

[a]$\omega_1 = (\pi/L)^2\sqrt{EI/m}$.

Table 2
Properties of various types of moving loads

Train type	$N+K-1$	K	d (m)	p (kN)	$v_{r,1}$ (km/h)	$v_{r,2}$ (km/h)
T-24	10	2	24	360	309	618
T-18	14	3	18	300	232	464
T-14	18	3	14	210	180	360

response of the beam subjected to the moving loads, the first 20 vibration modes will be considered in the following examples.

6.1. Phenomena of resonance and sub-resonance

In order to illustrate the phenomena of resonance and sub-resonance on the beam due to successive moving loads, let us first neglect the effect of damping of the beam and consider first the moving loads of the T-24 type over the beam. Fig. 2 shows the time histories of the mid-point deflection of the beam subjected to the moving loads at the resonant speeds of $v_{r,1}$ and $0.5v_{r,1}$. As can be seen, both responses continue to build up as there are more loads passing the beam. Moreover, the mid-point deflection response shows a single cycle of vibration between the passage duration Δt of two adjacent moving loads, where $\Delta t = t_N - t_{N-1} = d/v$, for the resonant speed $v_{r,1}$ and two cycles of vibration for the speed $v_{r,1/2}$. These are characteristic of the resonance and sub-resonance of the first mode of the beam due to a series of equidistant moving loads.

6.2. Maximum acceleration of an undamped beam due to various moving loads

The speed parameter, which is dimensionless and defined as $S = \pi v/\omega_1 L$, is a useful parameter for studying the dynamic responses of beams subjected to moving loads of various speeds. If the ratio of d/L is given for a problem, then the resonant speed of the moving loads over the beam can be deduced from the non-dimensional resonant speed parameter $S_{n,r}$ presented in Section 4. For the present purposes, the three sets of moving loads, i.e., T-24, T-18, and T-14, as specified in Table 2, for moving loads of long, medium, and short axle distances, respectively, will be employed to investigate the maximum acceleration responses of an undamped beam. By letting each set of moving loads traverse the beam at different speeds, the maximum acceleration a_{max} has been plotted against the speed parameter S and the position x/L in Figs. 3(a)–(c) for each set of moving loads. As can be seen, whenever the speed parameter is equal to any of the resonant speed parameters associated with the vibration modes to be excited, that is, when the running speed of

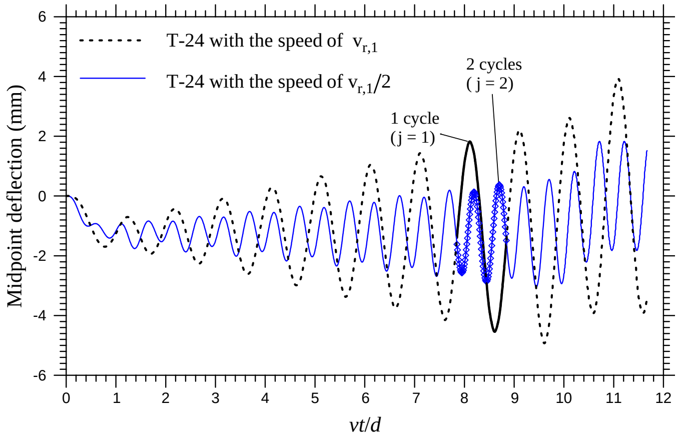

Fig. 2. Time history responses at resonant speeds.

the moving loads coincides with any of the resonant speeds of the vibration modes, the maximum acceleration responses of the beam will be significantly amplified. Moreover, the shorter the interval of the moving loads, the easier the resonance will be induced. Unlike the case for the maximum deflection [3], the maximum acceleration of the beam need not necessarily occur at the mid-point, as indicated in Fig. 3(b) for the T-18 moving loads. Further studies will be presented later on concerning the occurrence of resonance at positions beyond the mid-point of the beam.

6.3. Effect of higher modes

Consider first the acceleration response of the beam subjected to the T-24 moving loads at the resonant speed $v_r = \omega_n d/2j\pi$ derived from Section 4. Correspondingly, the resonant speed parameters of the vibration modes that may be excited can be deduced as follows:

$$S_r = \frac{\pi v_r}{\omega_1 L} = \frac{\pi \times (\omega_n d/2j\pi)}{\omega_1 L} = \frac{n^2 d}{2jL}, \tag{34}$$

where the number of (j, n) represent the jth resonance associated with the nth mode of the simple beam. The maximum acceleration $a_{\max}$ solved along the axis x/L of the beam have been plotted against the resonant speed parameter in Fig. 4(a). This figure indicates that for the undamped beam considered, as the first resonance of the first mode is excited, the resonance associated with other higher modes will be excited as well, resulting in some secondary peaks on the acceleration response curve. Because of this, the maximum acceleration does not occur at the mid-point of the beam. For instance, the resonant speed parameter of the sixth resonance ($j = 6$) of the third mode ($n = 3$) is $3^2 \times 24/(2 \times 6 \times 40) = 0.45$. As indicated by the dashed line in Fig. 4(a), there exist three response peaks for the third mode resonance at the resonant speed parameter $S_r = 0.45$ under the T-24 moving loads. Thus, it is possible to count the mode number from the number of peaks appearing on the acceleration response curve. Similar observations can be made from Figs. 4(b) and (c) for the beam under the action of the T-18 and T-14 moving loads at their resonant speeds, respectively. Noteworthy is the fact that the maximum acceleration of the beam for the

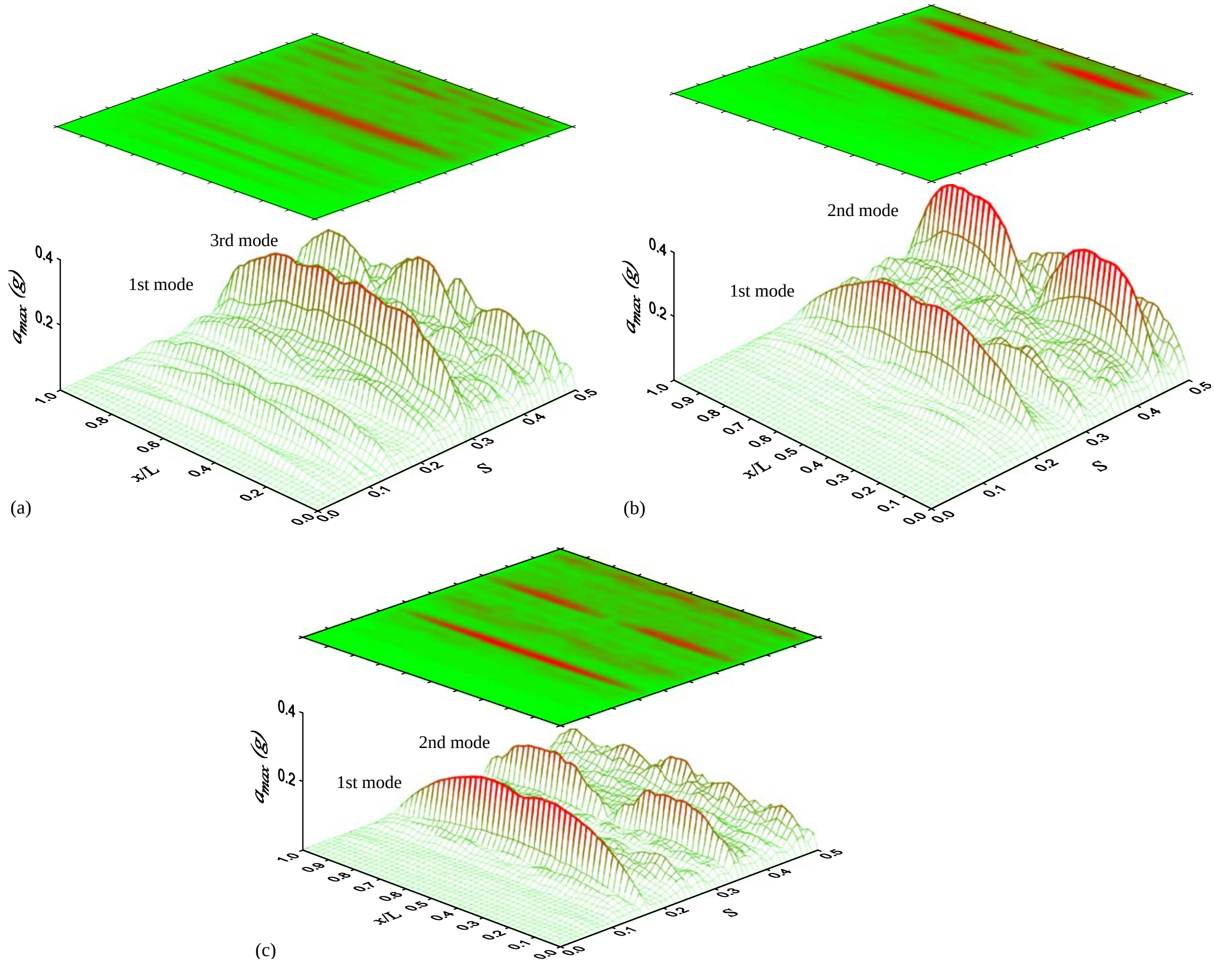

Fig. 3. $a_{\max}$–S–x/L diagram: (a) $d/L = 0.6$; (b) $d/L = 0.45$ and (c) $d/L = 0.35$.

second resonance of the second mode due to the T-18 moving loads, as indicated in Fig. 4(b), is larger than that of the first resonance of the first mode, for which the reason will be given in the following.

As indicated in Figs. 4(a)–(c), once the nth mode of the beam is excited, the number of peak responses of the maximum acceleration along the beam is equal to the mode number of the vibration mode excited. From Eqs. (18) and (21), it can be observed that under the condition of resonance, the residual responses are proportional to the number of loads passing the beam, i.e., $(N-1)$, as indicated in Eqs. (18c) and (21c). For the case with sufficiently large value of N, the maximum response of the beam may be governed by the residual response. For the extreme case when the number $(N-1)$ of moving loads passing the beam under resonance is so large that the acceleration of the beam induced by the moving loads directly acting on the beam can be neglected, as represented by the term with $H(t-t_k)$ in Eqs. (18)

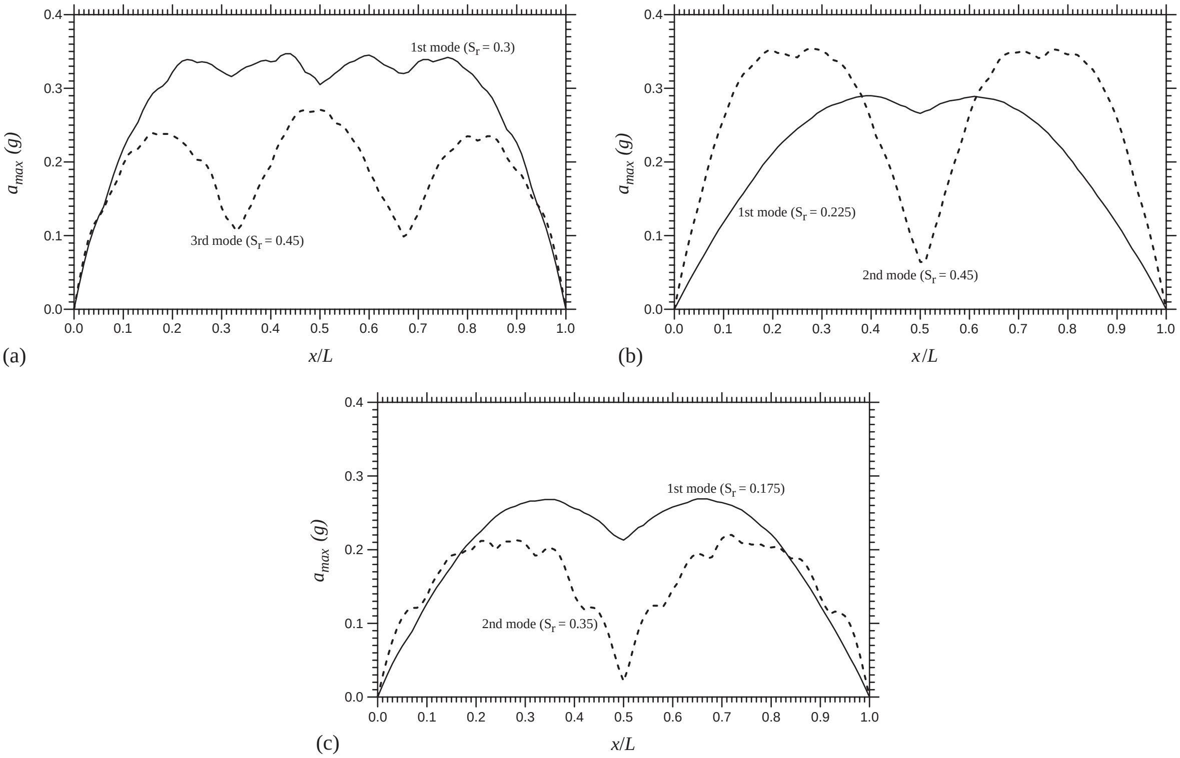

Fig. 4. Maximum acceleration of an undamped beam to various types of moving loads at resonance: (a) T-24; (b) T-18 and (c) T-14.

and (21), the maximum acceleration response of the beam is governed by the vibration mode that has been excited.

It follows that if the first resonant acceleration response of the first mode is considered, the mid-point acceleration response of the beam as given in Eq. (18) can be approximated as

$$\ddot{u}_{r,1}\left(\frac{L}{2},t\right) \simeq \frac{2p}{mL}\frac{S_{1,r1}}{1-S_{1,r1}^2} \times \left[2(N-1)\cos\frac{\omega_1 L}{2v_{r,1}}\sin\omega_1\left(t-\frac{L}{2v_{r,1}}\right)\right] \times H\left(t-t_{N-1}-\frac{L}{v_{r,1}}\right), \quad (35)$$

where $S_{r,1} = d/2L$ and $v_{r,1} = \omega_1 d/2\pi$. Similarly, if the second resonance of the second mode is excited, the maximum acceleration amplitude will occur at the first or third quarter-points of the beam. Consider the resonant acceleration at the first quarter-point of the beam. The acceleration response for the second mode as given in Eq. (21) can be approximated as

$$\ddot{u}_{r,2}\left(\frac{L}{4},t\right) \simeq \frac{2p}{mL}\frac{S_{2,r2}}{1-S_{2,r2}^2} \times \left[2(N-1)\sin\frac{\omega_2 L}{2v_{r,2}}\cos\omega_2\left(t-\frac{L}{2v_{r,2}}\right)\right] \times H\left(t-t_{N-1}-\frac{L}{v_{r,2}}\right), \quad (36)$$

where $S_{r,2} = d/2L$ and $v_{r,2} = \omega_2 d/4\pi = 2v_{r,1}$.

Let us consider the condition when the mid-point maximum acceleration is smaller than the quarter-point maximum acceleration, i.e., $\ddot{u}_{r,1}(L/2,t)_{\max} < \ddot{u}_{r,2}(L/4,t)_{\max}$, or

$$\left|\frac{S_{1,r1}}{1-S_{1,r1}^2}\times\cos\frac{\omega_1 L}{2v_{r,1}}\right| < \left|\frac{S_{2,r2}}{1-S_{2,r2}^2}\times\sin\frac{\omega_2 L}{2v_{r,2}}\right|. \tag{37}$$

Substituting the relations $S_{r,1} = S_{r,2} = d/2L$, $v_{r,1} = \omega_1 d/2\pi$ and $v_{r,2} = 2v_{r,1}$ mentioned earlier into Eq. (37), one can get the following relation:

$$\left|\cos\frac{L\pi}{d}\right| < \left|\sin\frac{2L\pi}{d}\right|. \tag{38}$$

The preceding equation can be solved to yield the following inequality:

$$\frac{1}{2J+1/6} > \frac{d}{L} > \frac{1}{2J+5/6}, \tag{39}$$

where J is a positive integer. For a beam simultaneously acted upon by multiple moving loads, namely, for beams that are not short. By letting $J = 1$, one can get $0.353 < d/L < 0.462$. Once the length ratio of d/L is located within this range, the maximum acceleration of the mid-point of the beam will be smaller than that of the quarter-span. In the present study, the length ratio d/L of the interval of the T-18 moving loads to the span of the beam is $18/40 = 0.45$, which satisfies the condition derived above from Eq. (39). Besides, the ratio d/L for the T-24 and T-14 moving loads are 0.6 and 0.35, respectively. Both are not within the range of $0.353 < d/L < 0.462$. Therefore, the dominant mode of the maximum acceleration of the beam to moving loads is the fundamental mode.

6.4. *Effect of damping*

No consideration was made for damping of the beam in Section 6.3. Once the damping of the beam is considered, all the resonant peaks of maximum acceleration will be significantly reduced, especially those of the higher modes, and the maximum acceleration will occur only at the mid-point of the beam, as indicated in Fig. 5(a). An observation from the acceleration responses shown in Figs. 5(a)–(c) is that damping plays an important role in reducing the resonant acceleration response of the beam to moving loads. Since the structural damping can quickly damp out the contributions of higher modes to the acceleration of the beam, the maximum acceleration response of the beam is generally dominated by the fundamental mode in most practical situations.

7. Concluding remarks

In this paper, the acceleration response of a simple beam subjected to successive moving loads with identical intervals has been studied by the mode superposition method. The analysis results indicate that the contributions of higher modes to the beam acceleration cannot be neglected for beams with light damping. In particular, when the characteristic frequency of the moving loads coincides with any of the resonant speeds associated with the higher mode frequencies, resonance

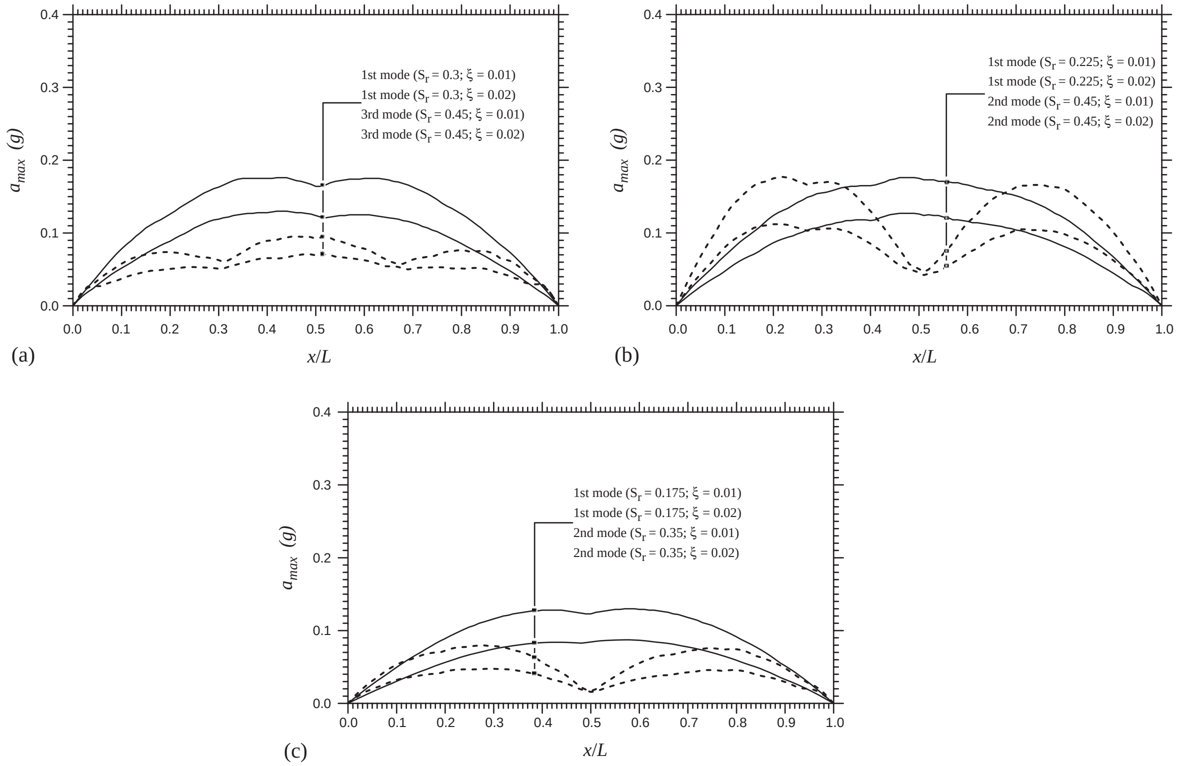

Fig. 5. Maximum acceleration of a damped beam to various types of moving loads at resonance: (a) T-24; (b) T-18 and (c) T-14.

may be excited for those frequencies on the beam. Because of this, the maximum acceleration of the beam need not occur at the mid-point.

The location of maximum acceleration on the beam depends on the vibration mode that has been excited. In this paper, the range of car/span length ratio is identified for the case where the maximum acceleration of the quarter-point is larger than that of the mid-point. For a damped beam subjected to moving loads at resonant speeds, the contributions of higher modes to the acceleration response can be neglected as they are quickly damped out. Therefore, increasing the structural damping of a ballasted bridge can help reduce the maximum acceleration on the bridge deck, which can be achieved through installation of additional damper devices to the bridge or by using materials with higher damping ratios.

Acknowledgements

The research reported herein was supported in part by grants from the National Science Council of the ROC through Nos. NSC 92-2211-E-002-046 and NSC 93-2211-E-032-010.

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