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Convergence studies on static and dynamic analyses of plates by using the U-transformation and the finite difference method

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Abstract

This paper presents the exact static and dynamic analyses of simply supported rectangular plates. The analytical solutions for displacements, buckling loads, natural frequencies and dynamic responses are obtained by using the double U-transformation method and the finite difference method. After an equivalent system with cyclic periodicity in two directions is established, the difference governing equation for such an equivalent system can be uncoupled by applying the double U-transformation. Then the exact analytical finite difference solutions, the exact error expressions and the exact convergence rates are derived. These results cannot be obtained if other methods are used instead.
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1. Introduction

As an important numerical computational method, the finite difference method has broad applications in various scientific research fields, e.g. physics, mechanics, astronomy and engineering technology. The study on the convergence of difference schemes always attracts the attention of computing mathematicians and dynamicists.

It is well known that the governing equations of cyclic periodic structures can be uncoupled in each mode subspace by using the U-transformation method [1]. This method may be employed to

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greatly reduce the computational effort in the numerical analysis of cyclic periodic [2], periodic [3], nearly periodic [4] and bi-periodic structures [5,6].

For another important application of the U-transformation method, it can be used to study the convergence of computational schemes for numerical methods. In 1993, Chan et al. applied the U-transformation method to study the convergence rates of the natural frequencies and the dynamic responses of simply supported plates by using the non-conforming rectangular element, and the explicit finite element solutions were derived [7]. Liu et al. extended this method to the convergence study of the finite difference method. The static analysis of a simply supported beam subjected to uniformly distributed loads was investigated and exact convergence rates of difference solutions were obtained by using the one-dimensional finite difference formula [8].

For the structures with cyclic periodicity in two directions, the double U-transformation method may be applied to the static and dynamic analyses [9]. The present paper is therefore to extend the application of the U-transformation method to the convergence study of two-dimensional difference formula. The analysis of two-dimensional periodic structures is more complex than that of one-dimensional structures. The static and dynamic analyses of a simply supported rectangular plate are studied by using the two-dimensional finite difference method. By adopting the double U-transformation technique, the governing equations in difference form can be uncoupled to a set of independent equations, and then the analytical expressions for the static displacements, buckling loads, natural frequencies and dynamic responses can be easily obtained. The convergence rate can be discussed simply and effectively from the explicit solutions. A few numerical results are given to demonstrate the proposed procedure.

2. Simply supported plate

A rectangular plate with all edges simply supported is considered. The structure may be divided into $n \times m$ equal rectangular elements, as shown in Fig. 1. L_1, L_2 denote the lengths of the plate and a, b denote the lengths of an element in the x and y directions, respectively. P is the load

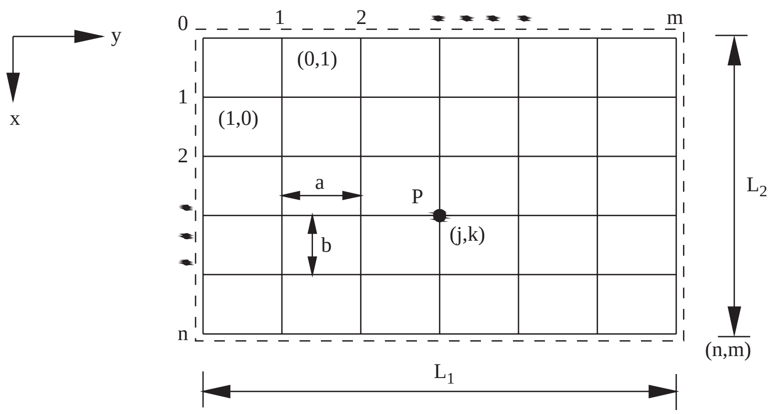

Fig. 1. Simply supported rectangular plate with $n \times m$ elements.

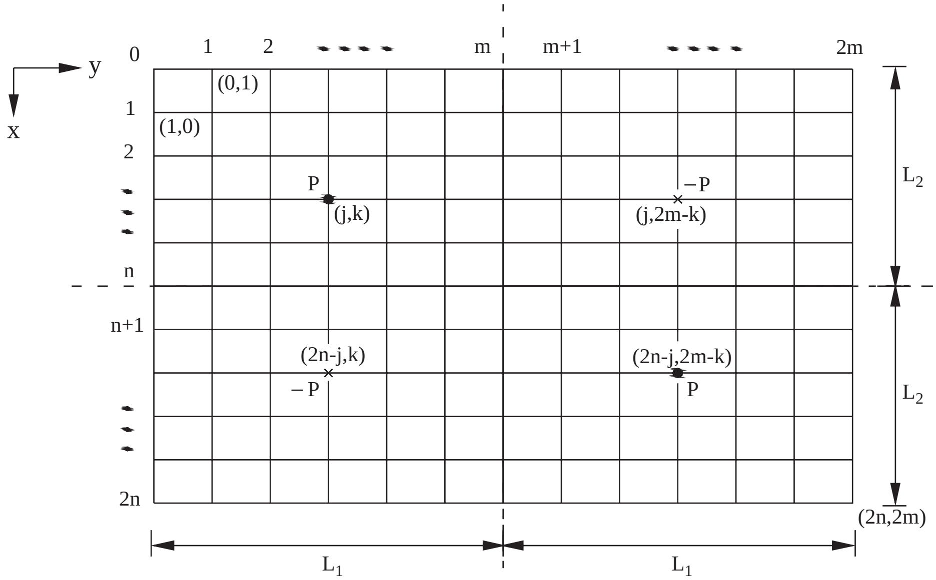

Fig. 2. Equivalent system with $2n \times 2m$ elements.

function acts on the node (j, k), and j, k are the node numbers. Now the plate may be regarded as a periodic structure in two directions.

At the outset, let us consider the extended plate (see Fig. 2) whose length and width are twice that of the actual ones. Moreover, the anti-symmetrical loads about the right side and lower side of actual plate must be applied on the corresponding extended part and then we regard the extended plate as one having cyclic periodicity in x and y directions [10]. The boundary conditions of the original plate can be satisfied automatically so that an equivalent system is produced.

3. Static displacements

Consider now the equivalent system with $2n \times 2m$ substructures and cyclic periodicity instead of the actual original structure. The governing equations for all substructures are of the same form, i.e.,

$$D\nabla^4 w_{(j,k)} = F_{(j,k)}, \quad j = 1, 2, \ldots, 2n, \;\; k = 1, 2, \ldots, 2m, \tag{1}$$

where D is the flexural rigidity, $w_{(j,k)}$ and $F_{(j,k)}$ denote the transverse displacement and loading for the node (j, k) and

$$F_{(j,k)} = -F_{(j,2m-k)} = -F_{(2n-j,k)} = F_{(2n-j,2m-k)}, \quad j = 1, 2, \ldots, n, \;\; k = 1, 2, \ldots, m. \tag{2}$$

From the cyclic periodicity in two directions, the following conditions are obtained:

$$w_{(2n+1,k)} = w_{(1,k)}, \quad w_{(2n+1,2m+1)} = w_{(1,1)}, \quad w_{(j,2m+1)} = w_{(j,1)},$$
$$j = 1, 2, \ldots, 2n, \;\; k = 1, 2, \ldots, 2m. \tag{3}$$

Substituting the difference formula for $\nabla^4 w_{(j,k)}$ into the governing equation (1) results in

$$\begin{aligned}&(6+6\alpha^2+8\alpha)w_{(j,k)}-4(1+\alpha)w_{(j,k-1)}-4(1+\alpha)w_{(j,k+1)}-4\alpha(1+\alpha)w_{(j-1,k)}\\&-4\alpha(1+\alpha)w_{(j+1,k)}+2\alpha w_{(j-1,k-1)}+2\alpha w_{(j+1,k+1)}+2\alpha w_{(j+1,k-1)}\\&+2\alpha w_{(j-1,k+1)}+\alpha^2 w_{(j-2,k)}+\alpha^2 w_{(j+2,k)}+w_{(j,k-2)}+w_{(j,k+2)}=a^4F_{(j,k)}/D\end{aligned}\tag{4}$$

in which $\alpha = a^2/b^2,\ a = L_1/m,\ b = L_2/n.$

In order to uncouple the difference equation (4), the double U-transformation need to be used, i.e., let

$$q_{(r,s)}=\frac{1}{\sqrt{2n}\sqrt{2m}}\sum_{j=1}^{2n}\sum_{k=1}^{2m}\mathrm{e}^{-\mathrm{i}(j-1)r\psi_1}\mathrm{e}^{-\mathrm{i}(k-1)s\psi_2}w_{(j,k)}\tag{5a}$$

or

$$w_{(j,k)}=\frac{1}{\sqrt{2n}\sqrt{2m}}\sum_{r=1}^{2n}\sum_{s=1}^{2m}\mathrm{e}^{\mathrm{i}(j-1)r\psi_1}\mathrm{e}^{\mathrm{i}(k-1)s\psi_2}q_{(r,s)}\tag{5b}$$

in which $\mathrm{i}=\sqrt{-1}$, and $\psi_1 = 2\pi/2n,\ \psi_2 = 2\pi/2m.$

Applying the double U-transformation (5) to the difference equation (4) results in

$$DAq_{(r,s)}=f_{(r,s)},\tag{6}$$

where

$$\begin{aligned}A=\frac{1}{a^4}[&6+6\alpha^2+8\alpha-4(1+\alpha)\mathrm{e}^{-\mathrm{i}s\psi_2}-4(1+\alpha)\mathrm{e}^{\mathrm{i}s\psi_2}-4\alpha(1+\alpha)\mathrm{e}^{-\mathrm{i}r\psi_1}\\&-4\alpha(1+\alpha)\mathrm{e}^{\mathrm{i}r\psi_1}+2\alpha\mathrm{e}^{-\mathrm{i}r\psi_1}\mathrm{e}^{-\mathrm{i}s\psi_2}+2\alpha\mathrm{e}^{\mathrm{i}r\psi_1}\mathrm{e}^{\mathrm{i}s\psi_2}+2\alpha\mathrm{e}^{\mathrm{i}r\psi_1}\mathrm{e}^{-\mathrm{i}s\psi_2}\\&+2\alpha\mathrm{e}^{-\mathrm{i}r\psi_1}\mathrm{e}^{\mathrm{i}s\psi_2}+\alpha^2\mathrm{e}^{-2\mathrm{i}r\psi_1}+\alpha^2\mathrm{e}^{2ir\psi_1}+\mathrm{e}^{-2\mathrm{i}s\psi_2}+\mathrm{e}^{2is\psi_2}],\end{aligned}\tag{7}$$

$$f_{(r,s)}=\frac{1}{\sqrt{2n}\sqrt{2m}}\sum_{j=1}^{2n}\sum_{k=1}^{2m}\mathrm{e}^{-\mathrm{i}(j-1)r\psi_1}\mathrm{e}^{-\mathrm{i}(k-1)s\psi_2}F_{(j,k)}.\tag{8}$$

Now, the governing equation (1) has been uncoupled into Eq. (6) successfully. From Eq. (6), $q_{(r,s)}$ can be expressed as

$$q_{(r,s)}=\frac{f_{(r,s)}}{AD}.\tag{9}$$

Consider the simple case of a square uniform plate with $n \times n$ uniform elements subjected to a uniform load with magnitude p_0, i.e.,

$$n=m,\quad L_1=L_2=L,\quad a=b=\frac{L}{n},\quad \alpha=1,\quad \psi_1=\psi_2=\frac{\pi}{n}=\psi,\tag{10a}$$

$$F_{(j,k)}=p_0,\quad j=1,2,\ldots,n,\ \ k=1,2,\ldots,m.\tag{10b}$$

Substituting Eq. (10) into Eqs. (7) and (8) yields

$$A=4(\cos r\psi+\cos s\psi-2)^2/a^4,\tag{11}$$

$$f_{(r,s)} = -\frac{2p_0 \mathrm{e}^{\mathrm{i}r\psi}\mathrm{e}^{\mathrm{i}s\psi}}{n}\frac{\sin r\psi \sin s\psi}{(1-\cos r\psi)(1-\cos s\psi)}, \quad r,s = 1,3,\ldots,2n-1,$$

$$f_{(r,s)} = 0, \quad r \text{ or } s = 2,4,\ldots,2n. \tag{12}$$

Inserting Eqs. (11) and (12) in Eq. (9), the generalized displacement $q_{(r,s)}$ becomes

$$q_{(r,s)} = -\frac{a^4 p_0}{nD} \times \frac{\mathrm{e}^{\mathrm{i}r\psi}\mathrm{e}^{\mathrm{i}s\psi}\sin r\psi \sin s\psi}{2(\cos r\psi + \cos s\psi - 2)^2(1-\cos r\psi)(1-\cos s\psi)}, \quad r,s = 1,3,\ldots,2n-1,$$

$$q_{(r,s)} = 0, \quad r \text{ or } s = 2,4,\ldots,2n. \tag{13}$$

Now every nodal displacement can be obtained from the double U-transformation (5) with Eq. (13), i.e.,

$$w_{(j,k)} = \frac{a^4 p_0}{n^2 D}\sum_{r=1,3}^{n}\sum_{s=1,3}^{n}\frac{\sin jr\psi \sin ks\psi \sin r\psi \sin s\psi}{(\cos r\psi + \cos s\psi - 2)^2(1-\cos r\psi)(1-\cos s\psi)}. \tag{14}$$

The maximum displacement occurs at the center of the square plate and its magnitude can be expressed as

$$w_{\max} = w_{(n/2,n/2)} = \frac{a^4 p_0}{n^2 D}\sum_{r=1,3}^{n-1}\sum_{s=1,3}^{n-1}\frac{\sin r(\pi/2)\sin s(\pi/2)\sin r\psi \sin s\psi}{(\cos r\psi + \cos s\psi - 2)^2(1-\cos r\psi)(1-\cos s\psi)}. \tag{15}$$

Expanding the right side of Eq. (15) into power series of ψ results in

$$w_{\max} = \frac{16 p_0 L^4}{D\pi^6}\sum_{r=1,3}^{n-1}\sum_{s=1,3}^{n-1}\frac{(-1)^{(r-1)/2}(-1)^{(s-1)/2}}{rs(r^2+s^2)^2}\left[1 + \frac{(r^2-s^2)^2}{12(r^2+s^2)}\cdot\frac{\pi^2}{n^2} + O(n^{-4})\right]. \tag{16}$$

Let

$$A(n) = \sum_{r=1,3}^{n-1}\sum_{s=1,3}^{n-1}(-1)^{(r-1)/2}(-1)^{(s-1)/2}\frac{1}{rs(r^2+s^2)^2},$$

$$B(n) = \sum_{r=1,3}^{n-1}\sum_{s=1,3}^{n-1}(-1)^{(r-1)/2}(-1)^{(s-1)/2}\frac{(r^2-s^2)^2}{12rs(r^2+s^2)^3}. \tag{17}$$

Introducing Eq. (17) into Eq. (16) yields

$$w_{\max} = \frac{16 p_0 L^4}{D\pi^6} \times [A(n) + B(n)\pi^2(n^{-2})] + O(n^{-4}). \tag{18}$$

By letting n approach infinity, the limit of $A(n)$ and $B(n)$ becomes the definite integral

$$\lim_{n\to\infty} A(n) = 0.244094, \quad \lim_{n\to\infty} B(n) = -0.0027984. \tag{19}$$

The first term on the right-hand side of Eq. (18) represents the limiting solution, which is in agreement with the analytical solution. The second term represents the main error of the displacement found by finite difference method. When n approaches infinity, the finite difference solution for displacement converges to the analytical solution at an asymptotic rate of n^{-2}. And

Table 1
The central displacement $w_{\max}$ of simply supported square plate

n	2	4	8	16	32	∞
$w_{\max}$	0.79438	0.94860	0.98715	0.99679	0.99920	1.00000
Multiplier			p_0L^4/D			

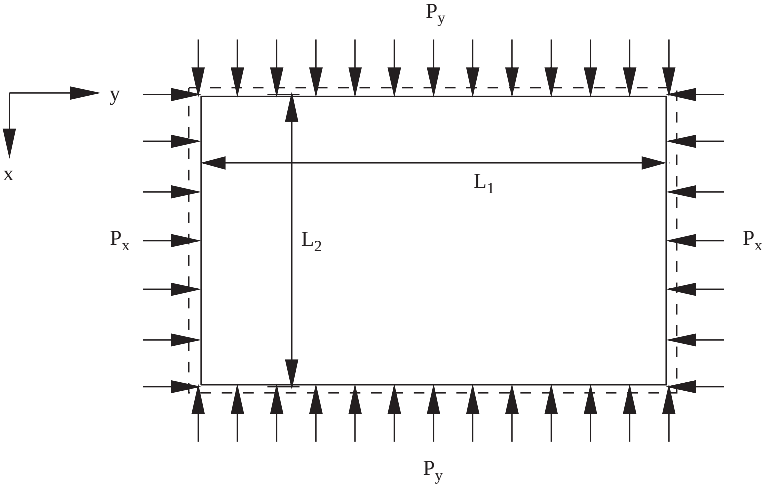

Fig. 3. Simply supported rectangular plate subjected to pressure in middle plane.

meanwhile the precise coefficient of the error term $-0.044774p_0L^4/D\pi^4$ is determined. Some numerical solutions for Eq. (18) are given in Table 1, from which one can know that the central displacement converges quickly.

4. Buckling loads

Simply supported rectangular plates subjected to uniformly distributed compression in x and y directions are considered. P_x and P_y denote the pressure per unit length in x and y directions as shown in Fig. 3.

The buckling equation can be expressed as

$$D\nabla^4 w_{(j,k)} + P_x\left(\frac{\partial^2 w_{(j,k)}}{\partial x^2} + \beta\frac{\partial^2 w_{(j,k)}}{\partial y^2}\right) = 0, \tag{20}$$

where $\beta = P_y/P_x$. Substituting the difference formulae for $\nabla^4 w_{(j,k)}$, $\partial^2 w_{(j,k)}/\partial x^2$, $\partial^2 w_{(j,k)}/\partial y^2$ and double U-transformation (5) into the buckling equation (20) results in

$$DAq_{(r,s)} + 2P_x\left[\frac{(\cos r\psi_1 - 1)}{a^2} + \frac{\beta(\cos s\psi_2 - 1)}{b^2}\right]q_{(r,s)} = 0, \tag{21}$$

where A has been defined as Eq. (7).

From Eq. (21) the critical load can be written as

$$P_x = \frac{-DA}{2((\cos r\psi_1 - 1)/a^2 + \beta(\cos s\psi_2 - 1)/b^2)}. \tag{22}$$

Obviously the buckling load depends on the ratios between p_x and p_y. Firstly, consider the case of bi-directional uniform compression, i.e., $p_x = p_y = p$. Substituting $\beta = 1$ into Eq. (22) results in

$$P_x = \frac{-DA}{2((\cos r\psi_1 - 1)/a^2 + (\cos s\psi_2 - 1)/b^2)}. \tag{23}$$

Consider the case of simply supported square plate with $n \times n$ uniform elements. Substituting Eqs. (10a) and (11) into Eq. (23) yields

$$P_x = \frac{-2D(\cos r\psi + \cos s\psi - 2)}{a^2}. \tag{24}$$

Expanding the right-hand side of Eq. (24) into a power series of ψ results in

$$P_x = \frac{D(r^2+s^2)\psi^2}{a^2}\left[1 - \frac{r^4+s^4}{12(r^2+s^2)}\psi^2 + O(\psi^4)\right]. \tag{25}$$

The minimum buckling load P_{cr} can be expressed, by substituting $r = 1$ and $s = 1$ into Eq. (25), as

$$P_{\mathrm{cr}} = \frac{2D\pi^2}{L^2} - \frac{D\pi^4}{6L^2}n^{-2} + O(n^{-4}). \tag{26}$$

The first term on the right-hand side of Eq. (26) represents the limiting solution which is in agreement with the analytical solution. Then the second term represents the main error of the buckling load found by the finite difference method. When n approaches infinity, the buckling load converges from below the analytical solution at an asymptotic rate of n^{-2}.

Secondly, consider the case of unidirectional compression, i.e., $p_x = p$ and $p_y = 0$. Substituting $\beta = 0$ into Eq. (22) results in

$$P_x = \frac{-DAa^2}{2(\cos r\psi_1 - 1)}. \tag{27}$$

Consider the case of simply supported square plate with $n \times n$ uniform elements. Substituting Eqs. (10a) and (11) into Eq. (27) yields

$$P_x = \frac{2D(\cos r\psi + \cos s\psi - 2)^2}{a^2(1-\cos r\psi)}. \tag{28}$$

Expanding the right-hand side of Eq. (28) into a power series of ψ results in

$$P_x = \frac{D(r^2+s^2)^2\psi^2}{a^2r^2}\left[1 + \frac{r^2s^2 - r^4 - 2s^4}{12(r^2+s^2)}\psi^2 + O(\psi^4)\right] \tag{29}$$

and the magnitude of buckling load is

$$P_{\mathrm{cr}} = \frac{4D\pi^2}{L^2} - \frac{D\pi^4}{3L^2}n^{-2} + O(n^{-4}). \tag{30}$$

The first term on the right-hand side of Eq. (30) is in agreement with the analytical solution and then the second term represents the main error of the buckling load found by the finite difference method. Making a comparison between Eqs. (30) and (26), leads to the conclusion that the rates of convergence of finite difference formulae for both unidirectional and bi-directional compression are the same.

5. Natural frequency

Consider the equivalent system with cyclic periodicity in x and y directions shown in Fig. 2. The dynamic equations for all substructures are of the same form, i.e.,

$$D\nabla^4 w_{(j,k)} + \rho \frac{\partial^2 w_{(j,k)}}{\partial t^2} = F_{(j,k)}, \tag{31}$$

where ρ denotes the mass per unit area of plate.

Substituting the difference formula for $\nabla^4 w_j$ into the governing equation (31) results in

$$\begin{aligned} &\frac{D}{a^4}[(6 + 6\alpha^2 + 8\alpha)w_{(j,k)} - 4(1+\alpha)w_{(j,k-1)} - 4(1+\alpha)w_{(j,k+1)} - 4\alpha(1+\alpha)w_{(j-1,k)} \\ &\quad - 4\alpha(1+\alpha)w_{(j+1,k)} + 2\alpha w_{(j-1,k-1)} + 2\alpha w_{(j+1,k+1)} + 2\alpha w_{(j+1,k-1)} + 2\alpha w_{(j-1,k+1)} \\ &\quad + \alpha^2 w_{(j-2,k)} + \alpha^2 w_{(j+2,k)} + w_{(j,k-2)} + w_{(j,k+2)}] + \rho \frac{\partial^2 w_{(j,k)}}{\partial t^2} = F_{(j,k)}. \end{aligned} \tag{32}$$

Applying the double U-transformation (5) into Eq. (32) results in

$$\ddot{q}_{(r,s)} + \frac{DA}{\rho} q_{(r,s)} = \frac{f_{(r,s)}}{\rho}, \tag{33}$$

where A and $f_{(r,s)}$ have been defined as Eqs. (7) and (8).

Firstly, consider the natural vibration. Inserting $q_{(r,s)}(t) = Q_{(r,s)}\mathrm{e}^{\mathrm{i}\omega t}$ and $f_{(r,s)} = 0$ into Eq. (33) yields the frequency equation

$$\left(\frac{DA}{\rho} - \tilde{\omega}^2_{(r,s)}\right) Q_{(r,s)} = 0, \tag{34}$$

where $\tilde{\omega}_{(r,s)}$ denotes the natural frequency found by the finite difference method. The natural frequencies can be obtained from Eq. (34) as

$$\tilde{\omega}^2_{(r,s)} = \frac{DA}{\rho}. \tag{35}$$

Consider the simple case of a square plate with $n \times n$ uniform elements. Substituting Eq. (11) into Eq. (35) yields

$$\tilde{\omega}^2_{(r,s)} = \frac{4D}{\rho a^4}(\cos r\psi + \cos s\psi - 2)^2. \tag{36}$$

Expanding the right-hand side of Eq. (36) into a power series of ψ results in

$$\tilde{\omega}_{(r,s)} = \omega_{(r,s)}\left[1 - \frac{r^4 + s^4}{12(r^2+s^2)}\pi^2 n^{-2} + O(n^{-4})\right], \tag{37}$$

Table 2
Natural frequencies $\tilde{\omega}_{(r,s)}$ for simply supported square plate

	n					
(r,s)	2	4	8	16	32	∞
(1, 1)	15.6805 (−0.206)[a]	18.7245 (−0.0514)	19.4855 (−0.0129)	19.6758 (−0.00321)	19.7234 (−0.000803)	19.7392
(1, 2), (2, 1)		40.7233 (−0.175)	47.1918 (−0.0437)	48.8090 (−0.0109)	49.2133 (−0.00273)	49.3480
(2, 2)		62.7220 (−0.206)	74.8981 (−0.0514)	77.9422 (−0.0129)	78.7032 (−0.00321)	78.9568
Multiplier			$L^{-2}\sqrt{D/\rho}$			

[a]The numbers in the parentheses denote the relative errors.

where $\omega_{(r,s)}$ denotes the analytical solution for natural frequency defined as

$$\omega_{(r,s)} = (r^2 + s^2)\frac{\pi^2}{L^2}\sqrt{\frac{D}{\rho}}. \tag{38}$$

The natural frequencies found by the finite difference method converge from below the exact answers at an asymptotic rate of n^{-2} when n approaches infinity. Some numerical results for Eq. (37) are given in Table 2.

6. Dynamic response

Let a concentrated load of magnitude $p(t)$ being time dependent act at the center of the simply supported rectangular plate with $n \times m$ uniform elements where n and m are even. For this case, the loading for equivalent system may be expressed as

$$F_{(n/2,m/2)} = F_{(n+n/2,m+m/2)} = \frac{p(t)}{ab}, \quad F_{(n/2,m+m/2)} = F_{(n+n/2,m/2)} = -\frac{p(t)}{ab} \tag{39}$$

with other nodal loadings being equal to zero.

Substituting Eq. (39) into Eq. (8) yields

$$f_{(r,s)} = -\frac{2p(t)}{ab\sqrt{nm}}\mathrm{e}^{\mathrm{i}r\psi_1}\mathrm{e}^{\mathrm{i}s\psi_2}\sin\frac{r\pi}{2}\sin\frac{s\pi}{2} \tag{40}$$

and then inserting Eq. (40) in Eq. (33) results in

$$\ddot{q}_{(r,s)} + \frac{DA}{\rho}q_{(r,s)} = -\frac{2p(t)\mathrm{e}^{\mathrm{i}r\psi_1}\mathrm{e}^{\mathrm{i}s\psi_2}}{\rho ab\sqrt{nm}}\sin\frac{r\pi}{2}\sin\frac{s\pi}{2}. \tag{41}$$

The dynamic equation (41) can be solved by means of Duhamel integral, i.e.,

$$q_{(r,s)} = -\frac{2\mathrm{e}^{\mathrm{i}r\psi_1}\mathrm{e}^{\mathrm{i}s\psi_2}}{\rho ab\sqrt{nm}\tilde{\omega}_{(r,s)}}\sin\frac{r\pi}{2}\sin\frac{s\pi}{2}\int_0^t p(\tau)\sin\tilde{\omega}_{(r,s)}(t-\tau)\,\mathrm{d}\tau, \tag{42}$$

where $\tilde{\omega}_{(r,s)}$ has been defined by Eq. (36). Now the response function of the deflection can be obtained from the double U-transformation (5) with Eq. (42)

$$\tilde{w}_{(j,k)} = \frac{1}{\sqrt{2n}\sqrt{2m}} \sum_{r=1,3}^{2n-1} \sum_{s=1,3}^{2m-1} e^{i(j-1)r\psi_1} e^{i(k-1)s\psi_2} q_{(r,s)}. \tag{43}$$

The response function of the displacement at the center of the plate may be found by substituting $j = n/2$ and $k = m/2$ into Eq. (43), i.e.,

$$\tilde{w}_c(t) = \frac{4}{\rho L_1 L_2} \sum_{r=1,3}^{n-1} \sum_{s=1,3}^{m-1} \frac{1}{\tilde{\omega}_{(r,s)}} \int_0^t p(\tau) \sin \tilde{\omega}_{(r,s)}(t-\tau)\, d\tau. \tag{44}$$

The analytical solution may be obtained by the double sine series method. The center deflection can be expressed as

$$w_c(t) = \frac{4}{\rho L_1 L_2} \sum_{r=1,3}^{\infty} \sum_{s=1,3}^{\infty} \frac{1}{\omega_{(r,s)}} \int_0^t p(\tau) \sin \omega_{(r,s)}(t-\tau)\, d\tau. \tag{45}$$

The finite difference solution shown in Eq. (44) converges but does not converge uniformly to the analytical solution when the number of elements approaches infinity. The convergence rate is dependent on the characteristic of the loading function.

7. Conclusions

Explicit finite difference solutions for displacements, buckling loads, natural frequencies and dynamic responses have been developed to study the static and dynamic convergence. The two-dimensional finite difference formulae are studied, and the solutions are in agreement with the exact analytical ones. At the same time, the exact coefficients of the main error terms are derived. It is shown that the U-transformation method has great advantages in exact structural analysis.

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