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Transient elastic waves in a transversely isotropic laminate impacted by axisymmetric load

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Abstract

A method of reverberation-ray matrix has been extended to the investigation of the field of wave propagation in a transversely isotropic laminate. By using the decomposition in a local coordinate system, any complicated waves can be separated into a departing part and an arriving part, which are expressed in the local scattering matrix at structural interface. Together with the local phase matrix, we obtain a certain wave transmitting from a layer to the neighboring one. Thus, the wave propagation in the whole laminate can be described when assembling the local information with global phase and global permutation matrices. This method is perfectly suitable for evaluating the transient waves involving a large number of generalized-rays. In this paper, the method is applied to laminate made of transversely isotropic material. Numerical results show the influence of the change of thickness and elastic constants of the layers on the wave propagation in laminate.

1. Introduction

Studies of the propagation of elastic waves in the layered media have long been of interest to researchers in the fields of geophysics, acoustics, and nondestructive evaluation [1–7]. Common to all of these studies is the investigation of the degrees of interaction among the layers, which manifest themselves in the forms of reflection and transmission agents and give rise to geometric dispersion. These interactions depend, among other factors, upon the mechanical properties of

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media, geometric arrangements, the number of layers, interfacial conditions, and loading conditions. In recent years, continued efforts have been expended upon modeling wave propagation interacted with layered anisotropic media mostly in fields other than seismology, such as nondestructive evaluation. This interest has been prompted by the recent expansion of the use of composite materials in a wide variety of applications [8]. In comparison with the reasonable rich literature on the interaction of plane harmonic waves in anisotropic media, very little work is available on the response of such media to concentrated source loadings. Following the classical work of Lamb [9] who obtained the exact solution for the disturbances that are generated by an impulsive, concentrated load applied a line on the free surface of homogeneous isotropic elastic half-space, Kraut [10] examined the influence of transverse isotropy on such a problem. Van der Hijden [11] used the Cagniard's de-Hoop method [12] to study unbounded anisotropic media. Taylor [13] studied Lamb's problem for semi-space that has as low as monoclinic symmetry. By using numerical transforms, Weaver et al. [14] studied the dynamic response of a thick plate, whose axis of transverse isotropy is normal to the plate surface.

In this paper, we extend the newly developed method of reverberation-ray matrix [15–18] to the evaluation of the propagation of elastic waves in a transversely isotropic laminate to simulate the wave propagation in the composite material. We intend to demonstrate the high accuracy, simple computing process of this method, and to study the effects aroused by the change of thickness and elastic constants in the interlayers. By applying Hankel transform and Laplace transform into the spatial variable and the time variable, respectively, the integrals of wave numbers can express the steady-state waves generated by point source to the receiver in axisymmetric problem. Using the method of separating elastic waves into two different parts, the wave departing from and arriving at the same interface, the process of transmission and reflection at the interface of two adjacent layers is represented by a local scattering matrix. Associating with local phase matrix, which expresses the relationship between departing and arriving of waves shown by two opposite local coordinate systems in the same layer, the global scattering matrix and phase matrix of the laminate can be expressed by assembling the local ones of all layers, respectively. Their product together with a global permutation matrix gives rise to the reverberation-ray matrix $\mathbf{R}$, which represents the multi-reflected and transmitted steady-state waves within the entire medium. The transient waves are then determined by another integration consisting of two inverse-numerical transforms, known as the ray-integrals that contain a power series of $\mathbf{R}$. After satisfying the convergence criteria of ray expansion by adjusting the real part of the complex number p, which is a parameter in the Laplace transform, we can approximately calculate the ray-integrals. This method is particularly suitable for evaluating the transient waves involving a large number of generalized-rays by calculating the double integrals numerically as illustrated by the examples of laminates, which are made of transversely isotropic material in this paper. Furthermore, we discuss the influence of the change of thickness and elastic constants in the layers on the wave propagation in laminate.

2. Global and local coordinate system

Consider a laminate consisting of m transversely isotropic layers impacted by axisymmetric load, we adopt capital letters such as $I, J, \ldots$ to denote the interfaces of the laminate, and then denote the layer by using the name of its surface, such as $IJ, JK, \ldots$. In each layer, we define two local cylindrical coordinate systems $(r,z)^{IJ}$ and $(r,z)^{JI}$, whose origins are on the upper and lower surface, respectively, shown in Fig. 1. If the thickness of a certain layer is h^{IJ}, we deduce the relationship equations as follows:

$$r^{IJ} = r^{JI}, \quad z^{IJ} = h^{IJ} - z^{JI}. \tag{1}$$

Using the definition of displacements and stresses, we get their relations between two local coordinate systems $(r,z)^{IJ}$ and $(r,z)^{JI}$.

$$u_r^{IJ}(r^{IJ}, z^{IJ}) = u_r^{JI}(r^{JI}, h^{IJ} - z^{IJ}), \tag{2}$$

$$u_z^{IJ}(r^{IJ}, z^{IJ}) = -u_z^{JI}(r^{JI}, h^{IJ} - z^{IJ}), \tag{3}$$

$$\tau_{zz}^{IJ}(r^{IJ}, z^{IJ}) = \tau_{zz}^{JI}(r^{JI}, h^{IJ} - z^{IJ}), \tag{4}$$

and

$$\tau_{zr}^{IJ}(r^{IJ}, z^{IJ}) = -\tau_{zr}^{JI}(r^{JI}, h^{IJ} - z^{IJ}). \tag{5}$$

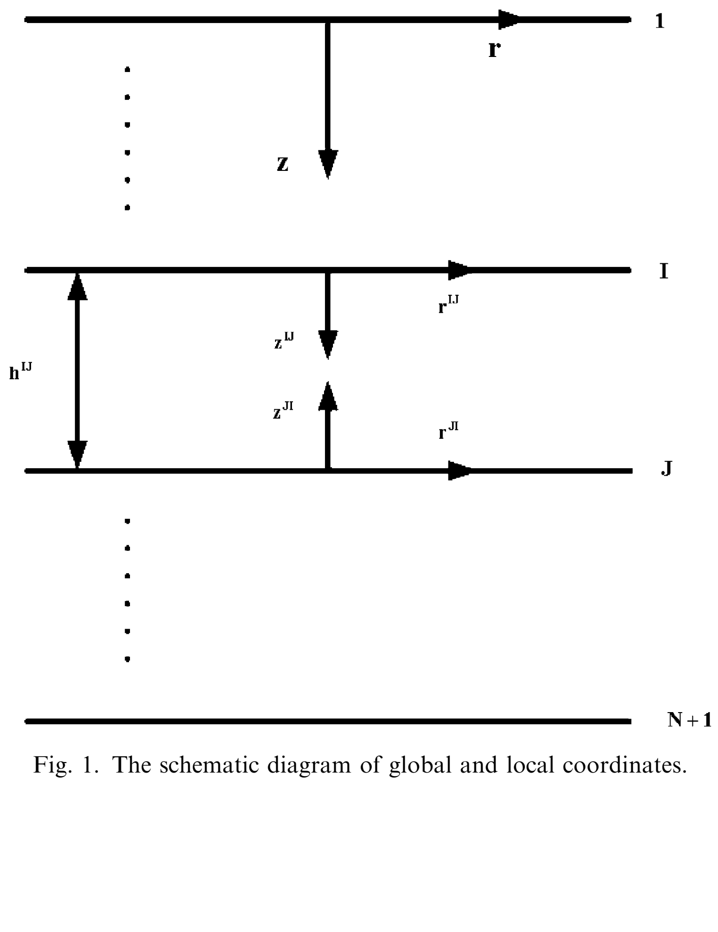

Fig. 1. The schematic diagram of global and local coordinates.

3. Elastic waves in a transversely isotropic laminate

The wave equations for an axisymmetric problem in a local coordinate system $(r,z)^{IJ}$ are expressed as

$$C_{55}\frac{\partial^2 u_r}{\partial z^2}+C_{11}\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r}\right)+(C_{13}+C_{55})\frac{\partial^2 u_z}{\partial z\,\partial r}-\rho\frac{\partial^2 u_r}{\partial t^2}=0, \tag{6}$$

$$(C_{13}+C_{55})\left(\frac{\partial^2 u_r}{\partial z\,\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+C_{33}\frac{\partial^2 u_z}{\partial z^2}+C_{55}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)-\rho\frac{\partial^2 u_z}{\partial t^2}=0, \tag{7}$$

where u_r and u_z are the displacements in r and z directions, respectively. ρ is the density of the material, and C_{11}, C_{55}, C_{13}, and C_{33} are the independent elastic constants of the transversely isotropic material. Then the stresses in the plate are expressed as

$$\tau_{zz}=C_{13}\left(\frac{u_r}{r}+\frac{\partial u_r}{\partial r}\right)+C_{33}\frac{\partial u_z}{\partial z}, \tag{8}$$

$$\tau_{zr}=C_{55}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right). \tag{9}$$

The Hankel transform and Laplace transform are defined, respectively, as

$$\hat{f}^{Hv}(k,z,t)=\int_0^\infty f(r,z,t)J_v(kr)r\,\mathrm{d}r, \tag{10}$$

$$\hat{f}(r,z,p)=\int_0^\infty f(r,z,t)\mathrm{e}^{-pt}\,\mathrm{d}t. \tag{11}$$

Application of Laplace transforms in time variable t into Eqs. (1) and (2), and then Hankel transforms of rank one and zero in spatial variable r into displacement u_r and u_z, respectively, yields

$$C_{55}\frac{\partial^2\hat{u}_r^{H1}}{\partial z^2}-C_{11}k^2\hat{u}_r^{H1}-(C_{13}+C_{55})k\frac{\partial\hat{u}_z^{H0}}{\partial z}-\rho p\hat{u}_r^{H1}=0, \tag{12}$$

$$(C_{13}+C_{55})k\frac{\partial\hat{u}_r^{H1}}{\partial z}+C_{33}\frac{\partial^2\hat{u}_z^{H0}}{\partial z^2}-C_{55}k\hat{u}_z^{H0}-\rho p\hat{u}_z^{H0}=0. \tag{13}$$

Furthermore, so do Eqs. (8) and (9), we obtain

$$\hat{\tau}_{zz}^{H0}=C_{13}k\hat{u}_r^{H1}+C_{33}\frac{\partial\hat{u}_z^{H0}}{\partial z}, \tag{14}$$

$$\hat{\tau}_{zr}^{H1}=C_{55}\frac{\partial\hat{u}_r^{H1}}{\partial z}+C_{55}k\hat{u}_z^{H0}. \tag{15}$$

Then the formal solutions of Eqs. (12) and (13) can be expressed as follows:

$$\hat{u}_r^{H1}=a_1\mathrm{e}^{s_1z}+a_2\mathrm{e}^{s_2z}+d_1\mathrm{e}^{-s_1z}+d_2\mathrm{e}^{-s_2z}, \tag{16}$$

$$\hat{u}_z^{H0}=a_1w_1\mathrm{e}^{s_1z}+a_2w_2\mathrm{e}^{s_2z}+d_1w_3\mathrm{e}^{-s_1z}+d_2w_4\mathrm{e}^{-s_2z}, \tag{17}$$

where s_1^2 and s_2^2 are the roots of quadratic algebraic equation

$$Ax^2 + Bx + C = 0, \tag{18}$$

whose coefficients are defined as

$$A = C_{33}C_{55}, \tag{19}$$

$$B = -C_{55}\rho p^2 - c_{11}k^2C_{33} - \rho p^2C_{33} + 2k^2C_{55}C_{13} + k^2C_{13}^2, \tag{20}$$

$$C = C_{11}k^4C_{55} + C_{11}k^2\rho p^2 + \rho p^2C_{55}k^2 + \rho^2p^4, \tag{21}$$

and

$$w_{1,2} = (C_{55}s_1^2 - C_{11}k^2 - \rho p^2)/((C_{55} + C_{13})ks_{1,2}), \tag{22}$$

$$w_{3,4} = -w_{1,2}. \tag{23}$$

Defining the arriving wave and departing wave displacement vectors $\hat{\mathbf{a}} = \{a_1, a_2\}^{\mathrm{T}}$ and $\hat{\mathbf{d}} = \{d_1, d_2\}^{\mathrm{T}}$, the displacement vector $\hat{\mathbf{U}}(k,z,p) = \{\hat{u}_r^{H1}(k,z,p), \hat{u}_z^{H0}(k,z,p)\}^{\mathrm{T}}$ can be denoted as

$$\hat{\mathbf{U}}(k,z,p) = \mathbf{A}_u(k,z,p)\hat{\mathbf{a}} + \mathbf{D}_u(k,z,p)\hat{\mathbf{d}}, \tag{24}$$

where $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are the unknown amplitudes,

$$\mathbf{A}_u = \begin{bmatrix} \mathrm{e}^{s_1z} & \mathrm{e}^{s_2z} \\ w_1\mathrm{e}^{s_1z} & w_2\mathrm{e}^{s_2z} \end{bmatrix}, \quad \mathbf{D}_u = \begin{bmatrix} \mathrm{e}^{-s_1z} & \mathrm{e}^{-s_2z} \\ w_3\mathrm{e}^{-s_1z} & w_4\mathrm{e}^{-s_2z} \end{bmatrix}$$

and the stress vector $\hat{\mathbf{F}}(k,z,p) = \{\hat{\tau}_{zr}^{H1}(k,z,p), \hat{\tau}_{zz}^{H0}(k,z,p)\}^{\mathrm{T}}$ is expressed as

$$\hat{\mathbf{F}}(k,z,p) = \mathbf{A}_f(k,z,p)\hat{\mathbf{a}} + \mathbf{D}_f(k,z,p)\hat{\mathbf{d}}, \tag{25}$$

where

$$\mathbf{A}_f = \begin{bmatrix} C_{55}(s_1 - kw_1)\mathrm{e}^{s_1z} & C_{55}(s_2 - kw_2)\mathrm{e}^{s_2z} \\ (C_{13}k + C_{33}w_1s_1)\mathrm{e}^{s_1z} & (C_{13}k + C_{33}w_2s_2)\mathrm{e}^{s_2z} \end{bmatrix},$$

$$\mathbf{D}_f = \begin{bmatrix} C_{55}(-s_1 - kw_3)\mathrm{e}^{-s_1z} & C_{55}(-s_2 - kw_4)\mathrm{e}^{-s_2z} \\ (C_{13}k - C_{33}w_3s_1)\mathrm{e}^{-s_1z} & (C_{13}k - C_{33}w_4s_2)\mathrm{e}^{-s_2z} \end{bmatrix}.$$

4. Scattering matrices for waves at interfaces

For simplification, we first consider a vertical point force $F(t) = -F_0H(t)$ acting on the location $r = 0$, and $z = z^J$ (interface J) of the plate with zero initial condition. This method is also suitable to more complicated cases. According to the interface conditions, we obtain four continuity conditions at the interface J.

$$\hat{u}_r^{J(J-1)}(\xi, 0, p) - \hat{u}_r^{J(J+1)}(\xi, 0, p) = 0, \tag{26}$$

$$\hat{u}_z^{J(J-1)}(\xi,0,p)+\hat{u}_z^{J(J+1)}(\xi,0,p)=0, \tag{27}$$

$$\hat{\tau}_{rz}^{J(J-1)}(\xi,0,p)+\hat{\tau}_{rz}^{J(J+1)}(\xi,0,p)=0, \tag{28}$$

$$\hat{\tau}_{zz}^{J(J-1)}(\xi,0,p)-\hat{\tau}_{zz}^{J(J+1)}(\xi,0,p)=-\hat{F}(p)/2\pi, \quad J=2\ldots m-1. \tag{29}$$

At the upper and lower surface of laminate, the continuity conditions are rewritten as

$$\hat{\tau}_{zz}^{12}=-\hat{F}(p)/2\pi, \tag{30}$$

$$\hat{\tau}_{zr}^{12}=0, \tag{31}$$

$$\hat{\tau}_{zz}^{m(m-1)}=\hat{F}(p)/2\pi, \tag{32}$$

$$\hat{\tau}_{zr}^{m(m-1)}=0. \tag{33}$$

Substituting Eqs. (24) and (25) into the above equations yields

$$\mathbf{A}^J\mathbf{a}^J+\mathbf{D}^J\mathbf{d}^J=\hat{\mathbf{g}}^J, \tag{34}$$

where $\mathbf{A}^J$ and $\mathbf{D}^J$ are 4×4 matrices. $\mathbf{a}^J$ and $\mathbf{d}^J$ are unknown vectors which represent amplitudes of waves departing from and arriving at interface J, respectively, which are expressed as

$$\mathbf{a}^J=\{a_1^{J(J-1)},a_2^{J(J-1)},a_1^{J(J+1)},a_2^{J(J+1)}\}^{\mathrm{T}},$$

$$\mathbf{d}^J=\{d_1^{J(J-1)},d_2^{J(J-1)},d_1^{J(J+1)},d_2^{J(J+1)}\}^{\mathrm{T}}.$$

For the surface of laminate, they are defined as

$$\mathbf{a}^1=\{a_1^{12},a_2^{12}\}^{\mathrm{T}}, \tag{35}$$

$$\mathbf{d}^1=\{d_1^{12},d_2^{12}\}^{\mathrm{T}}, \tag{36}$$

$$\mathbf{a}^m=\{a_1^{m(m-1)},a_2^{m(m-1)}\}^{\mathrm{T}}, \tag{37}$$

$$\mathbf{d}^m=\{d_1^{m(m-1)},d_2^{m(m-1)}\}^{\mathrm{T}}. \tag{38}$$

$\hat{\mathbf{g}}^J=\{0,0,0,-\hat{F}(p)/2\pi\}^{\mathrm{T}}$ is a source vector, and for the surface of laminate, it is denoted as

$$\hat{\mathbf{g}}^J=\{0,-\hat{F}(p)/2\pi\}^{\mathrm{T}}.$$

Solving $\mathbf{d}^J$ in terms of unknown vector $\mathbf{a}^J$ and a given source vector $\hat{\mathbf{g}}^J$ in Eq. (34) leads to

$$\mathbf{d}^J=\mathbf{S}^J\mathbf{a}^J+\mathbf{s}^J, \tag{39}$$

where $\mathbf{S}^J=-(\mathbf{D}^J)^{-1}\mathbf{A}^J$ is called the scattering matrix at Jth interface, and $\mathbf{s}^J=-(\mathbf{D}^J)^{-1}\hat{\mathbf{g}}^J$ is called the source wave vector at Jth interface.

5. Reverberation matrix

In the global coordinate system, we construct a system of equations for the laminate in compact notation

$$\mathbf{d} = \mathbf{S}\mathbf{a} + \mathbf{s}, \tag{40}$$

where the global arriving wave amplitude vector $\mathbf{a} = \{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^N\}^{\mathrm{T}}$ represents all waves arriving at all interfaces, and the global departing wave amplitude vector $\mathbf{d} = \{\mathbf{d}^1, \mathbf{d}^2, \ldots, \mathbf{d}^N\}^{\mathrm{T}}$ represents all waves departing from all interfaces. The square matrix $\mathbf{S}$ is called the global scattering matrix. The vector $\hat{\mathbf{s}}$ is called the global source wave vector.

Since both vectors $\mathbf{a}$ and $\mathbf{d}$ are unknown quantities, we need an additional equation related to $\mathbf{a}$ and $\mathbf{d}$. Consider a wave arriving at the interface J in the local coordinate $(r, z)^{J(J+1)}$, which is expressed as

$$\hat{\mathbf{U}}^{J(J+1)} = \mathbf{A}_u^J(\xi, z^{J(J+1)}, p)\hat{\mathbf{a}}^{J(J+1)}. \tag{41}$$

But in the local coordinate $(r, z)^{(J+1)J}$, the above wave is also considered to be the wave departing from the other surface of the same layer, which is denoted as

$$\hat{\mathbf{U}}^{(J+1)J} = \mathbf{D}_u^{(J+1)J}(\xi, z^{(J+1)J}, p)\hat{\mathbf{d}}^{(J+1)J}. \tag{42}$$

Substituting Eqs. (41) and (42) into Eqs. (2) and (3) yields

$$\hat{\mathbf{a}}^{J(J+1)} = \mathbf{P}^{J(J+1)}\hat{\mathbf{d}}^{(J+1)J}, \tag{43}$$

where

$$\mathbf{P}^{J(J+1)} = \begin{bmatrix} \mathrm{e}^{-s_1 h^{J(J+1)}} & 0 \\ 0 & -\mathrm{e}^{-s_2 h^{J(J+1)}} \end{bmatrix}$$

is the local phase matrix.

For interface J, we stack Eq. (43) into

$$\mathbf{a}^J = \mathbf{P}^J\tilde{\mathbf{d}}^J, \tag{44}$$

where

$$\mathbf{P}^J = \begin{bmatrix} \mathbf{P}^{J(J-1)} & \\ & \mathbf{P}^{J(J+1)} \end{bmatrix}, \quad \text{and} \quad \tilde{\mathbf{d}}^J = \{\mathbf{d}^{(J-1)J}, \mathbf{d}^{(J+1)J}\}^{\mathrm{T}}.$$

In the global coordinate system, we obtain

$$\mathbf{a} = \mathbf{P}\tilde{\mathbf{d}}, \tag{45}$$

where $\mathbf{d} = \{[\mathbf{d}^1]^{\mathrm{T}}, [\mathbf{d}^2]^{\mathrm{T}}, \ldots, [\mathbf{d}^n]^{\mathrm{T}}\}^{\mathrm{T}}$, and

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}^1 & & & \\ & \mathbf{P}^2 & & \\ & & \ddots & \\ & & & \mathbf{P}^n \end{bmatrix}$$

is called global phase matrix.

The global vectors $\tilde{\mathbf{d}}$ and $\mathbf{d}$ contain the same elements but sequenced in different order. We may express this equivalence through a permutation matrix $\mathbf{H}$,

$$\tilde{\mathbf{d}} = \mathbf{H}\mathbf{d}, \tag{46}$$

where $\mathbf{U}$ is a $4m \times 4m$ matrix composed of only one element whose value is one in each line and each row and others are all zero. For example, in matrix $\mathbf{d}$, if d_i^{JK} and d_i^{KJ} are in the positions p and q, respectively, then the elements H_{pq} and H_{qp} in the matrix $\mathbf{H}$ have the same value one.

Substituting Eqs. (46) into (45) deduces

$$\mathbf{a} = \mathbf{PHd}. \tag{47}$$

Solving Eqs. (47) and (40) leads to

$$\mathbf{d} = [\mathbf{I} - \mathbf{R}]^{-1}\mathbf{s}, \tag{48}$$

$$\mathbf{a} = \mathbf{PH}[\mathbf{I} - \mathbf{R}]^{-1}\mathbf{s}, \tag{49}$$

where $\mathbf{I}$ is an identity matrix, and $\mathbf{R} = \mathbf{SPH}$ is called reverberation matrix.

6. Transient waves in the laminate

Once the vectors $\mathbf{d}$ and $\mathbf{a}$ are known from Eqs. (48) and (49), the complete list of displacements in frequency domain will be denoted as

$$\hat{\mathbf{U}} = (\mathbf{A}_u\mathbf{PH} + \mathbf{D}_u)(\mathbf{I} - \mathbf{R})^{-1}\mathbf{s}. \tag{50}$$

Applying the inverse Laplace transform and the inverse Hankel transform, the transient responses are expressed as

$$\mathbf{U}(r, z, t) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \int_{Br} \hat{\mathbf{U}}(k, z, p) \mathrm{e}^{pt} \mathbf{J}_v(kr) k \, \mathrm{d}p \, \mathrm{d}k. \tag{51}$$

In order to simplify the numerical computation, the inverse of the matrix $[\mathbf{I} - \mathbf{R}]$ in the integrand of the double integral is replaced by the power series $[\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \cdots + \mathbf{R}^N + \cdots]$ through the Neumann expansion, and the original double integral that is singular at the poles of $\det[\mathbf{I} - \mathbf{R}] = 0$ is then converted into a series of double integrals, known as the ray-integrals. So Eq. (51) is rewritten as

$$\mathbf{U}(r, z, t) = \sum_{K=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_0^\infty \int_{Br} [\mathbf{A}_u\mathbf{PH} + \mathbf{D}_u]\mathbf{R}^K \mathbf{s} \mathbf{J}_v(kr) k \mathrm{e}^{pt} \, \mathrm{d}p \, \mathrm{d}k. \tag{52}$$

Here, each term in the above integral containing $\mathbf{R}^K$ can be defined as a group of rays, which represents the set of K times reflections and transmissions of the source waves arriving at receivers at (r, z) in the laminate. When $K = 0$, the group of rays shows the waves from sources to the receivers directly, it is called as source wave. Here, every group of rays contains a series of generalized rays [19], and the number of generalized rays increases exponentially when the number of layer or K is increased. The inverse Laplace transform and Hankel transform of Eq. (52) are computed numerically by a fast algorithm based on fast Fourier transform (FFT) [20].

7. Numerical examples

7.1. Five-layer transversely isotropic laminate with equal-thickness layers

Firstly, we consider a laminate including five transversely isotropic layers with the same thickness shown in Fig. 2. This laminate consists of two kinds of material (materials A and B), whose lay-up is [ABABA]. The elastic constants are listed, respectively:

$$\text{A:}\, C_{11}/C_{55} = 159.4/38.9, \quad C_{12}/C_{55} = C_{13}/C_{55} = 73.9/38.9, \quad C_{33}/C_{55} = 126.1/38.9, \tag{53}$$

$$\text{B:}\, C_{11}/C_{55} = 318.8/38.9, \quad C_{12}/C_{55} = C_{13}/C_{55} = 147.8/38.9, \quad C_{33}/C_{55} = 252.2/38.9. \tag{54}$$

A vertical force, $-F_0H(t)$, acts at a point of the top surface, $r = 0$ and $z = 0$. Two receivers are set at points C $(2.39h, 0.05h)$ and D $(2.39h, 4.95h)$, respectively. In the subsequent analysis, the normalized vertical displacement U_z and the normalized time τ are defined as

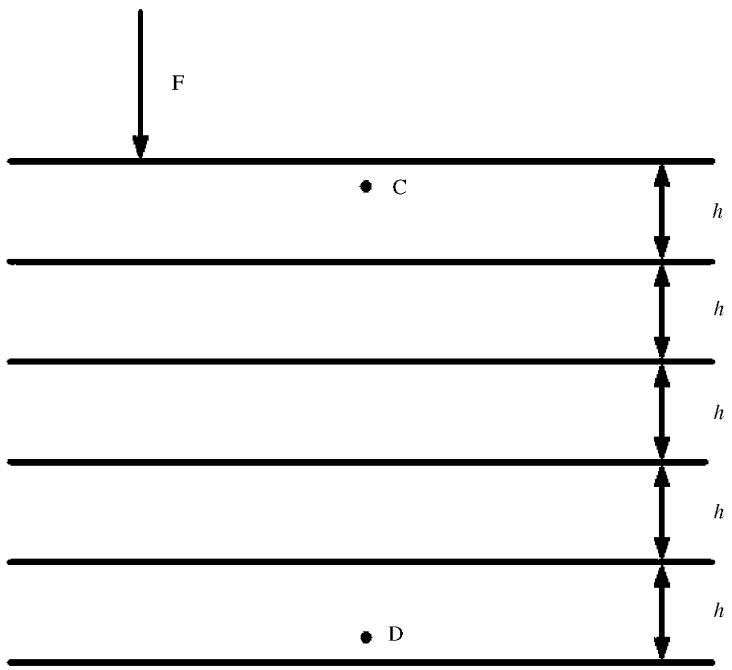

Fig. 2. Position of two receivers in the five-layered transversely isotropic laminate.

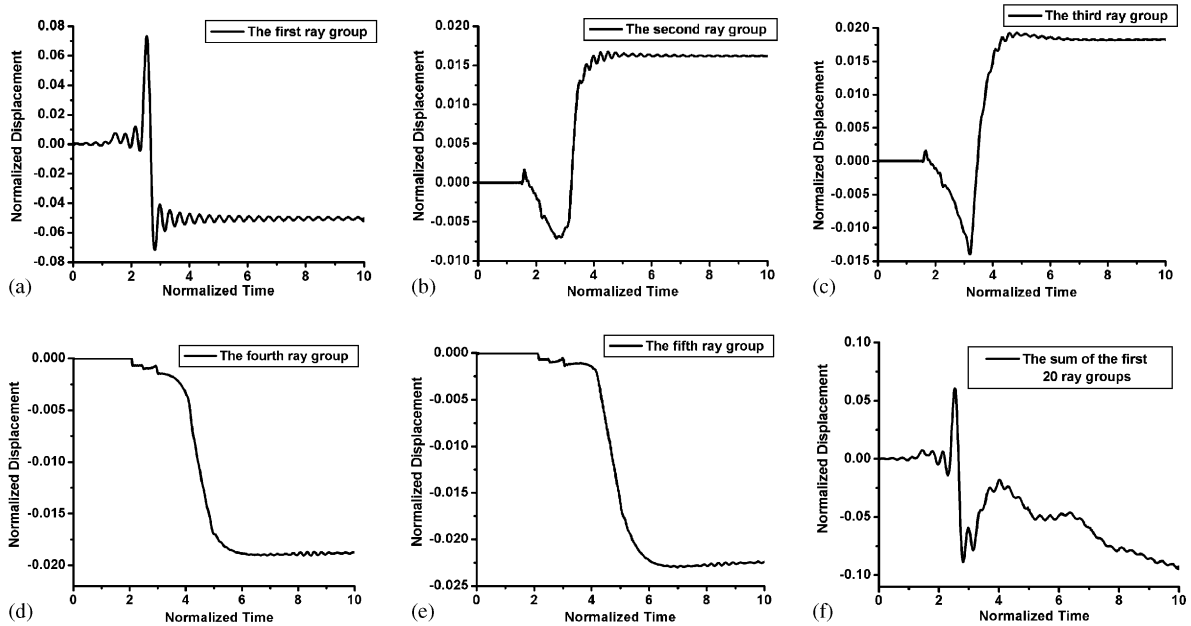

Fig. 3. The vertical displacement at receiver C in the five-layered transversely isotropic laminate. (a) The first ray group, (b) the second ray group, (c) the third ray group, (d) the fourth ray group, (e) the fifth ray group, (f) the sum of the first 20 ray groups.

$U_z = u_z/(F_0/(C_{55}h))$ and $\tau = \sqrt{C_{11}/\rho}t/h$, respectively. The wave velocity is normalized by $c_s^2 = C_{55}/\rho$, where c_s is the velocity of S-wave in the isotropic material.

The first ray group arrived at point C is shown in Fig. 3(a). For isotropic material, the elastic waves are decoupled into P- and S-wave, but for transversely isotropic material, the velocity of elastic wave is determined by the direction of wave propagation, and P- and S-wave are coupled with each other, which is called quasi-P-wave and quasi-S-wave. The quasi-P-wave and the quasi-S-wave arrive at point A at $\tau \approx 1.18$ and 2.28, and the surface wave arrives at $\tau \approx 2.48$, all of which are close to the theoretical analysis. The second ray group, the third ray group, the fourth ray group, the fifth ray group, and the sum of first 20 ray groups of the vertical displacement at receiver C are given in Fig. 3(b)–(f), respectively.

The normalized vertical displacement at the receiver D is shown in Fig. 4. Because the first four ray groups do not arrive at receiver D, the fifth ray group shown in Fig. 4(a), will generate the response at receiver D firstly. For the complexity of elastic wave in transversely isotropic material, we cannot obtain the exact time of the quasi-P-wave arriving at the receiver. By using theoretical method, the approximate time is $\tau \approx 2.5$, which is the same as that shown in this figure. The sum of the first 20 ray groups of the vertical displacement at receiver D is given in Fig. 4(b).

7.2. Five-layer transversely isotropic laminate with different thickness and elastic constants of layers

Secondly, in order to discuss the influence of elastic constants and thickness of the layers, we take two five-layer laminates for example similar to the first case. Each laminate consists of five

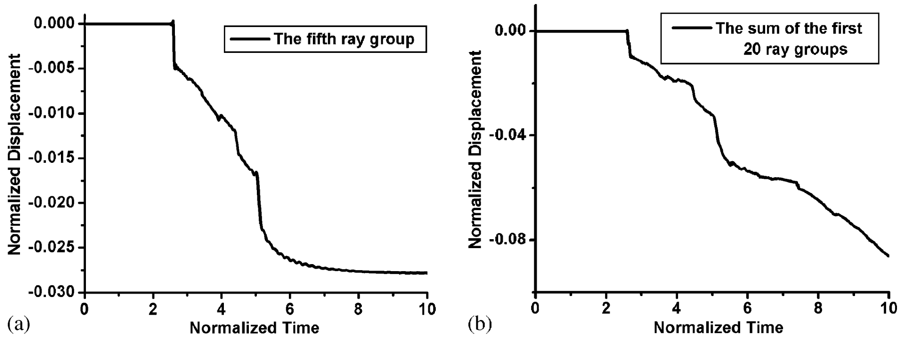

Fig. 4. The vertical displacement at receiver D in the five-layered transversely isotropic laminate. (a) The fifth ray group, (b) the sum of the first 20 ray groups.

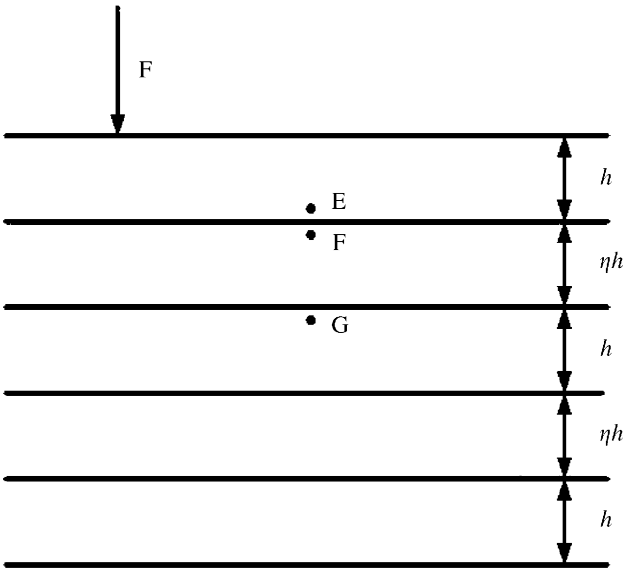

Fig. 5. Position of the three receivers in the five-layered transversely isotropic laminate with different thickness and elastic constants.

layers of transversely isotropic lamina; in the odd layers, the thickness is $h_1 = h$, the density is ρ_1, and the elastic constants are $C_{11}/C_{55} = 159.4/38.9$, $C_{12}/C_{55} = C_{13}/C_{55} = 73.9/38.9$, and $C_{33}/C_{55} = 126.1/38.9$; the thickness of the even layers is $h_2 = \eta h_1 = \eta h$, the density is $\rho_2 = \rho_1$, and ξ is expressed as the ratio of the elastic constants of the even layers to that of the odd layers. Being similar to the first example, a vertical force, $-F_0H(t)$, acts at the central point of the top surface, $r = 0$ and $z = 0$. Three receivers are set at points E $(2.39h, 0.05h)$ in the first layer, F $(2.39h, 1.05h)$ in the second layer, and G $(2.39h, 2.05h)$ in the third layer, respectively, as shown in Fig. 5.

Fig. 6 shows the comparison of the vertical displacement aroused by the first ray group at receiver E. We conclude from the four cases shown in Fig. 6 that the thickness and elastic

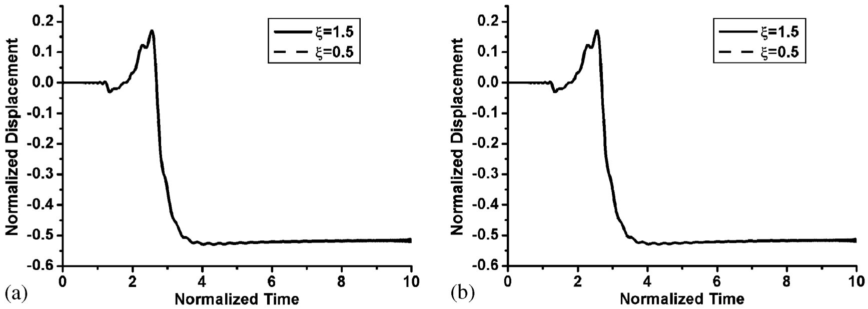

Fig. 6. The vertical displacement of the first ray group at receiver E. (a) $\eta = 1$, (b) $\eta = \frac{1}{10}$.

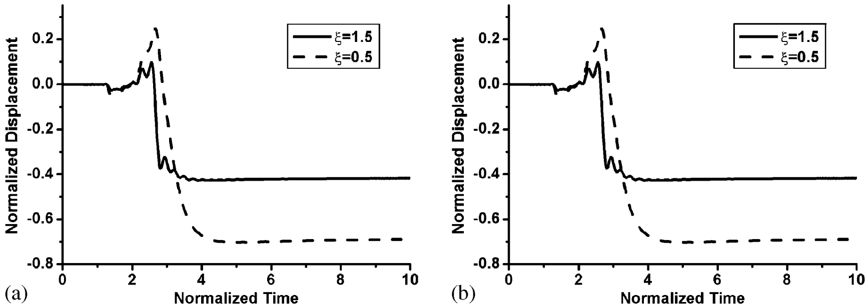

Fig. 7. The vertical displacement of the second ray group at receiver F. (a) $\eta = 1$, (b) $\eta = \frac{1}{10}$.

constants of the even layers have little influence on the wave propagation for the source wave. Due to the transversely isotropic material, there are also quasi-P-wave and quasi-S-wave in the laminate. Using the theoretical analysis [8], we deduce the velocities of quasi-P-wave and quasi-S-wave to be approximately equal to 2.02 and 1, so quasi-P-wave and quasi-S-wave will arrive at receiver E at $\tau_p \approx 1.29$ and $\tau_s \approx 2.61$, respectively, which are consistent with those shown in the figures.

Fig. 7 shows the comparison of the vertical displacement caused by the second ray group at receiver F. We notice the influence of the elastic constants in the even layers. However the thickness of even layers does not affect the wave propagation because this ray group cannot arrive at the bottom of the second layer, so there is no distinction between two parts of the figure. We can compute the wave velocity and the arriving time in these cases by using the theoretical analysis [8]. In the case of $\xi = 1.5$, the velocity of quasi-P-wave and quasi-S-wave are $C_p \approx 2.47$ and $C_s \approx 1.22$; in the case of $\xi = 0.5$, they are $C_p \approx 1.43$ and $C_s \approx 0.71$, respectively. In the two cases, the fastest wave, P–P wave arrives at $\tau \approx 1.28$ and 1.32 wave, respectively, and that of the slowest wave, S–S wave at $\tau \approx 2.59$ and 2.66, respectively. We notice from the figures that the peaks appear when the S–S waves arrive, which are similar to the theoretical analysis.

Fig. 8 shows the comparison of the vertical displacements caused by the third ray group at receiver G. Here, the thickness and elastic constants of the even layers affect the wave propagation more complicatedly, so the two figures are obviously different. In the case that the thickness of

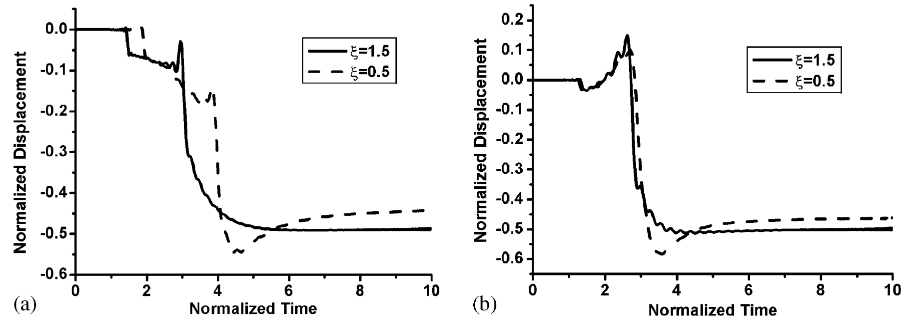

Fig. 8. The vertical displacement of the third ray group at receiver G. (a) $\eta = 1$, (b) $\eta = \frac{1}{10}$.

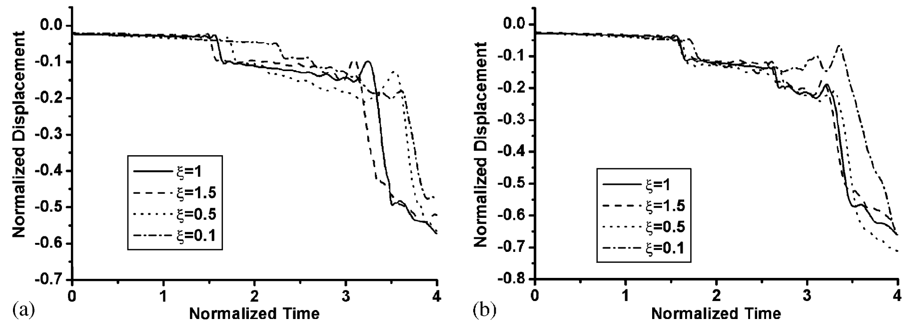

Fig. 9. The vertical displacement of the sum of the first ten ray groups at receiver G. (a) $\eta = 1$, (b) $\eta = \frac{1}{10}$.

even layers is $\frac{1}{10}$ of that of the odd ones, the delay caused by the decrease of the elastic constants of the interlayer is not obvious. On the contrary, in the other case the delay is more obvious, which cannot be ignored.

Fig. 9 shows the comparison of the vertical displacement aroused by the sum of the first ten ray groups at receiver G with two kinds of thickness ($\eta = 1, \frac{1}{10}$) and four kinds of elastic constants ($\xi = 1, 1.5, 0.5, 0.1$). We note that when the thickness of interlayer is comparatively small enough (such as $\frac{1}{10}$ of the thickness of the other layers), the change of its elastic constants affects less on the wave propagation so that it can be omitted in these instances. This is important for wave propagation in binding layer or dope layer in the laminate. If we approximately omit the effects of the folium, we will save the more computing time. But the data of $\xi = 0.1$ give some limits of the theory. If the elastic constants of the interlayer are extraordinarily small, just like interspaces, bugs and cracks in the layer, the effect is comparatively distinct so that we must consider their influences.

8. Conclusions

The method of reverberation-ray matrix is extended to investigate elastic waves in the transversely isotropic laminate. The axisymmetric waves in the time and spatial field are expressed via the Fourier–Hankel transform and the method of reverberation-ray matrix. With the advance of digital computer and numerical method, the ray-integrals, double integral in frequency and wave number, are evaluated numerically by applying an algorithm in fast Hankel transform and fast Fourier transform.

In the example of a five-layered laminate, the summation of the first eight ray groups containing more than 10 000 rays, can be obtained by this method in only one procedure. It would be unthinkable to handle such a large number of generalized-ray integral by applying the Cagniard's de-Hoop method. So this method is more efficient to reduce the workload of numerical evaluation than the classical generalized-ray method.

Using this theory, we have also discussed the effects aroused by the change of thickness and elastic constants in the layers. The results show that when the thickness of interlayer is comparatively small enough, the change of its elastic constants, if in the same magnitude, barely affects the wave propagation and can be omitted in some instances. But if the material constant of the interlayer is extraordinarily small, the effect is comparatively distinct, which cannot be ignored.

Acknowledgements

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