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Multiple scattering of a spherical acoustic wave from fluid spheres

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Abstract

The multiple scattering of a spherical acoustic wave from an arbitrary number of fluid spheres is investigated theoretically. The tool to attack the multiple scattering problem is a kind of addition formulas for the spherical wave functions, which are presented in the paper, based on the bicentric expansion form of Green function in the spherical coordinates. For an arbitrary configuration of N fluid spheres, the kind of addition formulas permits the field expansions (all referred to the center of each sphere). With these the sound fields scattered by each sphere can be described by a set of N equations. The interactions between any two fluid spheres are taken into account in these equations exactly and their coefficients are coupled through double sums in the spherical wave functions. By truncating the infinite series in the equations depending on certain calculation accuracy and solving the coefficients matrix by using the Gauss–Seidel iteration method, we can obtain the scattered sound field by the configuration of the fluid spheres. Finally, the scattering calculations by using the kind of addition formulas are carried out.

1. Introduction

Acoustic scattering by multiple objects is an important problem with various practical applications. The simplest realistic problem of multiple scattering by finite bodies

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appears to be that by two spheres. Many publications on this subject can be found in Refs. [1–6].

In 1967 Liang and Lo [1] considered the scattering of electromagnetic waves by two spheres using a translational addition theorem by Stein [2] and Cruzan [3]. Even with the help of several previous theoretical studies and the availability of high-speed computers, their numerical evaluation had to be limited to the spheres of radii less than three quarters of wavelength and wide spacing, due to the complexity of the addition theorem. Twersky [4] considered a more general problem with many scatterers, using dyadic Green function approach. Besides, Bruning and Lo [5] obtained an exact solution to the scattering of a plane electromagnetic wave by two spheres using the translational addition theorems for vector spherical wave functions given by Cruzan [3].

Recently, Gaunaurd et al. [6] used the addition theorems for the spherical wave functions to consider two spheres insonified by plane waves at arbitrary angles of incidence.

More recently the backscattering of sound from two regularly arranged bubbles is studied theoretically and experimentally by Kapodistrias and Dahl [7], whose scattering calculations are carried out using a closed-form solution derived from the multiple scattering series.

In this paper a kind of addition formulas for the spherical wave functions by using the bicentric expansion of Green function is presented first. Further, the multiple scattering of a spherical acoustic wave from an arbitrary number of fluid spheres is considered. The treatment of the multiple-scattering problem is analytical and exact. The kind of addition formulas that we generate in the text is valid and important for the acoustic problem. The accuracy of the solution depends on the accuracy used to determine the coefficients that appear as the solutions of a series of infinite equations. Finally, the computing results are validated by comparison to Fig. 8 in Ref. [7], which was obtained for backscattering from two bubbles of radius 585 μm at a frequency of 110 kHz and variable distance between the centers of each bubble.

2. Bicentric expansion of Green function in the spherical coordinates

The bicentric expansion form of Green function in the spherical coordinates was mentioned in Refs. [8,9]. Here, we present the bicentric expansion of Green function in detail.

The bicentric Green function in this paper is defined in the following non-homogeneous Helmholtz's equation:

$$(\nabla_\alpha^2 + K^2)G(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{R}_{\alpha\beta}; K) = \delta(\mathbf{r}_\alpha - \mathbf{r}_\beta - \mathbf{R}_{\alpha\beta}), \tag{1}$$

where K is the wavenumber of the medium. As shown in Fig. 1, α and β are the two different centers and $R_{\alpha\beta}$ denotes the distance between them. Let the origin of the spherical coordinate system (r, θ, φ) located at the center α and the coordinates of the point β are $(R_{\alpha\beta}, \theta_{\alpha\beta}, \varphi_{\alpha\beta})$ with reference to the origin α. A source point A is located at $(r_\alpha, \theta_\alpha, \varphi_\alpha)$ with reference to the center α, and a field point B is at $(r_\beta, \theta_\beta, \varphi_\beta)$ with reference to β. The bicentric expansion form of Green function $G(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{R}_{\alpha\beta}; K)$ is derived in detail in Appendix A. Therefore only the final form is listed here.

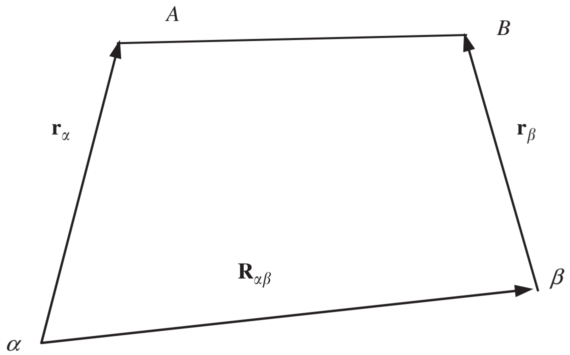

Fig. 1. The relation among $\mathbf{r}_\alpha$, $\mathbf{r}_\beta$ and $\mathbf{R}_{\alpha\beta}$.

The bicentric expansion form of Green function has different forms in three different regions, i.e.

(A) $r_\alpha + r_\beta \leqslant R_{\alpha\beta}$

$$G^{\mathrm{I}}(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{R}_{\alpha\beta}; K) = \mathrm{i}K \sum_L \sum_{L'} F^{(1)}_{LL'}(K\mathbf{R}_{\alpha\beta}) j_l(Kr_\alpha) j_{l'}(Kr_\beta) Y_{lm}(\theta_\alpha, \varphi_\alpha) Y^*_{l'm'}(\theta_\beta, \varphi_\beta), \tag{2}$$

where $L \equiv (l, m)$ and $\sum_L = \sum_{l=0}^{\infty} \sum_{m=-l}^{l}$ and

$$F^{(1)}_{LL'}(K\mathbf{R}_{\alpha\beta}) = \sum_{L''} 4\pi(-1)^l \mathrm{i}^{l+l'+l''} Q_{L''}(LL') h^{(1)}_{l''}(KR_{\alpha\beta}) Y^*_{l''m''}(\theta_{\alpha\beta}, \varphi_{\alpha\beta}), \tag{3}$$

$$Q_{L''}(L') = (-1)^m \left[\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}\right]^{1/2} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & -m'' \end{pmatrix}, \tag{4}$$

j_l and $h^{(1)}_l$ are the spherical Bessel function and the first kind spherical Hankel function, respectively. The spherical harmonic $Y_{lm}(\theta, \varphi)$ is given by

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P^m_l(\cos\theta) \mathrm{e}^{\mathrm{i}m\varphi} \quad (m = 0, \pm 1, \ldots, \pm l), \tag{5}$$

where P^m_l is the associated Legendre function. Y^*_{lm} is the complex conjugate of Y_{lm}.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

in expression (3) is the Wigner (3-j) symbol.

(B) $r_\beta + R_{\alpha\beta} \leqslant r_\alpha$

$$G^{\mathrm{II}}(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{R}_{\alpha\beta}; K) = \mathrm{i}K \sum_L \sum_{L'} F^{(2)}_{LL'}(K\mathbf{R}_{\alpha\beta}) h^{(1)}_l(Kr_\alpha) j_{l'}(Kr_\beta) Y_{lm}(\theta_\alpha, \varphi_\alpha) Y^*_{l'm'}(\theta_\beta, \varphi_\beta), \tag{6}$$

where

$$F^{(2)}_{LL'}(K\mathbf{R}_{\alpha\beta}) = \sum_{L''} 4\pi(-1)^l \mathrm{i}^{l+l'+l''} Q_{L''}(LL') j_{l''}(KR_{\alpha\beta}) Y^*_{l''m''}(\theta_{\alpha\beta}, \varphi_{\alpha\beta}). \tag{7}$$

(C) $r_\alpha + R_{\alpha\beta} \leqslant r_\beta$

$$G^{\mathrm{III}}(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{R}_{\alpha\beta}; K) = \mathrm{i}K \sum_L \sum_{L'} F^{(2)}_{LL'}(K\mathbf{R}_{\alpha\beta}) j_l(Kr_\alpha) h^{(1)}_{l'}(Kr_\beta) Y_{lm}(\theta_\alpha, \varphi_\alpha) Y^*_{l'm'}(\theta_\beta, \varphi_\beta). \tag{8}$$

Let $R_{\alpha\beta} \to 0$. Then the following classical expansions of Green function can be obtained from the above expansion forms of G^{II} and G^{III}, respectively:

$$\text{(a)} \quad r \geqslant r', \; G^{(1)}(\mathbf{r}, \mathbf{r}') = \mathrm{i}K \sum_L h^{(1)}_l(Kr) j_l(Kr') Y_{lm}(\theta, \varphi) Y^*_{lm}(\theta', \varphi'), \tag{9}$$

$$\text{(b)} \quad r \leqslant r', \; G^{(2)}(\mathbf{r}, \mathbf{r}') = \mathrm{i}K \sum_L j_l(Kr) h^{(1)}_l(Kr') Y_{lm}(\theta, \varphi) Y^*_{lm}(\theta', \varphi'). \tag{10}$$

3. A kind of addition formulas for the spherical wave functions

As shown in Fig. 2, O_1 is the origin of the spherical coordinate system (r, θ, φ), and a point sound source is located at $A(r_1, \theta_1, \varphi_1)$, and the coordinates of a field point B are $(r_0, \theta_0, \varphi_0)$ with reference to the center O_1. Because of

$$\mathbf{r}_0 = \mathbf{R}_{12} + \mathbf{r}_2, \tag{11}$$

Green function can be expressed as

$$G(\mathbf{r}_0, \mathbf{r}_1) = \frac{\mathrm{e}^{\mathrm{i}K|\mathbf{r}_1 - \mathbf{r}_0|}}{4\pi|\mathbf{r}_1 - \mathbf{r}_0|} = \frac{\mathrm{e}^{\mathrm{i}K|\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{R}_{12}|}}{4\pi|\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{R}_{12}|} = G(\mathbf{r}_1, \mathbf{r}_2, \mathbf{R}_{12}; K). \tag{12}$$

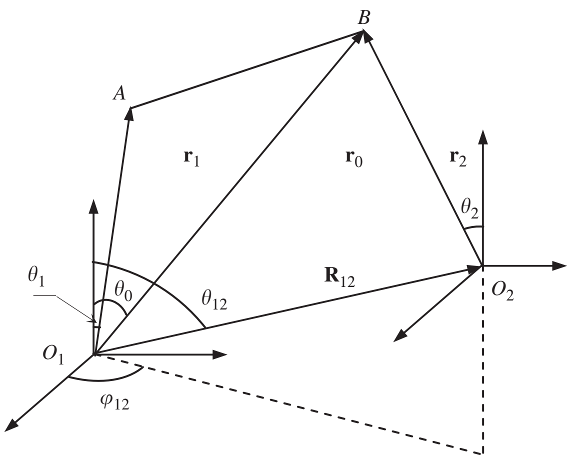

Fig. 2. Sketch of vector analysis.

On the other hand, $G(\mathbf{r}_0, \mathbf{r}_1)$ can be expanded into the following series forms:

$$G(\mathbf{r}_0, \mathbf{r}_1) = \begin{cases} \mathrm{i}K \sum_L h_l^{(1)}(Kr_1) j_l(Kr_0) Y_{lm}(\theta_1, \varphi_1) Y_{lm}^*(\theta_0, \varphi_0), & r_0 < r_1, \\ \mathrm{i}K \sum_L j_l(Kr_1) h_l^{(1)}(Kr_0) Y_{lm}(\theta_1, \varphi_1) Y_{lm}^*(\theta_0, \varphi_0), & r_0 > r_1. \end{cases} \tag{13}$$

According to the orthotropic character of Y_{lm}, the following addition formulas can be obtained from expressions (2), (6), (8) and (13).

(a) When $r_2 \leqslant R_{12}$,

$$\begin{aligned} h_l^{(1)}(Kr_0) Y_{lm}^*(\theta_0, \varphi_0) = & \sum_{L'} \sum_{L''} 4\pi(-1)^l \mathrm{i}^{l+l'+l''} Q_{L''}(LL') \\ & \times h_{l''}^{(1)}(KR_{12}) j_{l'}(Kr_2) Y_{l''m''}^*(\theta_{12}, \varphi_{12}) Y_{l'm'}^*(\theta_2, \varphi_2), \end{aligned} \tag{14}$$

where

$$r_0 = \sqrt{R_{12}^2 + r_2^2 + 2R_{12} r_2 \cos \gamma},$$

$$\cos \gamma = \cos \theta_{12} \cos \theta_2 + \sin \theta_{12} \sin \theta_2 \cos(\varphi_{12} - \varphi_2).$$

(b) When $R_{12} \leqslant r_2$,

$$\begin{aligned} h_l^{(1)}(Kr_0) Y_{lm}^*(\theta_0, \varphi_0) = & \sum_{L'} \sum_{L''} 4\pi(-1)^l \mathrm{i}^{l+l'+l''} Q_{L''}(LL') \\ & \times j_{l''}(KR_{12}) h_{l'}^{(1)}(Kr_2) Y_{l''m''}^*(\theta_{12}, \varphi_{12}) Y_{l'm'}^*(\theta_2, \varphi_2). \end{aligned} \tag{15}$$

(c) For any r_2 and R_{12},

$$\begin{aligned} j_l(Kr_0) Y_{lm}^*(\theta_0, \varphi_0) = & \sum_{L'} \sum_{L''} 4\pi(-1)^l \mathrm{i}^{l+l'+l''} Q_{L'}(LL') \\ & \times j_{l''}(KR_{12}) j_{l'}(Kr_2) Y_{l''m''}^*(\theta_{12}, \varphi_{12}) Y_{l'm'}^*(\theta_2, \varphi_2). \end{aligned} \tag{16}$$

Expressions (14)–(16) are a kind of addition formulas presented for the spherical wave functions.

In Appendix B, we can derive from Eqs. (14)–(16) the expansions for the spherical wave functions $h_l^{(1)}(Kr_0) P_n^m(\cos \theta_0) \exp(\mathrm{i}m\varphi_0)$ and $j_l(Kr_0) P_n^m(\cos \theta_0) \exp(\mathrm{i}m\varphi_0)$, which are in full agreement with the previous derivation by another method in Ref. [10]. However, the form of the addition formulas (14)–(16) is more suitable for dealing with the multiple-scattering problem here.

4. Sound scattering from an arbitrary number of fluid spheres

The geometry for sound scattering of a spherical wave by an arbitrary number of spheres is shown in Fig. 3. This is a combined system with N fluid spheres, whose centers $O_1, O_2, \ldots, O_N$ are positioned by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_q, \ldots, \mathbf{r}_s, \ldots, \mathbf{r}_n$, respectively. The radius of the qth sphere is a_q $(q = 1, 2, \ldots, N)$. The coordinates of a field point B are $\mathbf{r}(r, \theta, \varphi)$ with reference to the origin O and are $\mathbf{r}_q(r'_q, \theta'_q, \varphi'_q)$ to $O_q(r_q, \theta_q, \varphi_q)$. Provided there is a point sound source with unit strength located at a point A, whose coordinates are $\mathbf{r}_0(r_0, \theta_0, \varphi_0)$, we calculate the scattered sound pressure at B by the combined system.

The free sound field $P_0(\mathbf{r}, \mathbf{r}_0)$ generated by the point sound source can be easily got according to expression (13),

$$P_0(\mathbf{r}, \mathbf{r}_0) = \begin{cases} \mathrm{i}K \sum\limits_L j_l(Kr) h_l^{(1)}(Kr_0) Y_{lm}(\theta, \varphi) Y^*_{lm}(\theta_0, \varphi_0), & r_0 > r, \\ \mathrm{i}K \sum\limits_L h_l^{(1)}(Kr) j_l(Kr_0) Y_{lm}(\theta, \varphi) Y^*_{lm}(\theta_0, \varphi_0), & r_0 < r. \end{cases} \tag{17}$$

The scattering sound field from the combined system can be expressed as

$$P_{\mathrm{SC}}(\mathbf{r}, \mathbf{r}_0) = \sum_{q=1}^{N} P_q(\mathbf{r}'_q), \tag{18}$$

where

$$P_q(\mathbf{r}'_q) = \sum_L C_L^{(q)} h_l^{(2)}(Kr'_q) Y_{lm}(\theta'_q, \varphi'_q), \tag{19}$$

where $C_L^{(q)}$ $(q = 1, 2, \ldots, N)$ are the coefficients to be determined from the boundary conditions and $h_l^{(2)}$ is the second kind spherical Hankel function.

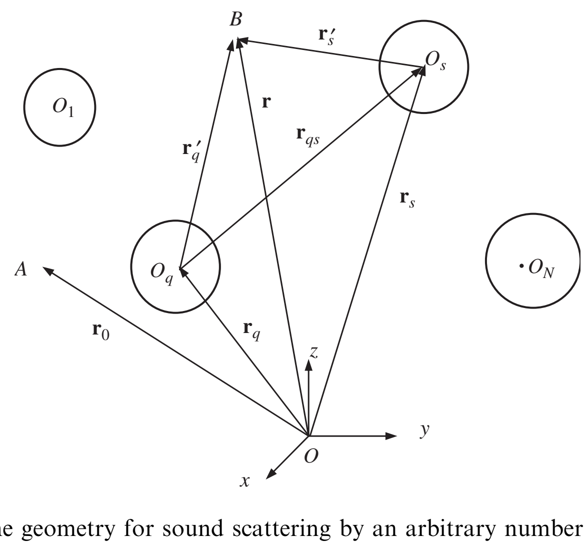

Fig. 3. The geometry for sound scattering by an arbitrary number of spheres.

Since the addition formulas (14) and (15) are expanded into the series in terms of $h_l^{(1)} Y_{lm}^*$, whose complex conjugate is $h_l^{(2)} Y_{lm}$, and the free sound field $P_0(\mathbf{r}, \mathbf{r}_0)$ in Eq. (17) is expanded into the form of Y_{lm}, the scattered field in Eq. (19) can be expanded into the series in terms of $h_l^{(2)} Y_{lm}$ to get the consistent form with the incident field.

In order to expand the sound field—all referred to the center of the qth sphere—the above addition formulas need to be applied in the triangle $BO_S O_q$. Because all these spheres are non-overlapped, $a_s + a_q \leqslant r_{qs}$. Considering the boundary conditions, we can get from expression (14) that

$$\begin{aligned} h_l^{(2)}(Kr'_q) Y_{lm}(\theta'_q, \varphi'_q) = \sum_{L'} \sum_{L''} & 4\pi(-1)^l (-\mathrm{i})^{l+l'+l''} Q_{L''}(LL') \\ & \times h_{l''}^{(2)}(Kr_{qs}) j_{l'}(Kr'_q) Y_{l''m''}(\theta_{qs}, \varphi_{qs}) Y_{l'm'}(\theta'_s, \varphi'_s), \end{aligned} \tag{20}$$

or

$$\begin{aligned} h_l^{(2)}(Kr'_s) Y_{lm}(\theta'_s, \varphi'_s) = \sum_{L'} \sum_{L''} & 4\pi(-1)^l (-\mathrm{i})^{l+l'+l''} Q_{L''}(LL') \\ & \times h_{l''}^{(2)}(Kr_{qs}) j_{l'}(Kr'_s) Y_{l''m''}(\pi - \theta_{qs}, \pi + \varphi_{qs}) Y_{l'm'}(\theta'_q, \varphi'_q). \end{aligned} \tag{21}$$

For the triangle OBO_q, if $r_0 > r$ the addition formula (16) can be used, or else when $r \leqslant r_q$ expression (21) can be used and when $r > r_q$ expression (15) can be used. Here, we only consider the situation $r_0 > r$, and the other situations can be dealt with in the same way. Thus,

$$P_0(\mathbf{r}, \mathbf{r}_0) = \sum_{L} \sum_{L'} G_{L''} j_{l'}(Kr'_q) Y_{l'm'}(\theta'_q, \varphi'_q), \tag{22}$$

where

$$G_{L''} = \mathrm{i}K h_l^{(1)}(Kr_0) Y_{lm}^*(\theta_0, \varphi_0) \sum_{L''} 4\pi(-1)^l (-\mathrm{i})^{l+l'+l''} Q_{L''}(LL') j_{l''}(Kr_q) Y_{l''m''}(\theta_q, \varphi_q). \tag{23}$$

Besides, the internal sound field of the qth sphere can be expressed as

$$P_{\mathrm{in}}^{(q)}(\mathbf{r}'_q) = \sum_{L} A_L^{(q)} j_l(K_q r'_q) Y_{lm}(\theta'_q, \varphi'_q), \tag{24}$$

where K_q is the wavenumber in the sound space inside the qth fluid sphere.

According to the continuous conditions of the stress and displacement on the boundary of the qth sphere, the following equations can be obtained:

$$(P_0(\mathbf{r}, \mathbf{r}_0) + P_{\mathrm{SC}}(\mathbf{r}, \mathbf{r}_0))\Big|_{r'_q = a_q} = P_{\mathrm{in}}^{(q)}(\mathbf{r}'_q)|_{r'_q = a_q}, \tag{25}$$

$$\frac{1}{\mathrm{i}\omega\rho_0} \frac{\partial(P_0(\mathbf{r}, \mathbf{r}_0) + P_{\mathrm{SC}}(\mathbf{r}, \mathbf{r}_0))}{\partial r'_q}\Bigg|_{r'_q = a_q} = \frac{1}{\mathrm{i}\omega\rho_q} \frac{\partial P_{\mathrm{in}}^{(q)}(\mathbf{r}'_q)}{\partial r'_q}\Bigg|_{r'_q = a_q}, \tag{26}$$

i.e.,

$$j_{l'}(Ka_q)\sum_L G_{L''} + C_{L'}^{(q)} h_{l'}^{(2)}(Ka_q) + \sum_{\substack{s=1\\ s\neq q}}^{N} \sum_L C_L^{(s)}\, T_{L''} j_{l'}(Ka_s) = A_{L'}^{(q)} j_{l'}(K_q a_q), \tag{27}$$

$$\frac{K}{\rho_0}\left[j'_{l'}(Ka_q)\sum_L G_{L''} + C_{L'}^{(q)} h_{l'}^{(2)'}(Ka_q) + \sum_{\substack{s=1\\ s\neq q}}^{N} \sum_L C_L^{(s)} T_{L''} j'_{l'}(Ka_s)\right] = \frac{K_q}{\rho_q} A_{L'}^{(q)} j'_{l'}(K_q a_q), \tag{28}$$

where

$$T_{L''} = \sum_{L''} 4\pi(-1)^l(-\mathrm{i})^{l+l'+l''} Q_{L''}(LL') h_{l''}^{(2)}(Kr_{qs}) Y_{l''m''}(\pi-\theta_{qs}, \pi+\varphi_{qs}) \tag{29}$$

and ρ_q is the medium density inside the qth fluid sphere. We can eliminate $A_{L'}^{(q)}$ from the above two equations and get

$$C_{L'}^{(q)} M_{l'}^{(2)} + \sum_{\substack{s=1\\ s\neq q}}^{N} \sum_L C_L^{(s)} T_{L''} M_{l'}^{(3)} = -M_{l'}^{(1)} \sum_L G_{L''} \quad (q = 1, 2, \ldots, N), \tag{30}$$

where

$$M_{l'}^{(1)} = j_{l'}(Ka_q) j'_{l'}(K_q a_q)\frac{K_q}{\rho_q} - j'_{l'}(Ka_q) j_{l'}(K_q a_q)\frac{K}{\rho_0}, \tag{31}$$

$$M_{l'}^{(2)} = h_{l'}^{(2)}(Ka_q) j'_{l'}(K_q a_q)\frac{K_q}{\rho_q} - h_{l'}^{(2)'}(Ka_q) j_{l'}(K_q a_q)\frac{K}{\rho_0}, \tag{32}$$

$$M_{l'}^{(3)} = j_{l'}(Ka_s) j'_{l'}(K_q a_q)\frac{K_q}{\rho_q} - j'_{l'}(Ka_s) j_{l'}(K_q a_q)\frac{K}{\rho_0}. \tag{33}$$

Because the superscript q can be taken for any integer from 1 to N and l' is variable from 0 to ∞, formula (30) is a series of equations with infinite dimensions $C_{L'}^{(q)}$. If $\sum_{l=0}^{n_1}\sum_{m=-l}^{l}$ is close enough to $\sum_L \equiv \sum_{l=0}^{\infty}\sum_{m=-l}^{l}$ depending on the calculation accuracy, the number of the coefficients to be determined is $(n_1+1)^2 N$. When the superscript q ranges from 1 to N and l' from 0 to n_1, we can obtain the same number of equations. The coefficients $C_{L'}^{(q)}$ can be determined by solving these equations, and the calculation precision can be improved by increasing n_1. In this paper, we have used the Gauss–Seidel iteration method to solve these equations.

According to expressions (18) and (19), the coefficients obtained from Eq. (30) bring the exact solution for the scattering sound field of the combined system by accounting for all interactions between any two spheres.

Every term in Eq. (30) has a specific physical meaning. The free term determines the scattering of a spherical wave by a single sphere without considering the multiple scattering. The double sums denote the interactions between any two spheres, in which $h_{l''}^{(2)}(Kr_{qs})$ in the factor $T_{L''}$ is relative to the distance between the two spheres O_q and O_s. And $T_{L''}$ is decreased with increasing Kr_{qs}, which indicates that the corresponding interactions become smaller.

5. Numerical simulation and discussions

In Ref. [7] the backscattering of sound from two regularly arranged bubbles is studied theoretically and experimentally. According to Ref. [7], the numerical results for the backscattering from the pair of bubbles can be reported in terms of target strength (TS), which is defined by

$$\mathrm{TS} = 10\,\log\left(\frac{|p_T|^2}{|p_{\mathrm{inc}}|^2}R_{\mathrm{BR}}^2\right), \tag{34}$$

where p_T is the total scattered pressure at the receiver from the pair of bubbles, p_{inc} is the incident pressure at the bubbles and R_{BR} is the distance from the bubbles to the receiver.

According to expressions (30) and (34), we calculate the target strength (TS) for the backscattering from two bubbles of radius 585 μm at a frequency of 110 kHz and variable d, which is the distance from the combined beam axis to the center of each bubble. The computed results, as shown in Fig. 4, were first validated by comparison to Fig. 8 in Ref. [7], which is obtained by using an exact, closed-form solution derived from the multiple scattering series. The oscillatory behavior is brought about by the interference of the waves scattered between the bubbles, and the straight dashed line represents the coherent scattering from the two identical bubbles without considering the interference between them. This result is in agreement with that in Ref. [7] except that the discrepancy of about 1 dB between them, which is caused by numerical

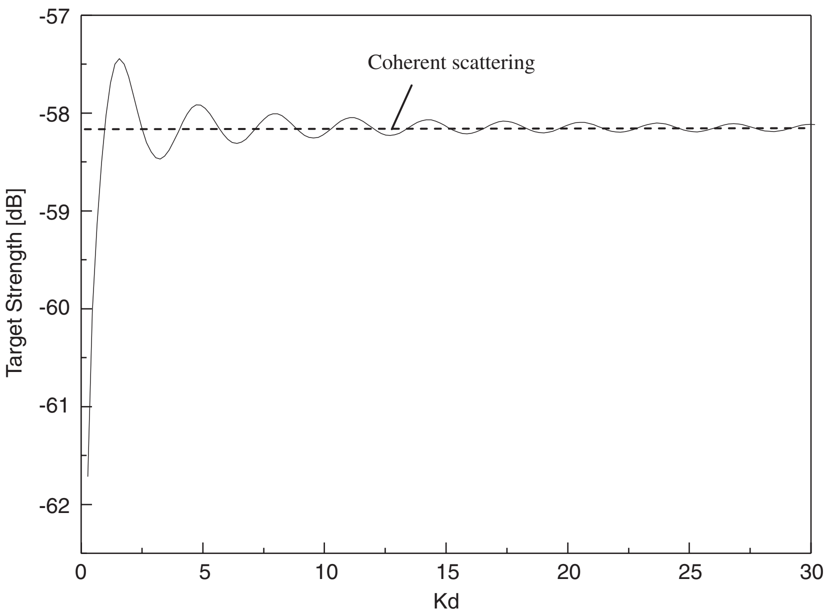

Fig. 4. Theoretical curve for backscattering from two bubbles of radius 585 μm at a frequency of 110 kHz versus variable distance d. The plot is generated by an analytical and exact method. It agrees very well with the theoretical result in Ref. [7, Fig. 8].

calculation with different methods, and may be improved further by increasing the calculation accuracy.

Next, we calculate the target strength (TS) for the backscattering from two bubbles of radius 585 μm with the distance $R = 0.02\,\mathrm{m}$ and plot it versus variable frequency in Fig. 5. Here the

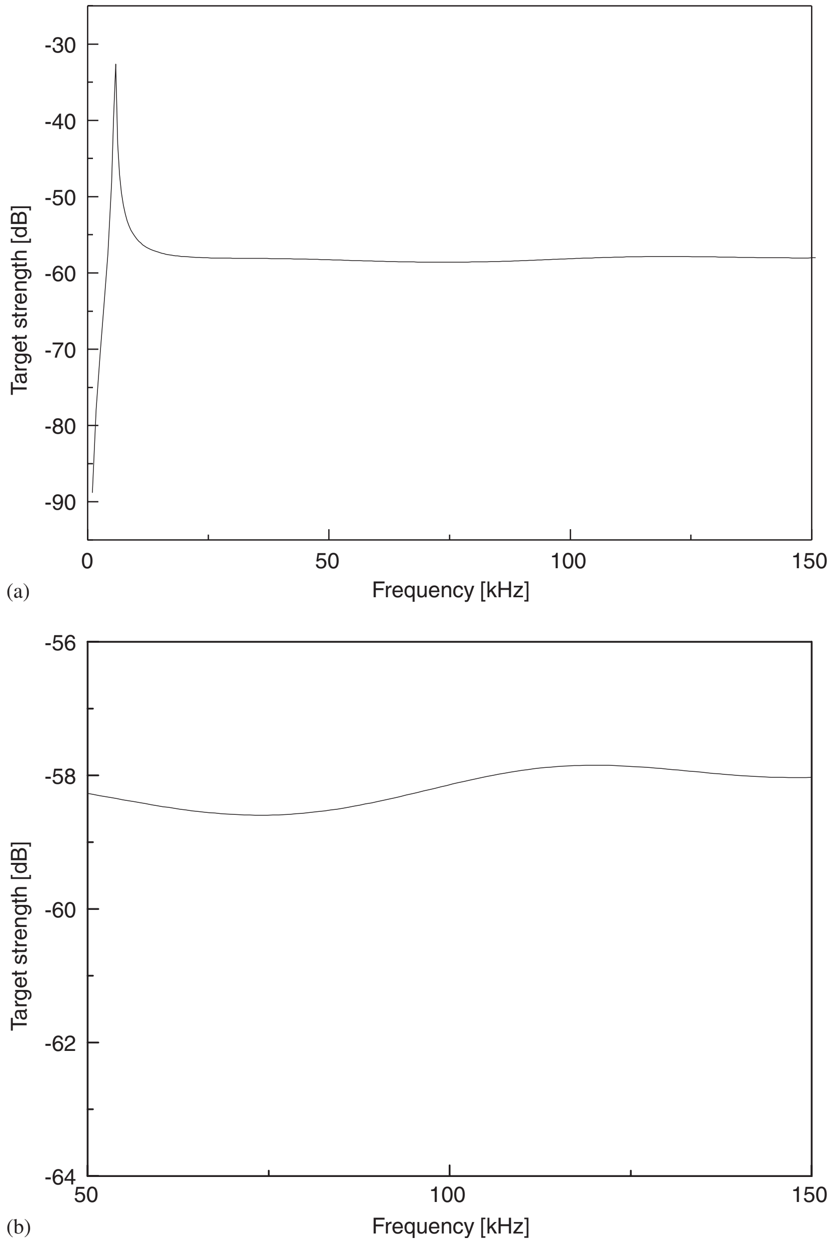

Fig. 5. Theoretical curve for backscattering from two bubbles of radius 585 μm versus variable frequency. Graph (b) is a detail of graph (a).

source and receiver are collocated at the same position and the distance from the source to set of bubbles is $R_0 = 0.58\,\mathrm{m}$. In Fig. 5 graph (b) is a detail of graph (a) and there is a small-amplitude oscillation in the frequency range 50–150 kHz in graph (b), which is coincident with Twersky's expression [11].

Besides, we also generate the target strength (TS) versus KR_0 in Fig. 6. Here R_0 is the distance from the source to the set of bubbles, $K = 110\,\mathrm{kHz}$ and the distance between the two bubbles $R = 0.02\,\mathrm{m}$. Fig. 6 shows that TS tends to a constant when R_0 is bigger than 0.05 m.

When the source and receiver are collocated at the same position, the two most important cases for the scattering from two bubbles are the backscattering, in which the source is perpendicular to the line along the centers of the two bubbles, and the end-on scattering, in which the source is on the drawn-out line along the centers of the two bubbles. The first case has already been plotted in Fig. 4, and we plot the second case in Fig. 7 at a frequency of 110 kHz, with the dashed line representing the TS value as the end-on scattering by the nearer bubble from the receiver. In Fig. 4 the solid line oscillates around the dashed line, and at the larger separations (i.e., $d \geqslant 0.065\,\mathrm{m}$ dashed and solid lines coincide. Meanwhile, in Fig. 7 some difference between the two lines is still observable at the large separations. Figs. 4 and 7 also show that the interactions between the two bubbles become weaker as the distance R between the centers of the two bubbles increases. When the distance R increases to some extent, the end-on scattering by the two bubbles will be close to the scattering only by the nearer bubble from the receiver.

At last, the multiple scattering of a spherical acoustic wave from an arbitrary number of spheres is further considered. For an arbitrary configuration of N fluid spheres, a set of N equations can

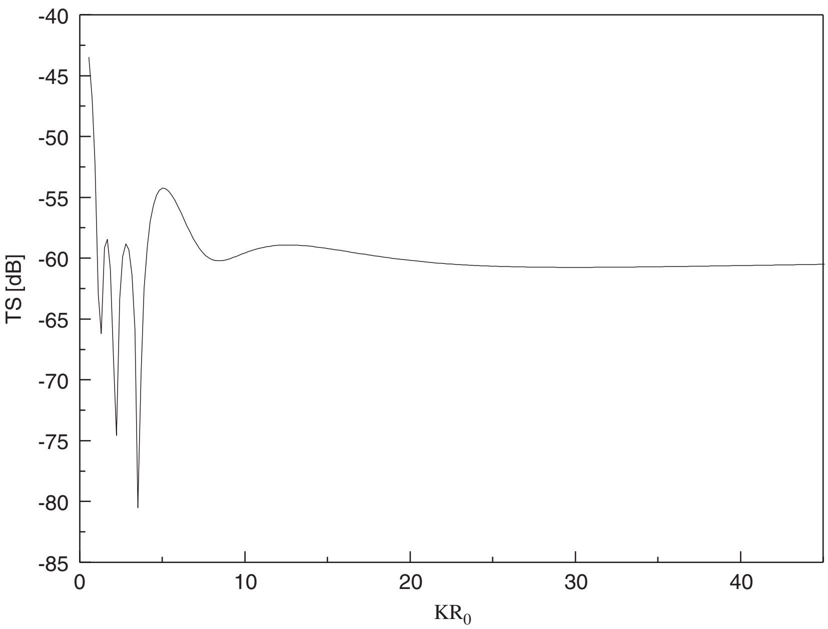

Fig. 6. Theoretical curve for backscattering from two bubbles of radius 585 μm versus variable distance R_0 from the source to set of bubbles at a frequency of 110 kHz.

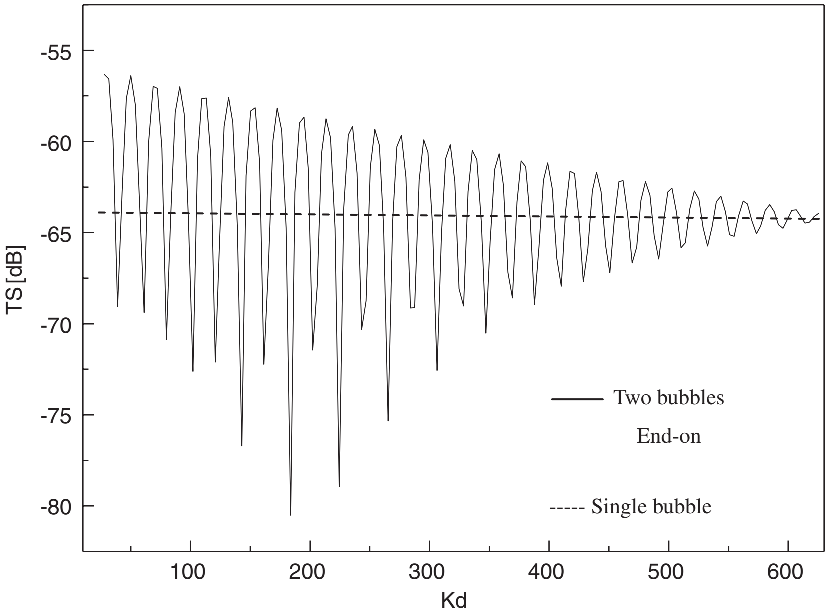

Fig. 7. Theoretical curve for end-on scattering from two bubbles of radius 585 μm at a frequency of 14 kHz versus variable distance d when the source is on the drawn-out line between the centers of the two bubbles.

be obtained by using Eq. (30). The interactions between any two fluid spheres are taken into account in these equations exactly and their coefficients are coupled through double sums in the spherical wave functions. By truncating the infinite series in the equations depending on certain calculation accuracy, and solving the coefficients matrix by using the Gauss–Seidel iteration method, we can obtain the scattered sound field by the configuration of the fluid spheres.

Here, we deal with three fluid spheres for an example. As shown in Fig. 8, three bubbles B_1, B_2, and B_3 with the same radius 585 μm are aligned by separation of d. The source and receiver are collocated at the same position $\mathbf{R}_0$ and the distance from the source to set of bubbles is still $R_0 = 0.58\,\mathrm{m}$.

For the three bubbles, three equations with respect to the scattering coefficients $C_{L'}^{(q)}$ ($q = 1, 2, 3$, and $L' \equiv (l', m')$) can be got from Eq. (30). These are a set of equations with infinite dimensions. Here, we truncate the infinite series with $n_1 = 20$. Thus the number of the scattering coefficients to be determined is 3×21^2. When l' ranges from 0 to n_1 for every equation, we can obtain the same number of equations. The coefficients $C_{L'}^{(q)}$ can be determined by solving these equations with the Gauss–Seidel iteration method.

In order to compare the numerical result of three-bubble cluster with that of two-bubble one, target strength (TS) in Eq. (34) is also used. However, here p_T is the total scattered pressure at the receiver from the three bubbles, p_{inc} is the incident pressure at the bubbles B_2 and B_3, and R_{BR} is the distance from the bubble B_2 or B_3 to the receiver.

Theoretical curve of the target strength for backscattering from three bubbles of radius 585 μm at a frequency of 110 kHz versus variable distance d is plotted in Fig. 9, in which the straight

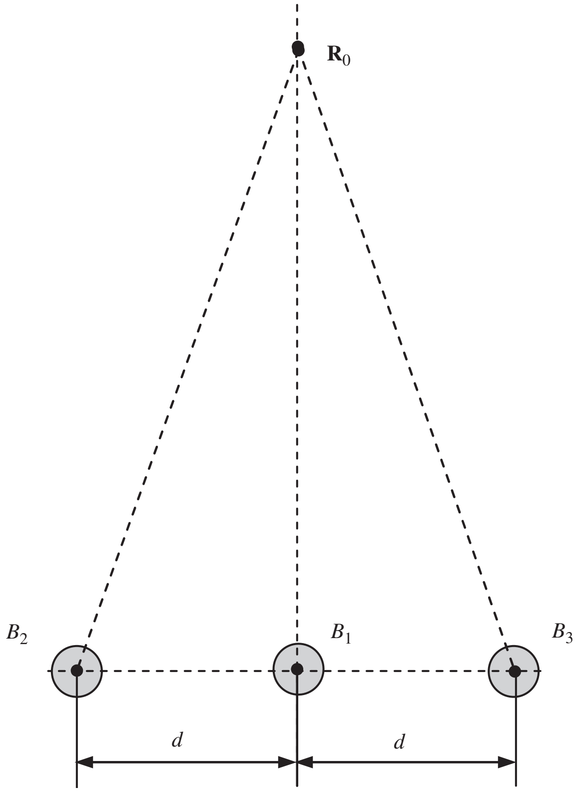

Fig. 8. Geometry of the backscattering from three bubbles (not to scale).

dashed line represents the coherent scattering from the three identical bubbles without considering the interference between them. On the whole, the behavior of the three-bubble cluster is similar to the two-bubble one, but the amplitude of the fluctuations of the interactive scattering is now somewhat greater.

6. Conclusions

Based on the bicentric expansion form of Green function in the spherical coordinates, we have presented the kind of addition formulas for the spherical wave functions and solved the problem of the multiple scattering of a spherical acoustic wave by an arbitrary number of fluid spheres. The kind of addition formulas allows us to express the exact analytical solution as an infinite set of equations, all referred to the center of each sphere. The accuracy of the solution will depend on the accuracy used to determine the coefficients of the equations. Numerical calculations were performed to validate the kind of addition formulas for the spherical wave functions. The exact approach can also be extended to the cases of the sound scattering by an arbitrary number of elastic spheres.

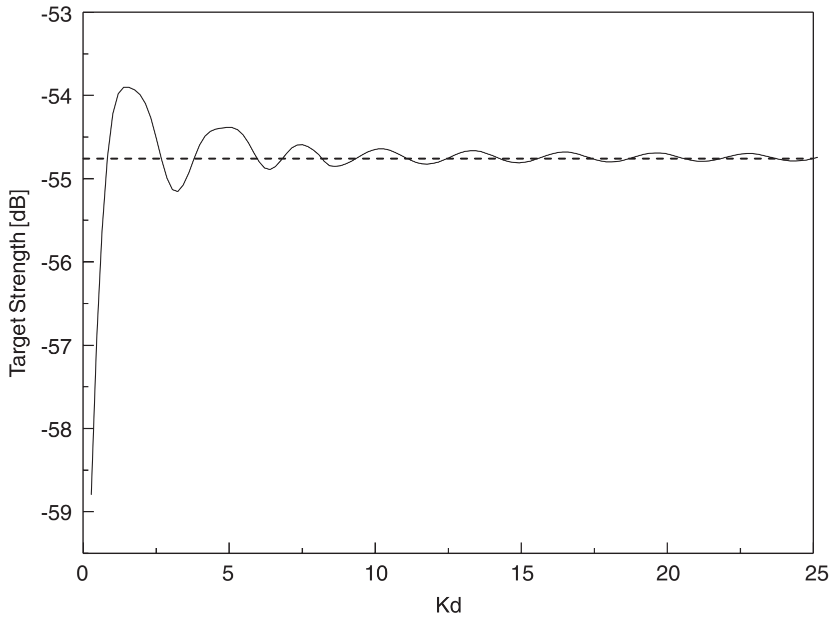

Fig. 9. Theoretical curve for backscattering from three bubbles of radius 585 μm at a frequency of 110 kHz versus variable distance d. The straight dashed line represents the coherent scattering without considering the interference between them.

Appendix A

In this appendix the bicentric expansion form of Green function of non-homogeneous Helmholtz equation in the spherical coordinates is derived.

It is known that Green function of Helmholtz equation can be expressed as [9]

$$G(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{R}_{\alpha\beta}; K) = -\frac{1}{(2\pi)^3} \int \mathrm{d}\mathbf{q}\, \frac{\mathrm{e}^{\mathrm{i}\mathbf{q}(\mathbf{r}_\alpha - \mathbf{r}_\beta - \mathbf{R}_{\alpha\beta})}}{q^2 - K^2}. \tag{A.1}$$

Substituting the following expansion form of plane wave into the integrand of expression (A.1):

$$\mathrm{e}^{\mathrm{i}qr\cos\zeta} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathrm{i}^l j_l(qr) Y_{lm}^*(\theta_q, \varphi_q) Y_{lm}(\theta_r, \varphi_r) \tag{A.2}$$

and applying the following integral formula [9]:

$$\begin{aligned} Q_{l''}(ll') &= \iint Y_{lm}^*(\theta_q, \varphi_q) Y_{l'm'}(\theta_q, \varphi_q) Y_{l''m''}(\theta_q, \varphi_q) \sin\theta_q \,\mathrm{d}\theta_q \,\mathrm{d}\varphi_q \\ &= (-1)^m \left[\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}\right]^{1/2} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & -m'' \end{pmatrix}, \end{aligned} \tag{A.3}$$

we can obtain that

$$G(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{R}_{\alpha\beta}; K) = 8 \sum_{L} \sum_{L'} \sum_{L''} (-1)^{l+1} \mathrm{i}^{l+l'+l''} Q_{L''}(LL')$$
$$\times F_{ll'l''}(r_\alpha, r_\beta, R_{\alpha\beta}; K) Y_{lm}(\theta_\alpha, \varphi_\alpha) Y^*_{l'm'}(\theta_\beta, \varphi_\beta) Y^*_{l''m''}(\theta_{\alpha\beta}, \varphi_{\alpha\beta}), \quad \text{(A.4)}$$

where ζ is the angle between vector $\mathbf{q}(q, \theta_q, \varphi_q)$ and vector $\mathbf{r}(r, \theta_r, \varphi_r)$, and the radial coefficient is given by

$$F_{ll'l''}(r_\alpha, r_\beta, R_{\alpha\beta}; K) = \int_0^\infty \frac{\mathrm{d}q\, q^2 j_l(qr_\alpha) j_{l'}(qr_\beta) j_{l''}(qR_{\alpha\beta})}{q^2 - K^2}, \quad \text{(A.5)}$$

where l, l' and l'' are restricted by Wigner (3-j) symbol [13] and satisfy the following conditions:

$$l + l' + l'' = \text{even (non-negative)}, \quad \text{(A.6)}$$

$$|l - l'| \leqslant l'' \leqslant l + l', \ |l'' - l| \leqslant l' \leqslant l + l'', \ |l' - l''| \leqslant l \leqslant l' + l''. \quad \text{(A.7)}$$

The integral in Eq. (A.5) has different results in the three non-overlapping regions $r_\alpha + r_\beta < R_{\alpha\beta}$, $r_\beta + R_{\alpha\beta} < r_\alpha$ and $r_\alpha + R_{\alpha\beta} < r_\beta$.

In the region $r_\alpha + r_\beta < R_{\alpha\beta}$, the integral in Eq. (A.5) can be calculated by applying the following contour integral:

$$\oint_{C_1} \frac{\mathrm{d}q q^2 j_l(qr_\alpha) j_{l'}(qr_\beta) h^{(1)}_{l''}(qR_{\alpha\beta})}{q^2 - (x + \mathrm{i}y)^2} = 0, \quad \text{(A.8)}$$

where x and y ($y > 0$) are the real numbers and the contour C_1 is shown in Fig. A.1.

Because $q = x + \mathrm{i}y$ must be a first-order singular point of the integrand in Eq. (A.8), the theorem of residues can be applied directly here.

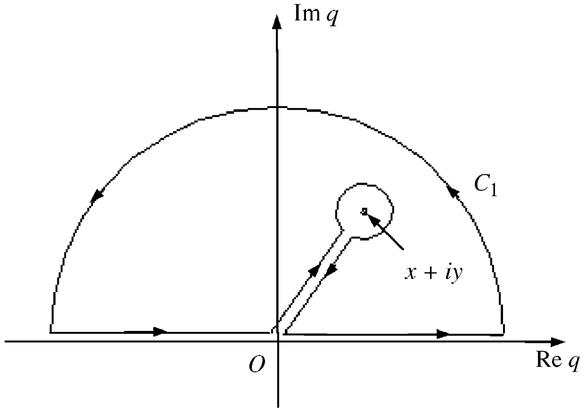

Fig. A.1. Sketch of the contour of the non-overlapping region.

Besides, considering the following condition:

$$\frac{q^2 j_l(qr_\alpha) j_{l'}(qr_\beta) h_{l''}^{(1)}(qR_{\alpha\beta})}{q^2-(x+\mathrm{i}y)^2} \to 0 \quad \text{if } q \to \infty \tag{A.9}$$

and the parity characters of the spherical function j_l and the second kind spherical Bessel function and Eq. (A.6), we can obtain the following result from Eq. (A.8) by using the theorem of residues,

$$\int_0^\infty \frac{\mathrm{d}q\, q^2 j_l(qr_\alpha) j_{l'}(qr_\beta) j_{l''}(qR_{\alpha\beta})}{q^2-(x+y\mathrm{i})^2} = \frac{\pi\mathrm{i}}{2}(x+\mathrm{i}y) j_l((x+\mathrm{i}y)r_\alpha) j_{l'}((x+\mathrm{i}y)r_\beta) h_{l''}^{(1)}((x+\mathrm{i}y)R_{\alpha\beta}). \tag{A.10}$$

Let $K = x + \mathrm{i}y$, then

$$F_{ll'l''}(r_\alpha, r_\beta, R_{\alpha\beta}; K) = \frac{\pi\mathrm{i}}{2} K j_l(Kr_\alpha) j_{l'}(Kr_\beta) h_{l''}^{(1)}(KR_{\alpha\beta}). \tag{A.11}$$

Therefore, $G(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{R}_{\alpha\beta}; K)$ in the region $r_\alpha + r_\beta < R_{\alpha\beta}$ can be obtained by (A.4).

It is easy to find that the difference between the regions $r_\beta + R_{\alpha\beta} < r_\alpha$ and $r_\alpha + r_\beta < R_{\alpha\beta}$ is only to exchange the positions of r_α and $R_{\alpha\beta}$, and the difference between the regions $r_\alpha + R_{\alpha\beta} < r_\beta$ and $r_\alpha + r_\beta < R_{\alpha\beta}$ is the exchange of r_β and $R_{\alpha\beta}$. Therefore, $G(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{R}_{\alpha\beta}; K)$ in the different regions can be obtained.

Besides, the radial coefficient of the expansion forms in the region boundary is proved to be continuous by applying the recursion method in Ref. [9].

Appendix B

The expansions for the spherical wave functions $h_l^{(1)}(Kr_0)P_n^m(\cos\theta_0)\exp(\mathrm{i}m\varphi_0)$ and $j_l(Kr_0)P_n^m(\cos\theta_0)\exp(\mathrm{i}m\varphi_0)$ have been obtained in Ref. [10] as follows:

$$\begin{aligned} j_l(Kr_0)P_l^m(\cos\theta)\exp(\mathrm{i}m\varphi) = \sum_{L'}\sum_p \{ & \mathrm{i}^{l'+p-l}(2l'+1)\{l',|m'|\}a(|m|,|m'|;p,l,l') \\ & \times j_p(KR_{12}) j_{l'}(Kr_2) P_p^{m'+m}(\cos\theta_{12}) P_{l'}^{m'}(\cos\theta_2) \\ & \times \exp[\mathrm{i}(m+m')\varphi_{12}]\exp(-\mathrm{i}m'\varphi_2)\}, \end{aligned} \tag{B.1}$$

$$\begin{aligned} h_l^{(1)}(Kr_0)P_l^m(\cos\theta)\exp(\mathrm{i}m\varphi) = \sum_{L'}\sum_p \{ & \mathrm{i}^{l'+p-l}(2l'+1)\{l',|m'|\}a(|m|,|m'|;p,l,l') \\ & \times h_p^{(1)}(Kr_>) j_{l'}(Kr_<) P_p^{m'+m}(\cos\theta_{12}) P_{l'}^{m'}(\cos\theta_2) \\ & \times \exp[\mathrm{i}(m+m')\varphi_{12}]\exp(-\mathrm{i}m'\varphi_2)\}, \end{aligned} \tag{B.2}$$

where, $p = l' + l, l' + l - 2, \ldots, l' - l$, $r_< = \min(R_{12}, r_2)$, $r_> = \max(R_{12}, r_2)$, and

$$a(|m|, |m'|; p, l, l') = \frac{(l + l' - p - 1)!!(2p + 1)}{(l + p - l')!!(l' + p - l)!!(p + l' + l + 1)!!} \times \sum_{j=0}^{\rho} \exp\{[(l + p - l')/2 + |m| + j]\pi \mathrm{i}\} \begin{pmatrix} \rho \\ j \end{pmatrix} \{l, -j - |m|\}\{l', |m| + j - p\}, \tag{B.3}$$

where, $\rho = p - |m| - |m'|$, $l' - l \leqslant p \leqslant l' + l$, $(s)!! = s(s - 2)(s - 4) \cdots 2$ or 1, and $(0)!! = (-1)!! = 1$.

According to expression (5) and the following relationships [12]:

$$Y^*_{lm} = (-1)^m Y_{l,-m}, \tag{B.4}$$

$$P_l^{-m}(x) = (-1)^m \frac{(l - m)!}{(l + m)!} P_l^m(x) \tag{B.5}$$

and [13]

$$a(m, m'; p, l, l') = (-1)^{m+m'}(2p + 1)\left[\frac{(l + m)!(l' + m')!(p - m - m')!}{(l - m)!(l' - m')!(p + m + m')!}\right]^{1/2} \times \begin{pmatrix} l & l' & p \\ m & m' & -(m + m') \end{pmatrix} \begin{pmatrix} l & l' & p \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.6}$$

applying the character of Wigner (3-j) symbol, we can then derive expansions (B.1) and (B.2) from the addition formulas (14)–(16). Meanwhile, this will validate the kind of broad addition formulas presented in this paper, based on the bicentric expansion form of Green function in the spherical coordinates.

References

[1] C. Liang, Y.T. Lo, *Radio Science* 2 (1967) 1481.
[2] S. Stein, *Quarterly of Applied Mathematics* 19 (1961) 15.
[3] O.R. Cruzan, *Quarterly of Applied Mathematics* 20 (1962) 33.
[4] V. Twersky, *Journal of Mathematical Physics* 8 (1967) 589.
[5] J.H. Bruning, Yuen T. Lo, *IEEE Transactions on Antennas and Propagation* AP-19 (1971) 378.
[6] G.C. Gaunaurd, H. Huang, H.C. Strifors, *Journal of the Acoustical Society of America* 98 (1995) 495.
[7] G. Kapodistrias, P.H. Dahl, *Journal of the Acoustical Society of America* 107 (2000) 3006.
[8] F. Herman, A.R. Williams, Keith H. Johnson, *Journal of Chemical Physics* 61 (1974) 3508.
[9] Su Zhan, *Scientia Sinica* 18 (1978) 171.
[10] B. Friedman, J. Russek, *Quarterly of Applied Mathematics* 12 (1954) 13.
[11] V. Twersky, *Journal of the Acoustical Society of America* 52 (1962) 162.
[12] Wang Zhuxi, Guo Dunren, *Introduction to Special Function*, Peking University Press, Beijing, 2000.
[13] Wang Zhiliang, Ren Wei, *Electromagnetic Scattering Theory*, Sichuan Science and Technology Press, Chendou, 1993.