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Repeated resonances in folded-back beam structures

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Abstract

This paper investigates the interesting behaviour of a structure composed of identical Euler beams, joined together in a fan-folded configuration with one end clamped and the other end completely free. It is shown that when the structure comprises an even number of beams, all resonances occur in perfect pairs. When the structure comprises an odd number of beams, some resonances occur in perfect pairs in addition to resonances that coincide exactly with the natural frequencies of a single clamped–free beam unit. Moreover, irrespective of the number of beams present in the structure, resonances occur in clusters each one of them having a closed boundary at a clamped–free beam resonance and an open boundary at a free–free beam resonance. For consecutive clusters, these boundaries are determined in alternation by the resonances of free–free and clamped–free beams.

1. Introduction

The analysis of structures having repeated resonances is both interesting and challenging. It is well known that eigenvector derivatives can approach infinity with respect to certain parameters where resonances coincide [1]. Structures that have almost perfect cyclic symmetry automatically produce (most) of their natural frequencies in perfect pairs but their eigenvector derivatives remain small. A structure is presented in this paper that comprises N beams, rigidly joined together at their ends such that a fan-folded arrangement is achieved. One of the free ends of this resulting structure is then clamped, see Fig. 1. This class of structures has a number of important

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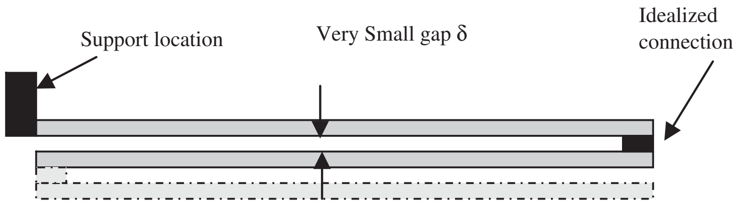

Fig. 1. Structure with two beams (the third, if any, is shown dashed).

and potentially-useful properties (when it is assumed that the beams behave as simple Euler beams). These structures are often found in the context of deployable structures for applications in space and the findings of this paper may have relevance to such structures. However, the main significance of this paper is that it provides a set of interesting theoretical test-cases of structures (other than cyclically-periodic ones) which have repeated resonances and structures which have very unusual distributions of modal densities. As far as the authors are aware, the first discussion of such a structure (with $N = 2$ only) was given in a presentation by Lallement in 1993 [2] on structures having repeated resonances, although the actual structure is not mentioned in the paper of the proceedings. This structure had been used extensively in the laboratory by colleagues of Lallement (Piranda and Fillod in particular) but there appears to be no other reference regarding this problem. For simplicity and without any loss of generality, the material of the structure is assumed to be steel, although any isotropic material can be used. The degrees of freedom (dofs) of the structure are assumed to be transverse deflections and in-plane rotations.

2. Analysis approach

The structure is analysed using the following methods:

(a) The Wittrick–Williams (W–W) algorithm [3], with a dynamic stiffness matrix obtained from exact beam functions, to determine the number of resonances below a trial frequency. This is useful especially in the case of structures having several beams as plotting of determinants of larger matrices is not a good way to determine resonances. The W–W algorithm stems from Rayleigh's theorem that states that if a constraint is imposed on a structure the resulting resonances are intermediate to those of the unconstrained structure. Using this algorithm, a dynamic stiffness matrix [4], say $K(\omega_t)$, is constructed for a trial frequency ω_t. This matrix is converted into an upper triangular form by Gaussian elimination without row interchanges. The number of resonances, N_t, is then obtained by $N_t = N_D + N_C$, where N_D is the number of negative elements on the diagonal of the triangular matrix (can also be determined from the number of negative eigenvalues of $K(\omega_t)$), and N_C is the number of resonances that are exceeded by the trial frequency when the structure is constrained in such a way that all dofs are zero. In this study, $K(\omega_t)$ is generated for the connections dofs of the structure. Only one element is taken along the length of each individual beam.

(b) The transfer matrix method, using exact beam functions to study the nature of the matrix that is used to determine the resonances. This is a 4×4 matrix, for any number of beams, very easy to handle and interpret. This approach is discussed in detail when looking into the nature of the problem.

3. Analytical derivations

The exact beam function for the transverse deflections of a plane beam, [5], is given by

$$v(x) = a\sin(\beta x/l) + b\cos(\beta x/l) + c\sinh(\beta x/l) + d\cosh(\beta x/l), \tag{1}$$

where a, b, c and d are constants to be determined, x is the distance from the origin as shown in Fig. 2 and $\beta^4 = \omega^2 l^4 \rho A/EI$. Here E, I, ρ and A are the elastic modulus, second moment of cross-sectional area, mass density and cross-sectional area, respectively, and ω is the natural frequency.

Assume that v_{Li}, θ_{Li} and v_{Ri} and θ_{Ri} are transverse displacements and rotations at the left and right ends of ith beam, respectively, as shown in Fig. 2. Similarly, M_{Li}, V_{Li} and M_{Ri}, V_{Ri} are the moments and shear forces at the left and right ends of the ith beam.

Let $z_{Ri} = \{v_{Ri}\ \ \theta_{Ri}\ \ M_{Ri}\ \ V_{Ri}\}^{\mathrm{T}}$ and $z_{Li} = \{v_{Li}\ \ \theta_{Li}\ \ M_{Li}\ \ V_{Li}\}^{\mathrm{T}}$ so that one can write

$$z_{Ri} = Uz_{Li}, \tag{2}$$

where $U = U(l)$ is a 4×4 transfer matrix [5], given by

$$U(l) = \begin{bmatrix} C_0 & -lC_1 & -\dfrac{l^2 C_2}{EI} & -\dfrac{l^3 C_3}{EI} \\ -\dfrac{\beta^4 C_3}{l} & C_0 & \dfrac{lC_1}{EI} & \dfrac{l^2 C_2}{EI} \\ -\dfrac{EI\beta^4 C_2}{l^2} & \dfrac{EI\beta^4 C_3}{l} & C_0 & lC_1 \\ -\dfrac{EI\beta^4 C_1}{l^3} & \dfrac{EI\beta^4 C_2}{l^2} & \dfrac{\beta^4 C_3}{l} & C_0 \end{bmatrix}. \tag{3}$$

Here

$$C_0 = \frac{\cosh(\beta) + \cos(\beta)}{2}, \quad C_1 = \frac{\sinh(\beta) + \sin(\beta)}{2\beta}, \quad C_2 = \frac{\cosh(\beta) - \cos(\beta)}{2\beta^2}$$

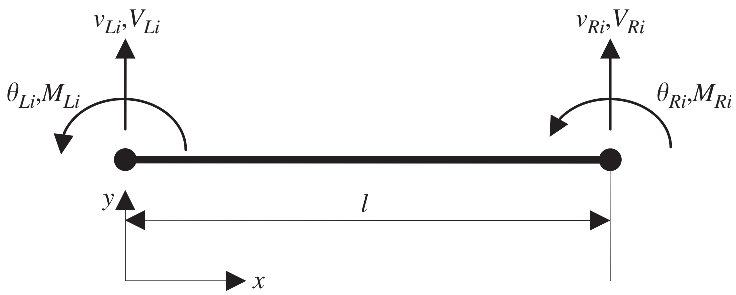

Fig. 2. Notation of the ith beam.

and

$$C_3 = \frac{\sinh(\beta) - \sin(\beta)}{2\beta^3}.$$

It should be noted here that the transfer matrix, that can be used to derive the resonances, does not depend on the constants coefficients (a, b, c and d) in Eq. (1). Once the resonances are determined, these constants can be determined, which will be different for different beams [5]. In this paper, however, mode shapes of the structures are not discussed.

Eq. (2) is used to derive the transfer matrix for structure having any number of beams, keeping in mind that deflections are the same at the joining point while the moments and shear forces have opposite signs at that point. Now the transfer matrices for structures having more than one beam can be derived by adopting the following notations:

$$z_{R1} = Uz_{L1},$$

$$z_{R2} = Uz_{L2},$$

$$z_{R3} = Uz_{L3} \tag{4}$$

and so on.

After defining

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so that $z_{R2} = Jz_{R1}$, and using Eq. (4) $Uz_{L2} = JUz_{L1}$, which gives

$$z_{L2} = U^{-1}JUz_{L1} = U_2 z_{L1}, \tag{5}$$

where $U_2 = U^{-1}JU$.

It should be noted that[1]

$$U(l)U(-l) = I \tag{6}$$

or

$$[U(l)]^{-1} = U(-l), \tag{7}$$

so that

$$U_2 = U(-l)JU(l). \tag{8}$$

[1] $z_{R1} = U(l)z_{L1}$ and $z_{L1} = U(-l)z_{R1}$ so that $z_{R1} = U(l)U(-l)z_{R1}$ or $U(l)U(-l) = I$.

Eqs. (5) and (8) imply that

$$\begin{bmatrix} v_{L2} \\ \theta_{L2} \\ M_{L2} \\ V_{L2} \end{bmatrix} = \begin{bmatrix} \cosh(\beta)\cos(\beta) & 0 & \dfrac{l^2}{EI\beta^2}\sinh(\beta)\sin(\beta) & \dfrac{l^3\Phi(\beta)}{EI\beta^3} \\ 0 & \cosh(\beta)\cos(\beta) & \dfrac{l\Psi(\beta)}{EI\beta} & \dfrac{l^2}{EI\beta^2}\sinh(\beta)\sin(\beta) \\ -\dfrac{EI\beta^2}{l^2}\sinh(\beta)\sin(\beta) & \dfrac{EI\beta\Phi(\beta)}{l} & -\cosh(\beta)\cos(\beta) & 0 \\ \dfrac{EI\beta^3\Psi(\beta)}{l^3} & -\dfrac{EI\beta^2}{l^2}\sinh(\beta)\sin(\beta) & 0 & -\cosh(\beta)\cos(\beta) \end{bmatrix} \times \begin{bmatrix} v_{L1} \\ \theta_{L1} \\ M_{L1} \\ V_{L1} \end{bmatrix}, \tag{9}$$

where $\Phi(\beta) = \sin(\beta)\cosh(\beta) - \cos(\beta)\sinh(\beta)$ and $\Psi(\beta) = \sin(\beta)\cosh(\beta) + \cos(\beta)\sinh(\beta)$.

Consider the structure having two beams first. Now, if the first beam is clamped at the left end then the moment and shear force at the left end of the second beam are zero and Eq. (9) becomes

$$\begin{bmatrix} v_{L2} \\ \theta_{L2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh(\beta)\cos(\beta) & 0 & \dfrac{l^2}{EI\beta^2}\sinh(\beta)\sin(\beta) & \dfrac{l^3\Phi(\beta)}{EI\beta^3} \\ 0 & \cosh(\beta)\cos(\beta) & \dfrac{l\Psi(\beta)}{EI\beta} & \dfrac{l^2}{EI\beta^2}\sinh(\beta)\sin(\beta) \\ -\dfrac{EI\beta^2}{l^2}\sinh(\beta)\sin(\beta) & \dfrac{EI\beta\Phi(\beta)}{l} & -\cosh(\beta)\cos(\beta) & 0 \\ \dfrac{EI\beta^3\Psi(\beta)}{l^3} & -\dfrac{EI\beta^2}{l^2}\sinh(\beta)\sin(\beta) & 0 & -\cosh(\beta)\cos(\beta) \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ M_{L1} \\ V_{L1} \end{bmatrix},$$

which gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cosh(\beta)\cos(\beta) & 0 \\ 0 & -\cosh(\beta)\cos(\beta) \end{bmatrix} \begin{bmatrix} M_{L1} \\ V_{L1} \end{bmatrix}.$$

Equating the determinant of the 2×2 matrix in the last equation to zero gives resonances of the two-beams structure. This matrix clearly shows that the structure has repeated resonances. From this matrix, the characteristic equation of the two beams structure can be written as

$$\cosh(\beta)\cos(\beta) = 0. \tag{10}$$

Roots of this last equation are given by

$$\frac{(2n-1)\pi}{2}, \quad \text{where } n = 1, 2, 3, \ldots .$$

Compare Eq. (10) with the characteristic equation for a free–free (F–F) beam, which is given by

$$\cosh(\beta)\cos(\beta) = 1 \tag{11}$$

and with that of a clamped–free (C–F) beam given by

$$\cosh(\beta)\cos(\beta) = -1. \tag{12}$$

Take the case when the function $\cosh(\beta)\cos(\beta)$ is decreasing. At $\beta = 0$, the value of the function is 1 after which it decreases to zero. It can be seen from Eq. (11) that $\beta = 0$ gives the resonance of an F–F beam. When this function passes through zero and reaches -1, the characteristic equation of a C–F beam is satisfied, which means that this resonance is bounded below by the resonance of an F–F beam and above by the resonance of a C–F beam. Similarly, when the function is increasing, the resonance of two beams structure is bounded below by the resonance of a C–F and above by that of an F–F. In this manner, an alternating behaviour for the bounds of resonances of the structure, in terms of the resonances of an F–F and a C–F beam, is obtained. This can also be seen in Fig. 3. In this figure, scaled values of the determinant of the dynamic stiffness matrix for the connection dofs, i.e. translation and rotation at right end joint, is plotted. Scaling is carried out using inverse sine hyperbolic function. It is seen that the resonances of the structure, zero values of the determinant showing a double root at ω_i, where ω_i is the ith resonance of the structure, are bounded below and above by the poles of the determinant which are resonances of F–F and C–F beam, respectively. These poles correspond to the situation in which all the connection dofs are zero [3], a case in which the structure converts to an F–F and a C–F beam, and hence the determinant of the dynamic stiffness matrix becomes infinite. At these poles, value of the characteristic polynomial changes from positive infinity to negative infinity and vice versa.

Next consider the case of a three-beam structure. Using the previous notations, the transfer matrix for this structure is given by

$$U_3 = U(l)JU_2. \tag{13}$$

After applying the boundary conditions, i.e. clamping one of the free ends, the 2×2 sub-matrix that determines the resonances of the three beams structure is

$$\begin{bmatrix} C_0(2\cos(\beta)\cosh(\beta)-1) & C_1(2\cos(\beta)\cosh(\beta)-1) \\ C_3(2\cos(\beta)\cosh(\beta)-1) & C_0(2\cos(\beta)\cosh(\beta)-1) \end{bmatrix}.$$

The determinant of this matrix is given by $(C_0^2 - C_1C_3)((2\cos(\beta)\cosh(\beta)-1)^2$. It can be seen that the first factor of this determinant corresponds to the resonances in a cantilever

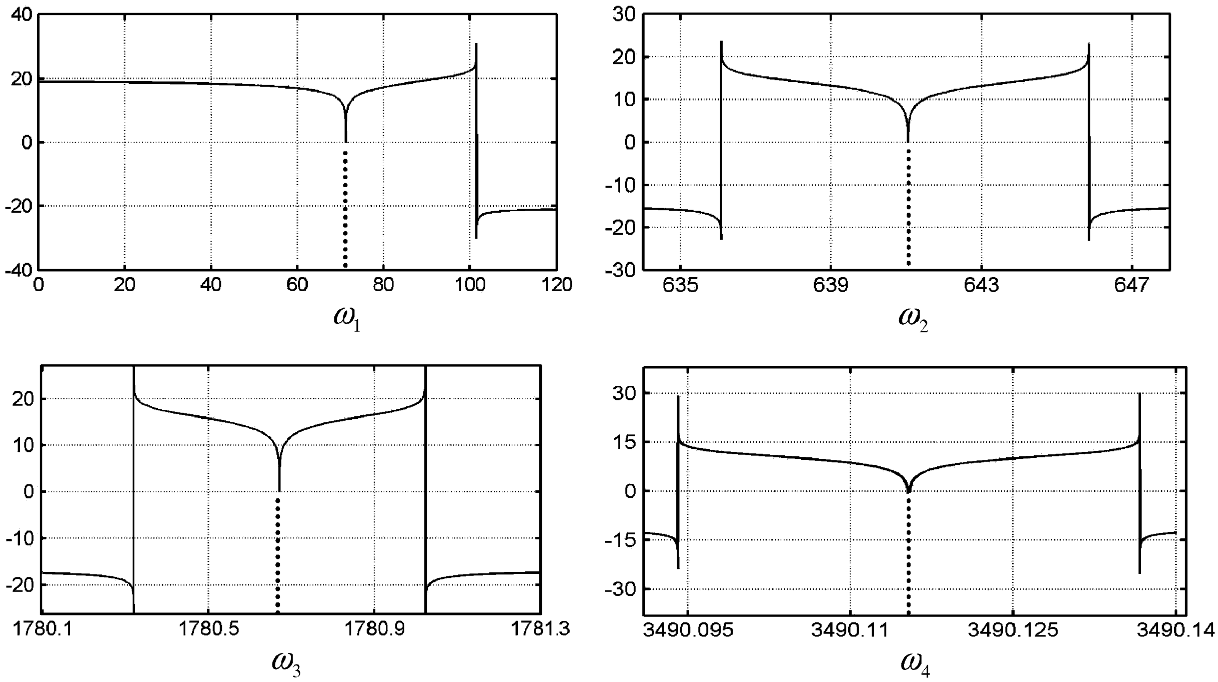

Fig. 3. Plot of the determinant for the two beams structure (scaled values of determinant against frequency (rad/s)).

beam while the second term gives double roots. It can also be deduced that the double roots are given by

$$1 - 2\cos(\beta)\cosh(\beta) = 0 \quad \text{or} \quad \cos(\beta)\cosh(\beta) = \tfrac{1}{2}.$$

After comparing this last equation with Eqs. (11) and (12), it is clear that the double roots are bounded by the resonances of an F–F and a C–F beam as discussed earlier for the case of two beams structure.

4. Generalization

4.1. Structures with even number of beams

In this section, the constant coefficients in the elements of the transfer matrix will be dropped for simplicity as they do not affect the form of the matrices obtained for structures with different numbers of beams.

Recall from Eq. (8) that

$$U_4 = U^{-1}JUJU^{-1}JU = U_2JU_2.$$

Since U_2, as obvious from Eq. (9), can be written as (after dropping out the constant coefficients)

$$U_2 = \begin{bmatrix} a & 0 & b & c \\ 0 & a & d & b \\ -b & c & -a & 0 \\ d & -b & 0 & -a \end{bmatrix}.$$

Hence

$$U_4 = \begin{bmatrix} a^2+b^2-cd & 0 & 2ab & 2ac \\ 0 & a^2+b^2-cd & 2ad & 2ab \\ -2ab & 2ac & -(a^2+b^2-cd) & 0 \\ 2ad & -2ab & 0 & -(a^2+b^2-cd) \end{bmatrix}.$$

This matrix has some special features:

- It has the same structure as U_2.
- It has double roots
- $\dfrac{U_4(2,4)}{U_4(1,4)} = \dfrac{U_2(2,4)}{U_2(1,4)}, \quad \dfrac{U_4(2,3)}{U_4(1,3)} = \dfrac{U_2(2,3)}{U_2(1,3)}.$

Now, it is assumed that the matrix, a modified form of the transfer matrix, for a structure with $2N$ beams, where $N = 1, 2, 3, \ldots$, is given by

$$U_{2N} = \begin{bmatrix} t & 0 & u & v \\ 0 & t & w & u \\ -u & v & -t & 0 \\ w & -u & 0 & -t \end{bmatrix} \quad \text{and}$$

- It has the same structure as U_2.
- It has double roots
- $\dfrac{U_{2N}(2,4)}{U_{2N}(1,4)} = \dfrac{U_2(2,4)}{U_2(1,4)}, \quad \dfrac{U_{2N}(2,3)}{U_{2N}(1,3)} = \dfrac{U_2(2,3)}{U_2(1,3)}.$

Using the same convention,

$$\begin{aligned} U_{2N+2} &= U_2 J U_{2N} = U_{2N} J U_2 \\ &= \begin{bmatrix} at+bu-dv & bv-cu & au+bt & av+ct \\ bw-du & at-cw+ub & aw+dt & au+bt \\ -(au+bt) & av+ct & -(at+bu-dv) & bv-cu \\ aw+dt & -(au+bt) & bw-du & -(at-cw+ub) \end{bmatrix}. \end{aligned} \tag{14}$$

The three properties for this matrix are now proved.

From third property of U_{2N}, $u/v = b/c \Rightarrow bv = cu, b/d = u/w \Rightarrow bw = du$ and hence $cw = dv$. It is also obvious that $(au + bt)/(av + ct) = b/c$, and $(aw + dt)/(au + bt) = d/b$. All these relations imply that U_{2N+2} also satisfies the three properties that are assumed for U_{2N}. This completes the generalization that structure having even number of beams have the repeated resonances.

4.2. Structures having odd number of beams

The modified transfer matrix for three-beam structure is given by

$$U_3 = UJU^{-1}JU = UJU_2 = \begin{bmatrix} AC_0 & -AC_1 & -BC_2 & -BC_3 \\ -AC_3 & AC_0 & BC_1 & BC_2 \\ -BC_2 & BC_3 & AC_0 & AC_1 \\ -BC_1 & BC_2 & AC_3 & AC_0 \end{bmatrix},$$

where C_i's have the same usual notation as for the original transfer matrix with some manipulation to the multipliers (stiffness and mass properties) to make the convention obvious. A, B are polynomials in β. It is interesting to observe that the derived transfer matrix looks like a Hadamard product[2] of the transfer matrix of a C–F beam and a matrix Ω, where

$$\Omega = \begin{bmatrix} A & A & B & B \\ A & A & B & B \\ B & B & A & A \\ B & B & A & A \end{bmatrix}$$

and A, B are the polynomials defined above.

The transfer matrix for $2N + 1$ number of beams, is given by

$$U_{2N+1} = UJU_N.$$

Here only one element of U_{2N+1} will be shown to be the product of the $U(1,1)$ and a function, $g(\beta)$, of β.

$$U_{2N+1}(1,1) = \left(\frac{\cosh(\beta) + \cos(\beta)}{2}\right)t - \left(\frac{\cosh(\beta) - \cos(\beta)}{2}\right)u + \left(\frac{\sinh(\beta) - \sin(\beta)}{2}\right)w,$$

where t, u and w are elements of U_{2N}.

$$\begin{aligned}\left(\frac{\sinh(\beta) - \sin(\beta)}{2}\right)w - \left(\frac{\cosh(\beta) - \cos(\beta)}{2}\right)u &= \frac{w}{2}\left\{\sinh(\beta) - \sin(\beta) - (\cosh(\beta) - \cos(\beta))\frac{u}{w}\right\} \\ &= \frac{w}{2}\left\{\sinh(\beta) - \sin(\beta) - (\cosh(\beta) - \cos(\beta))\frac{b}{d}\right\} \\ &= f(\beta)(\cosh(\beta) + \cos(\beta)).\end{aligned}$$

[2]Corresponding entries of two matrices are multiplied together in such product.

Hence

$$U_{2N+1}(1,1) = \left(\frac{\cosh(\beta) + \cos(\beta)}{2}\right) g(\beta) = C_0 g(\beta).$$

In this manner U_{2N+1} comes out to be

$$U_{2N+1} = \begin{bmatrix} g(\beta)C_0 & -g(\beta)C_1 & -h(\beta)C_2 & -h(\beta)C_3 \\ -g(\beta)C_3 & g(\beta)C_0 & h(\beta)C_1 & h(\beta)C_2 \\ -h(\beta)C_2 & h(\beta)C_3 & g(\beta)C_0 & g(\beta)C_1 \\ -h(\beta)C_1 & h(\beta)C_2 & g(\beta)C_3 & g(\beta)C_0 \end{bmatrix}.$$

Resonances of the structure having $(2N+1)$ beams are then given by the determinant of

$$U_{2N+1}(1{:}\,2, 1{:}\,2) \text{ or } U_{2N+1}(3{:}\,4, 3{:}\,4) \text{ i.e. } (g(\beta))^2(C_0^2 - C_1C_3).$$

It is obvious from this matrix that a structure having $(2N+1)$ beams will have double roots together with resonances coinciding with those of a cantilever beam.

5. Discussion

The results of this study demonstrate valid patterns of the distribution of resonances for different number of beams. The first group or cluster of resonances is bounded below by the first resonance of an F–F beam and above by the first resonance of a C–F beam while the second cluster is bounded below by the second resonance of a C–F beam while above by that of an F–F beam. In this way the lower and upper bounds for the clusters are determined alternatively by the resonances of F–F and C–F beam. It is also observed that these groups or clusters occur in groups of two for structures having even number of beams, while for the structures with odd number of

Table 1
Distribution of resonances in clusters bounded by resonances of F–F and C–F

No. of Beams	Distribution of resonances in clusters determined by F-F and C-F														
	A1		B1		B2		A2		A3		B3		B4		A4
2		2				2				2				2	
3		2	1		1	2				2	1		1	2	
4		2 2				2 2				2 2				2 2	
5		2 2	1		1	2 2				2 2	1		1	2 2	
6		2 2 2				2 2 2				2 2 2				2 2 2	

Table 2
Resonances of F–F and C–F beams (for reference)

Beam type	First	Second	Third	Fourth
F–F (A*i*)	645.8	1780.3	3490.2	5769.4
C–F (B*i*)	101.5	636.1	1781.1	3490.1

beams one resonance of each cluster occurs exactly at the resonance of a cantilever beam. These results are illustrated in Table 1 for structures having up to six beams and are derived by the W–W algorithm. Here, A*i*, B*i* are the *i*th resonances of an F–F and a C–F beam, respectively, and bold "1" and "2" mean a single and a double resonance, respectively. There are forbidden zones that are bounded by the resonances of the F–F and C–F. For reference, the natural frequencies for F–F and C–F beams are given in Table 2 [6] for a beam having length equal to 1 m and a square cross section having sides equal to 20 mm.

6. Conclusions

(a) The structures studied in this paper have repeated resonances that lie in well-defined clusters. The total number of resonances in these clusters is equal to the total number of beams in the structure.
(b) The clusters consist of resonances of multiplicity two for even number of beams. There are solitary resonances at the resonances of C–F beam for structures with odd number of beams.
(c) All resonance frequencies are bounded by the resonances of F–F and C–F beams in an alternating manner.
(d) The resonances tend to converge together as the frequency increases.
(e) A split is observed in the resonances if either a Timoshenko formulation of beam is used or the beams are connected through an element of finite stiffness resulting in unequal deflections for the two beams at the joining point.
(f) This class of structures can be considered as a test case for structures which have repeated resonances and structures which have very unusual distribution of modal densities.

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