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Journal of Sound and Vibration 290 (2006) 529–537

JOURNAL OF SOUND AND VIBRATION

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Short Communication

An approximate approach to study the free vibration of a string with an arbitrary variation in mass density and tension, with attached concentrated masses, and its application to hanging chain and rotating cord

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Received 16 July 2004; received in revised form 14 April 2005; accepted 30 April 2005
Available online 1 August 2005

Abstract

A multi-stepped approximation has been developed to study the free vibration of a string with arbitrary variation in the tension and the mass density. To achieve this objective, a novel exact approach is developed to study free vibration of a string with N uniform sections of different mass densities under N different tensions with $N+1$ attached concentrated masses and springs at the ends. The two advantages of this approach are that it leads to a single frequency equation for any number of sections and the same computer program can be used to solve different problems using pertinent data. This method is applied to cases where tension and/or mass density changes continuously such as hanging chain, rotating chain, or inhomogeneous strings vibrating in either horizontal or vertical position. In all cases, the results obtained using this approach agreed with previous results obtained using exact and approximate methods. It was found that natural frequencies of a fixed string under constant tension T with linear variation in the mass density, $\rho(x) = \rho_0(1 + \alpha x/L)$, can be approximated using a very simple equation, $\omega_n = (n\pi/L)[T/\{\rho_0(1 + 0.5\alpha)\}]^{1/2}$ and results reasonably agree with previous results for a wide range of α. New results for many other cases are also presented in what follows.

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doi:10.1016/j.jsv.2005.04.015

1. Introduction

Often transverse vibration of uniform string has been used as an example to introduce vibration of continuous systems [1]. Exact closed-form solutions under constant tension can be obtained with classical boundary conditions and with end masses and springs. Vibrations of strings under constant tension with stepped changes in mass density are investigated to find natural frequencies and to obtain harmonic spectra [2,3]. Vibrations of string with continuous variation in mass density under constant tension are also investigated [4–6]. Vibration of a uniform fixed string with attached concentrated mass is also investigated [7].

In all of the above cases tension is constant. However, in cases such as uniform hanging cord with or without tip mass [8–10] and rotating cord [11] tension changes from the free end to the fixed end. Such problems are solved using Bessel functions of zero order [8,9], method of Frobenius [10] and Legendre function [11]. However, solutions are not available for cases where the tension and the mass density vary arbitrarily along the length.

Here an approximate approach is developed to study vibration of string where mass density and tension varies arbitrarily using a multi-stepped representation. To achieve this goal, an exact approach has been developed in this manuscript to study the free vibration of a string with N uniform sections with different mass densities and tensions and with $N+1$ attached concentrated masses and with elastic springs at ends. A transfer matrix method developed previously by the author [12,13] is used here as it leads to a single-frequency equation for any number of sections. Exact closed-form solutions are developed for simple cases. It is found that natural frequencies of a fixed string under constant tension T with linear variation in the mass density, $\rho(x) = \rho_0(1 + \alpha x/L)$, can be approximated using a very simple equation, $\omega_n = (n\pi/L)[T/\{\rho_0(1 + 0.5\alpha)\}]^{1/2}$. The predictions of the proposed theory agree in all cases with previous results and new results are presented in the following.

2. Theory

The differential equation of motion of a string with variable tension $T(x)$ and mass per unit length $\rho(x)$ is given as

$$\partial/\partial x[T(x)\partial y(x,t)/\partial x] = \rho(x)[\partial^2 y(x,t)\partial^2 t]. \qquad (1)$$

In a multi-stepped method the T_i and ρ_i of the ith section are determined as, $T_i = (T_{i\text{exact}} + T_{(i+1)\text{exact}})/2$ and $\rho_i = (\rho_{i\text{exact}} + \rho_{(i+1)\text{exact}})/2$. Using this approximation, a transfer matrix approach is developed, and is presented in what follows.

Consider a string shown in Fig. 1 which consists of N different but uniform sections with end points of the ith section, i and $i+1$, length L_i and attached concentrated mass at point i, m_i and two springs at end points 1 and $N+1$ as K_1 and K_{N+1}, respectively. The differential equation for the small transverse vibration of the uniform ith section can be written as,

$$T_i\partial^2 y_i(x_i,t)/\partial^2 x_i = \rho_i\partial^2 y_i(x_i,t)/\partial^2 t. \qquad (2)$$

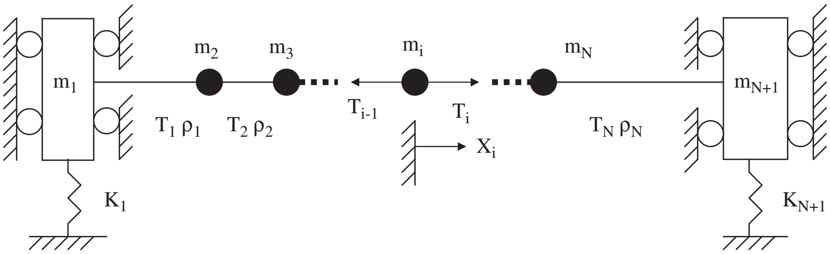

Fig. 1. Model of a string with N different tensions on different uniform sections with attached concentrated masses.

The $y_i(x_i, t)$ represents the deflection of the ith section at distance x_i measured from the ith point at time t. The pth mode during the harmonic motion of the ith section can be expressed as

$$y_{ip}(x_i, t) = (A_{ip} \cos \lambda_{ip} x_i + B_{ip} \sin \lambda_{ip} x_i) \exp(j\omega_p t), \tag{3}$$

where $\lambda_{ip} = \omega_p / a_i$, $a_i^2 = (T_i/\rho_i)$ and ω_p is the natural frequency to be determined. There are N sections and N values of A_{ip} and B_{ip} and this lead to $2N$ unknowns. Here an efficient approach, based on the transfer matrix method, is developed to express $(2N-2)$ unknowns in terms of A_{1p} and B_{ip} [12,13]. A novel general single-frequency equation in only one unknown ω_p was obtained by eliminating A_{1p} and B_{1p} using boundary conditions at the end points 1 and $N+1$. The required transfer matrix M_i relates A_{ip} and B_{ip} of the ith section to $A_{(i-1)p}$ and $B_{(i-1)p}$ of the $(i-1)$ th section and is obtained next. Considering the continuity of transverse displacement and force balance at point i leads to

$$y_{(i-1)p|x_{(i-1)}=L_{(i-1)}} = y_{ip|x_i=0} \tag{4}$$

and

$$(T_i \delta y_i(x_i, t)/\delta x_i)_{|x_i=0} - (T_{(i-1)} \delta y_{i-1}(x_{i-1}, t)/\delta x_{i-1})_{|x_i-1=L_i-1} \\ = (m_i \delta^2 y_i(x_i, t)/\delta^2 t)_{|x_i=0.} \tag{5}$$

Substituting and evaluating expressions of $y_{ip}(x_i, t)$ from Eq. (3) into Eqs. (4) and (5) leads to

$$A_{ip} = A_{(i-1)p} C_{i-1} + B_{(i-1)p}, S_{i-1}, C_{i-1} = \cos[\lambda_{(i-1)p} L_{(i-1)}] \text{ and } S_{i-1} = \sin[\lambda_{(i-1)p} L_{(i-1)}], \tag{6}$$

$$T_i \lambda_{ip} B_{ip} + \omega_p^2 m_i A_{ip} = T_{i-1} \lambda_{(i-1)p} \ (-A_{(i-1)p} S_{i-1} + B_{(i-1)p} C_{i-1}). \tag{7}$$

Eqs. (6) and (7) can be solved to express A_{ip} and B_{ip} in terms $A_{(i-1)p}$ and $B_{(i-1)p}$ using a transfer matrix M_i as

$$[A_{ip}, B_{ip}]^{\mathrm{T}} = M_i [A_{(i-1)p}, B_{(I-1)p}]^T, \tag{8}$$

where

$$M_i(1,1) = C_{i-1}, M_i(1,2) = S_{i-1}, \quad M_i(2,1) = (-T_{i-1} \lambda_{(i-1)p} S_{i-1} - \omega_p^2 m_i C_{i-1})/T_i \lambda_{ip}, \\ M_i(2,2) = (T_{i-1} \lambda_{(i-1)p} C_{i-1} - \omega_p^2 m_i S_{i-1})/T_i \lambda_{ip}. \tag{9}$$

Eq. (8) can be repeatedly used to relate the coefficients of the Nth span to those of the first span as,

$$[A_{Np}, B_{Np}]^{\mathrm{T}} = M_{TN} [A_{1p}, B_{1p}]^{\mathrm{T}} \text{ where } M_{TN} = M_N M_{N-1} M_{N-2}, \ldots, M_2. \tag{10}$$

The boundary conditions at points 1 and $N+1$ can be obtained by using force balance at these points as

$$(T_1 \delta y_1(x_1,t)/\delta x_1)_{|x_i=0} = (m_1 \delta^2 y_1/\delta^2 t + K_1 y_1)_{|x_i=0} \tag{11}$$

and

$$(-T_N \delta y_N/\delta x_N)\big|(x_N = L_N) = (m_{N+1}\delta^2 y_N/\delta^2 t) + K_{N+1} y_N))\big|(x_N = L_N). \tag{12}$$

Evaluating and substituting values of derivatives from Eq. (2) into Eqs. (11) and (12) leads to

$$(K_1 - m_1\omega_p^2)A_{1p} - T_1\lambda_{1p}B_{1p} = 0 \text{ and } R_{1N}A_{Np} + R_{2N}B_{Np} = 0, \tag{13a,b}$$

where

$$R_{1N} = T_N\lambda_{Np}S_N + (-K_{N+1} + m_{N+1}\omega_p^2)C_{NP} \text{ and } R_{2N} = -T_N\lambda_{Np}C_N + (-K_{N+1} + m_{N+1}\omega_p^2)S_N$$

Values of $A_{Np} = M_{TN}(1,1)\,A_{1p} + M_{TN}(1,2)\,B_{1p}$ and $B_{Np} = M_{TN}\,(2,1)\,A_{1p} + M_{TN}\,(2,2)\,B_{1p}$ are obtained from Eq. (10) and are substituted into Eq. (13a) to get

$$[R_{1N}M_{TN}(1,1) + R_{2N}M_{TN}(2,1)]A_{1p} + [R_{1N}M_{TN}(1,2) + R_{2N}M_{TN}(2,2)]B_{1p} = 0. \tag{14}$$

Eliminating A_{1p} and B_{1p} from Eqs. (13a) and (14) leads to a general frequency equation

$$[K_1 - m_1\omega_p^2][R_{1N}M_{TN}(1,2) + R_{2N}M_{TN}(2,2)] + T_1\lambda_{1p}[R_{1N}M_{TN}(1,1) + R_{2N}M_{TN}(2,1)] = 0. \tag{15}$$

Closed-form solutions can be obtained from Eq. (15) for large N, however, it is not practical. It is also possible to get results for fixed ends using extremely large values of spring stiffnesses K_1 and K_{N+1}. However, to avoid numerical problems following equations are used. Frequency equation of a string with fixed left end (i.e. $K_1 \to \infty$) and with both ends fixed (i.e. $K_1 \to \infty$ and $K_{N+1} \to \infty$) could be obtained as

$$R_{1N}M_{TN}(1,2) + R_{2N}M_{TN}(2,2) = 0 \text{ and } C_N M_{TN}(1,2) + S_N M_{TN}(2,2) = 0. \tag{16a,b}$$

The elements of M_i for a string under constant tension with all $m_i = 0$, are,

$$M_i(1,1) = C_{i-1},\;\; M_i(1,2) = S_{i-1},\;\; M_i(2,1) = (-a_{i-1}/a_i)S_{i-1},\;\; M_i(2,2) = (a_{i-1}/a_i)C_{i-1}. \tag{17}$$

The frequency equation for single span ($N = 1$) is obtained by substituting M_{TN} as an identity matrix as

$$\begin{aligned}&[(K_1 - m_1\omega_p^2)(K_2 - m_2\omega_p^2) - T_1^2\lambda_{1p}^2]\sin(\lambda_{1p}L_1)\\ &\quad + T_1\lambda_{1p}[K_1 + K_2 - \omega_p^2(m_1 + m_2)]\cos(\lambda_{1p}L_1) = 0.\end{aligned} \tag{18}$$

Next, closed-form solutions under constant tension ($T_1 = T_2 = T$) for $N = 2$ and 3 are developed. The frequency equation of a fixed–fixed uniform string ($\rho_1 = \rho_2 = \rho$), with concentrated mass m_2 at the center ($N = 2$) and length $2L$ ($L_1 = L_1 = L$) is obtained using $C_1 = C_2 = \cos(\lambda_{1p}L)$, $S_1 = S_2 = (\sin\lambda_{1p}L)$, $M_{TN} = M_2$, and $M_{TN}(1,2) = S_1$ and $M_{TN}(2,2) = C_1 - \omega_p^2 m S_1/T_1\lambda_{1p}$ as

$$\sin(\lambda_{1p}L)[2\cos(\lambda_{1p}L) - (\omega_p^2 m/T\lambda_{1p})\sin(\lambda_{1p}L)] = 0. \tag{19}$$

The second part of this equation is $\tan(\lambda_{1p}L) = 2T/(m\,a_1\,\omega_p)$ and agrees with that given in Ref. [7]. However, Eq. (19) is preferred. The frequency equation of a fixed–fixed string under constant tension with $N = 2$, $\rho_1 \neq \rho_2$ can be obtained using $M_{TN} = M_2$, with elements

$M_{T2}(1,2) = S_1$, $M_{T2}(2,2) = +(a_1/a_2)C_1$ given by Eq. (17), into Eq. (16b) as

$$C_2S_1 + (a_1/a_2)C_1S_2 = 0, \quad (20)$$

and is the same as given in Ref. [2]. Similarly, frequency equation of a fixed–fixed three part string ($N = 3$) under constant tension with different densities can be obtained by using $M_{TN} = M_3 M_2$. Substituting $M_{TN}(1,2) = C_2 S_1 + (a_2/a_1)C_1 S_2$ and $M_{TN}(2,2) = -(a_3/a_2)S_1 S_2 + (a_3/a_2)C_1 C_2$ into Eq. (16b) leads to frequency equation

$$-(a_3/a_2)S_1S_2S_3 + C_3C_2S_1 + (a_2/a_1)C_3C_1S_2 + (a_3/a_1)S_3C_1C_2 = 0 \quad (21)$$

and is the same as that given in Ref. [3].

3. Results and discussion

Natural frequencies are determined by solving the pertinent equations numerically. The iterative computations are terminated when the value of the expression changed sign due a change of 10^{-7} in the value of ω_p. The mode shape of a string can be obtained using Eqs. (3) and (10). The values of A_{1p} and B_{1p} are selected as follows: (a) left end fixed: $A_{1p} = 0$, $B_{1p} = 1$; (b) left end free: $A_{1p} = 1$ and $B_{1p} = 0$ and (c) left end with mass and spring: $A_{1p} = 1$ and $B_{1p} = (K_1 - m_1\omega_p^2)/T_1\lambda_{1p}$. All other A_{ip}'s and B_{ip}'s are obtained by repeatedly using Eq. (11).

3.1. Constant tension cases

The results obtained for a single span case agreed with previous results [1]. In Table 1 results of two and three part strings ($N = 2$ and $N = 3$) under constant tension without attached masses are presented and they agreed with previous results [2,3]. Additional results with attached masses are also presented. Results are also obtained for strings under constant tension for four non-homogeneous mass density functions given in Refs. [3,4,6] using parameter $\alpha = 2.5$ and are presented in Table 2. Values of exact results given in Table 2 agree with those obtained using theory presented in Section 2. The results for a string under constant tension with linear variation in mass density, $\rho(x) = \rho_0(1 + \alpha x/L)$, is considered by approximating it as a multi-stepped system and are presented in the last two blocks of Table 2 and they agree with exact results for $\alpha = 0.1$ and 60. Additional results obtained using 300 steps ($N = 300$) indicate that they are identical at least up to the 5th decimal with the exact results obtained using equation given in Ref. [4]. Additionally, comparison of results obtained using one section (i.e. $N = 1$, $\rho_1 = (\rho_{1\text{exact}} + \rho_{2\text{exact}})/2 = \rho_0(1 + 0.5\alpha)$ for a wide range of α to exact results also indicated good agreement. These results indicated that approximate values of ω_n for this case could be obtained as

$$\omega_n = (n\pi/L)[T/\{\rho_0(1 + 0.5\alpha)\}]^{1/2}. \quad (22)$$

More results are obtained using values of parameters given in Ref. [5] ($N = 100$, $L = 1\,\text{m}$, $\rho_0 = 0.01\,\text{kg/m}$, $\alpha = 3$, $T = 10\,\text{N}$). The values of the first three natural frequencies were $\omega_1 = 9.92428\,\text{Hz}$, $\omega_2 = 20.16531\,\text{Hz}$ and $\omega_3 = 30.37080\,\text{Hz}$ and they agree with previous results $\omega_1 = 9.9242 - 9.9244\,\text{Hz}$, $\omega_2 = 20.1653{-}20.1654\,\text{Hz}$ and $\omega_3 = 30.3707{-}30.3708\,\text{Hz}$ which are given in Ref. [5]. The results for variable tension cases are presented next.

Table 1
Natural frequencies of a two and three ($N = 2,3$) step strings with and without masses

Parameters	ω_1	ω_2	ω_3
Fixed–fixed, $N = 2$, $L_1 = 0.2$, $L_2 = 0.8$, $\rho_1 = 1.8$, $\rho_2 = 1$, [2] and $m_2 = 0.5$	3.077423 2.492889	5.879167 4.561965	8.685548 8.101929
Fixed–fixed, $N = 2$, $L_1 = 0.8$, $L_2 = 0.2$, $\rho_1 = 1.4$, $\rho_2 = 1$, [2] and $m_2 = 0.5$	2.673238 2.315033	5.422299 4.156389	8.230085 6.998833
Fixed–fixed, $N = 2$, $L_1 = 0.4$, $L_2 = 0.6$, $\rho_1 = 2.25$, $\rho_2 = 1$, [2] and $m_2 = 0.5$	2.617993 1.987431	5.235987 5.235987	7.853981 6.347937
Free–fixed, $N = 2$, $L_1 = 0.4$, $L_2 = 0.6$, $\rho_1 = 2.25$, $\rho_2 = 1$, [2] and $m_2 = 0.5$	1.141199 0.982672	4.049789 3.549103	6.377186 5.724327
Fixed–fixed, $N = 3$, $L_1 = 0.6$, $L_2 = 0.25$, $L_3 = 0.15$, $\rho_1 = 1$, $\rho_2 = 15.44448$, $\rho_3 = 17.64$, [3] and $m_2 = 0.5$ $m_3 = 0.5$	1.745328 1.462919	3.49066 2.908384	5.235988 5.231816
Fixed-free, $N = 3$, $L_1 = 0.36$, $L_2 = 0.24$, $L_3 = 0.4$, $\rho_1 = 1$, $\rho_2 = 2.25$, $\rho_3 = 0.81$, [3] and $m_2 = 0.5$ $m_3 = 0.5$	1.454441 1.060270	4.363323 3.359966	7.272205 5.326169
Free-free, $N = 3$, $L_1 = 0.6$, $L_2 = 0.3$, $L_3 = 0.1$, $\rho_1 = 1$, $\rho_2 = 4$, $\rho_3 = 36$, [3] and $m_2 = 0.5$ $m_3 = 0.5$	1.745329 1.223339	3.490658 2.425975	5.235987 4.058853
Free-free, $N = 3$, $L_1 = 0.6$, $L_2 = 0.2774456$, $L_3 = 0.125533$, $\rho_1 = 1$, $\rho_2 = 4.77921$, $\rho_3 = 22.841125$, [3] and $m_2 = 0.5$ $m_3 = 0.5$	1.813704 1.268549	3.422283 2.411818	5.235987 3.924170

Most parameters are chosen to match the previous results [2,3].

3.2. Variable tension—hanging chain

Variation in tension in three cases of hanging chain considered here are: (a) a uniform chain: $T(x) = \rho g(L - x)$, $0 \leqslant x \leqslant L$; (b) a uniform chain with the tip mass, m_{N+1} : $T(x) = \rho g(L - x) + gm_{N+1}$, $0 \leqslant x \leqslant L$ and (c) a uniform chain with the tip mass, m_{N+1} and other mass, m_P at distance L_p from the top: $T(x) = \rho g\,(L - x) + gm_{N+1}$, $L_P \leqslant x \leqslant L$, $T(x) = \rho g\,(L - x) + gm_{N+1} + gm_P$ $0 \leqslant x \leqslant L_P$, respectively. In Table 3, the results obtained for first two cases are presented in the rows 3 and 4 and they agree with previous results [9,10]. A few new results are presented in the same table for a fixed–free chain with two attached masses, fixed–fixed hanging string and a fixed–free hanging chain with linearly increasing mass density without and with attached mass.

3.3. Rotating cord

The variation in tension for uniform cord with end mass m_{N+1} is given as $T(x) = \Omega^2[m_{N+1}(L + R) + (\rho/2)(L^2 - x^2) + \rho R(L - x)]$, $0 \leqslant x \leqslant L$. The R is the distance from the center of rotation to the point where string is attached in a rotating frame. One additional example of string with linear variation in mass density, $\rho(x) = \rho_0(1 + \alpha x/L)$, with a tip mass is considered and tension in this case is given as $T(x) = \Omega^2 m_{N+1}(L + R) + \rho_0\Omega^2[(\alpha/(3L)(L^3 - x^3) + 0.5(1 + \alpha R/L)(L^2 - x^2) + R(L - x)]$. If additional mass m_{81} is attached at point 81,

Table 2
Effect of continuously changing mass density on the natural frequencies of a fixed–fixed string under constant tension

	ω_1	ω_2	ω_3
$(\alpha+2)^2/[4(\alpha^2+2\alpha)x+4]$			
Exact [6]	3.304943	6.380517	9.492739
Present results, $N=160$	3.304778	6.380015	9.491874
$[\ln(\alpha+1)/\alpha]^2(\alpha+1)^{2x}$			
Exact [6]	3.084686	6.250411	9.402039
Present results, $N=160$	3.084624	6.250285	9.401847
$(\alpha+1)^2\,(\alpha+1-\alpha x)^{-4}$			
Exact [3,6]	3.141592	6.283185	9.424777
Present results, $N=160$	3.141043	6.282036	9.423039
$\rho(x)=\rho_0(1+\alpha x/L)$, $\alpha=0.1$			
Exact [4]	0.975852	1.951914	2.927930
Present results, $N=300$	0.975852	1.951914	2.927934
Present results, $N=1$	0.975900	1.951800	2.927700
$\rho(x)=\rho_0(1+\alpha x/L)$, $\alpha=60$			
Exact [4]	0.176220	0.365537	0.555140
$N=300$	0.176220	0.365537	0.555140
$N=1$	0.179605	0.359210	0.538816

Length $L=1$, $L_i=1/N$, $T_i=1$ and $\alpha=2.5$.
Exact results are obtained using Refs. [3,4,6].

Table 3
Comparison of present results with previous once in case of a hanging chain with a tip mass

Parameters	ω_1		ω_2		ω_3	
	Previous results exact [9]	Present results $N=160$	Previous results exact [9]	Present results $N=160$	Previous results exact [9]	Present results $N=160$
Fixed–free, $m_{N+1}=0$	1.20241	1.20235	2.76003	2.75904	4.32686	4.32244
Fixed–free, $m_{N+1}=1$	1.05642	1.05642	4.08168	4.08169	7.73750	7.73751
Fixed–free, $m_{N+1}=10$	1.00793	1.00793	10.2764	10.2764	20.4041	20.4041
Fixed–free with $m_{N+1}=0.5$, $m_{1+N/2}=0.5$		1.12561		2.67139		6.15452
Fixed–fixed		1.47446		3.11977		4.75875
Fixed–free $\rho(x)=(1+\alpha x)$, $\alpha=2.5$		1.16761		2.93068		4.66068
Fixed-free $\rho(x)=(1+\alpha x)$, $\alpha=5$, $m_{n+1}=0.5$, $m_{1+N/2}=0.5$		1.13856		2.85959		4.98911

A few additional results are also presented. Total length $L=1$, $L_i=1/N$, $\rho_i=1$ and $g=1$.

Table 4
Natural frequencies of a rotating cord with $\rho(x) = \rho_0(1 + \alpha x/L)$ and with and without attached masses

Parameters		Exact [11] ω_1^2	Present ω_1^2	Present ω_2^2	Present ω_3^2
$\alpha = 0.0$	$M_{N+1} = 0$	0.3654	0.3654	2.6863	6.8736
$R = -0.4$	$M_{N+1} = 1$	0.5448	0.5468	8.2626	29.9162
	$M_{N+1} = 10$	0.5934	0.5934	61.4925	243.0856
	$M_{N+1} = 1$ and $M_{81} = 1$		0.4822	3.1959	31.26664
$\alpha = 0.0$	$M_{N+1} = 0$	1.000	0.9999	5.9968	14.9774
$R = 0.0$	$M_{N+1} = 1$	1.000	0.9999	15.0000	54.1220
	$M_{N+1} = 10$	1.000	0.9999	103.7459	409.6591
	$M_{N+1} = 1$ and $M_{81} = 1$		0.9989	6.0099	55.0183
$\alpha = 0.0$	$M_{N+1} = 0$	15.4935	15.4921	82.2891	202.2143
$R = 10.0$	$M_{N+1} = 1$	12.1674	12.1674	181.6799	653.0876
	$M_{N+1} = 10$	11.1594	11.1594	1159.810	4593.000
	$M_{N+1} = 1$ and $M_{81} = 1$		13.3916	76.6202	628.9602
$\alpha = -0.5$	$M_N = 0.0$		0.5378	1.4389	2.3024
$R = -0.4$	$M_{N+1} = 1.0$		0.7456	3.1114	6.0484
	$M_{81} = 1.0$		0.4940	1.2593	2.6804
$\alpha = 1.0$	$M_{N+1} = 0.0$		0.9999	2.5908	4.1206
$R = 0.0$	$M_{N+1} = 1.0$		0.9999	3.5145	6.5472
	$M_{81} = 1.0$		0.9987	2.0450	3.9393
$\alpha = 1.0$	$M_{N+1} = 0.0$		1.5516	3.8742	6.1400
$R = 1.0$	$M_{N+1} = 1.0$		1.4669	5.1136	9.5309
	$M_{81} = 1.0$		1.5999	3.0841	5.7252

Total length $L = 1$, $N = 160$, $L_i = 1/N$, $\rho_0 = 1.0$ and $\Omega = 1$.

which is located at the distance $L/2$ from the left end, then tension in this case is obtained by adding $\Omega^2 m_{81}(L/2 + R)$ to the tension that is given by the last equation to all sections before point 81. The results obtained for the case of a rotating uniform cord with $R = -0.04$, 0 and 10 without and with attached mass at the free end are presented in Table 4 and the values of ω_1^2 agree with the previous results [11]. Additional new results of a rotating uniform and non-uniform cord with $\rho(x) = \rho_0(1 + \alpha x/L)$ with and without attached masses are also presented.

4. Conclusions

An approximate approach is presented to study the free vibration of a string with arbitrary variation in the tension and the mass density using a multi-stepped approximation. To achieve this a novel exact approach has been developed to study free vibration of a string with N uniform sections of different mass densities under N different tensions and with $N + 1$ attached concentrated masses and springs at the ends. The two advantages of this approach are that it leads to a single-frequency equation for any number of sections and the same computer program can be used to solve different problems using pertinent data. It is found that natural frequencies of a fixed–fixed string under constant tension T with mass density, $\rho(x) = \rho_0(1 + \alpha x/L)$, can be approximated using a very simple equation, $\omega_n = (n\pi/L)[T/\{\rho_0(1 + 0.5\alpha)\}]^{1/2}$ and gives reasonable results for a wide range of α. In all cases, the results obtained using presented approach agreed with previous results. Author believes that this approach can be used in other areas of science and engineering.

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