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Parameter estimations of parametrically excited pendulums based on chaos feedback synchronization

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Abstract

In this paper, an iterate parameter estimation scheme based on chaos feedback synchronization is applied for parametrically excited pendulums. The response pendulum can exactly reproduce the system parameters and bifurcation diagram of the drive pendulum, thus allowing two chaotic pendulums to be synchronized even though their initial parameter values are quite different. The effects of feedback function and feedback weight are investigated. Feedback and parameter controls suppress the perturbations resulting from noise and parameter mismatches. This parameter estimation method based on feedback synchronization is robust to large parameter mismatches.

1. Introduction

Chaos in parametrically excited pendulums as well as its control and synchronization have been intensively studied through theoretical analyses and numerical simulations [1–6]. The experimental study of parametrically excited pendulums has also received considerable interest [7]. In practical applications, parameter mismatches inevitably exist and degrade synchronization [8,9]. The synchronization of two chaotic systems with initial parameter mismatches becomes an important

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question. In addition, a model equation can usually be specified, but direct and accurate measurements of all system parameters may be difficult. The estimation of chaotic system parameters is another important issue [10–12]. Applying chaos synchronization to parameter estimates presents the solution for these two questions, and it has recently received considerable attention [13–15].

The continuous feedback method [16] has been applied for chaos synchronization because of its characteristic virtues under the conditions of parameter mismatches and noise perturbations. It requires neither analysis of the dynamic behavior nor for the system to be divided into stable and unstable subsystems, such as in Pecora–Carroll synchronization [8]. In addition, feedback synchronization is especially robust to parameter mismatches and noise [4,5,16]. Thus the feedback method can be experimentally applied to a variety of dynamic systems in noisy environments [17]. In this paper, we study a parameter estimation scheme based on feedback synchronization. An iterative procedure of parameter control is applied to the response pendulum in order to approach the drive pendulum parameters. Two chaotic pendulums with initial parameter mismatches are finally synchronized. Also, the effects of feedback weight and noise are investigated. It is found that this parameter estimation method based on feedback synchronization is robust to large initial parameter mismatches.

2. Parameter estimation based on chaos feedback synchronization

To investigate the parameter estimation scheme based on feedback synchronization, we consider a pair of parametrically excited pendulums connected through variable feedback. The dynamics of the drive pendulum can be described as,

$$\dot{\mathbf{x}}_d = \mathbf{P}(\boldsymbol{\mu}_d, \mathbf{x}_d), \tag{1}$$

where the state vector is $\mathbf{x}_d = \{x_d, y_d\}^{\mathrm{T}}$, the parameter vector is $\boldsymbol{\mu}_d = \{\beta_d, p_d, \omega\}^{\mathrm{T}}$, and $\mathbf{P}(x, y; \beta, p, \omega) = \{y, -\beta y - (1 + p\cos(\omega t))\sin(x)\}^{\mathrm{T}}$. We assume that the response pendulum with feedback control has the same driving frequency ω as the drive pendulum, but has a different driving force p and a different damping constant β. The dynamic equation of the response pendulum can be written as:

$$\dot{\mathbf{x}}_r = \mathbf{P}(\boldsymbol{\mu}_r, \mathbf{x}_r) + K\mathbf{I}(\mathbf{x}_d - \mathbf{x}_r), \tag{2}$$

where $\mathbf{x}_r = \{x_r, y_r\}^{\mathrm{T}}$, $\boldsymbol{\mu}_r = \{\beta_r, p_r, \omega\}^{\mathrm{T}}$, $\mathbf{I} = \left\{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right\}$ is the feedback function, and K is the feedback weight. When $\Delta\boldsymbol{\mu} = \boldsymbol{\mu}_d - \boldsymbol{\mu}_r$, for an appropriate value of K, $\delta\dot{\mathbf{x}} = [\mathbf{P}_{\mathbf{x}_r}(\boldsymbol{\mu}_r, \mathbf{x}_r) - K\mathbf{I}]\delta\mathbf{x}$ has a stable zero solution $\delta\mathbf{x}_s(t) \to \mathbf{0}$ with $t \to \infty$, and thus the drive and response pendulums are synchronous, $\mathbf{x}_d = \mathbf{x}_r$ [4,5,16]. However, when $\|\boldsymbol{\mu}_d - \boldsymbol{\mu}_r\|$ is large, the synchronization error $\delta\mathbf{x}$ does not approach zero. For small synchronization error $\delta\mathbf{x} = \mathbf{x}_d - \mathbf{x}_r = \{\delta x, \delta y\}^{\mathrm{T}}$ and parameter difference $\|\Delta\boldsymbol{\mu}\|/\|\boldsymbol{\mu}_r\| \ll 1$, we have the linearized difference dynamics,

$$\delta\dot{\mathbf{x}} = [\mathbf{P}_{\mathbf{x}_r}(\boldsymbol{\mu}_r, \mathbf{x}_r) - K\mathbf{I}]\delta\mathbf{x} + \mathbf{P}_{\beta_d}\Delta\beta_d + \mathbf{P}_{p_d}\Delta p_d. \tag{3}$$

Eq. (3) can be integrated as

$$\delta\mathbf{x}(t) = \int_0^t \delta\mathbf{x}_s(t - \tau)(\mathbf{P}_{\beta_d}\Delta\beta_d + \mathbf{P}_{p_d}\Delta p_d)\,\mathrm{d}\tau. \tag{4}$$

We then have the proportional relationship between the synchronization error $\delta\mathbf{x}$ and parameter difference $\Delta\boldsymbol{\mu}$ as

$$\delta\mathbf{x}(t) = \mathbf{B}_1\Delta\beta_d + \mathbf{B}_2\Delta p_d, \tag{5}$$

where $\mathbf{B}_1 = \int_0^t \delta\mathbf{x}_s(t-\tau)\mathbf{P}_{\beta_d}\,\mathrm{d}\tau$ and $\mathbf{B}_2 = \int_0^t \delta\mathbf{x}_s(t-\tau)\mathbf{P}_{p_d}\,\mathrm{d}\tau$. In the pendulum systems, the synchronization error $\delta\mathbf{x}$ is measurable. Thus, if $\mathbf{B}_{\mathrm{i}}$ has been determined, then the parameter differences $\Delta\beta_d$ and Δp_d can be estimated. However, the analysis solution of $\mathbf{B}_i$ is difficult to obtain for chaotic pendulum systems. Here, we generalize the iterative procedure, which has been previously applied to the active-passive decomposition [15], to the feedback synchronization in order to numerically approach $\mathbf{B}_i$ and $\Delta\boldsymbol{\mu}$. Briefly, considering the following two additional subsystems with the variable feedback,

$$\dot{\mathbf{x}}_1 = \mathbf{P}(\beta_r + \varDelta, p_r; \mathbf{x}_1) + K\mathbf{I}(\mathbf{x}_d - \mathbf{x}_1), \tag{6a}$$

and

$$\dot{\mathbf{x}}_2 = \mathbf{P}(\beta_r, p_r + \varDelta; \mathbf{x}_2) + K\mathbf{I}(\mathbf{x}_d - \mathbf{x}_2), \tag{6b}$$

with $\delta\mathbf{x}^{(1)} = \mathbf{x}_1 - \mathbf{x}_r$, $\delta\mathbf{x}^{(2)} = \mathbf{x}_2 - \mathbf{x}_r$, and $\varDelta \ll 1$, we can approximately estimate $\mathbf{B}_i$ by $\tilde{\mathbf{B}}_1 = \delta\mathbf{x}^{(1)}/\varDelta$ and $\tilde{\mathbf{B}}_2 = \delta\mathbf{x}^{(2)}/\varDelta$. For a time series $\delta\mathbf{x}(j\tau)$, $j = 1, 2, \ldots, N$, and τ is the time interval), $\Delta\boldsymbol{\mu}$ then can be approximated by using the least-squares fit

$$\sum_{j=0}^{N} \left\| \delta\mathbf{x}(j\tau) - \tilde{\mathbf{B}}_1\Delta\tilde{\beta}_d - \tilde{\mathbf{B}}_2\Delta\tilde{p}_d \right\|^2 = \text{minimum}. \tag{7}$$

The drive parameters $\boldsymbol{\mu}_r$ an be approached by iteratively solving $\Delta\tilde{\boldsymbol{\mu}}(k)$ based on Eq. (7) and adapting the response parameters as $\boldsymbol{\mu}_r(k+1) = \boldsymbol{\mu}_r(k) - \Delta\tilde{\boldsymbol{\mu}}(k)$. When the parameter difference $\Delta\boldsymbol{\mu} = \boldsymbol{\mu}_d - \boldsymbol{\mu}_r$ is within a certain range, with the increase of the iterative number k, $\Delta\tilde{\boldsymbol{\mu}}(k)$ approaches zero and the response parameters $\boldsymbol{\mu}_r(k)$ asymptotically converge to the drive parameters $\boldsymbol{\mu}_d(k)$. Two chaotic pendulum systems with initial parameter mismatches can be finally synchronized. The convergence of this numerical procedure can be demonstrated using the similar method presented in Ref. [15]. However, when the parameter difference $\Delta\boldsymbol{\mu}$ is excessively large, this iterative procedure of parameter estimation may have a divergent solution of $\Delta\tilde{\boldsymbol{\mu}}(k)$, and then the parameters of the drive pendulum cannot be correctly estimated. In contrast to previous studies that were based on Pecora–Carroll decomposition and active–passive decomposition [13,15], our method of parameter estimation is based on feedback synchronization. It must be pointed out that although this study focuses on parametrically excited pendulums, the presented method can be applied to general chaotic systems, such as the Lorenz equation, Chua circuit, and the vocal fold model [18]. In addition, this parameter estimation method is applicable for multi-variable feedback as well as single-variable feedback.

3. Numerical calculation

To show the parameter estimation scheme based on feedback synchronization, we let the initial parameter values of the drive and response pendulums be ($\beta_d = 0.1$, $p_d = 1.28$) and ($\beta_r(0) = 0.15$, $p_r(0) = 2$), respectively. Fig. 1(a) and (b) show the strange attractors of the drive and response pendulums using a Poincare map, in which $T = 2\pi/\omega$ and $\omega = 0.3$. The drive pendulum, with Lyapunov exponents $\lambda_1 = 0.035$ and $\lambda_2 = -0.135$ and the response pendulum, with $\lambda_1 = 0.008$ and $\lambda_2 = -0.158$, behave as chaos. A time series $x(i\tau)\,(i = 1, 2, \ldots, N)$ of the angular displacement is produced at the discrete time interval $\tau = \pi/200$. Fig. 2 shows the results of using the parameter estimation procedure to the response system. In Figs. 2(a) and (b), when the iteration number $k < 10$, the feedback control is not active, and the drive and response pendulums are not synchronous. As a result, the drive parameters cannot be estimated. However, when the iteration number $k \geqslant 10$, the feedback and parameter controls are applied to the response pendulum, where $N = 10\,000$, $\Delta = 10^{-4}$, and the feedback weight $K = 0.4$. The parameter and

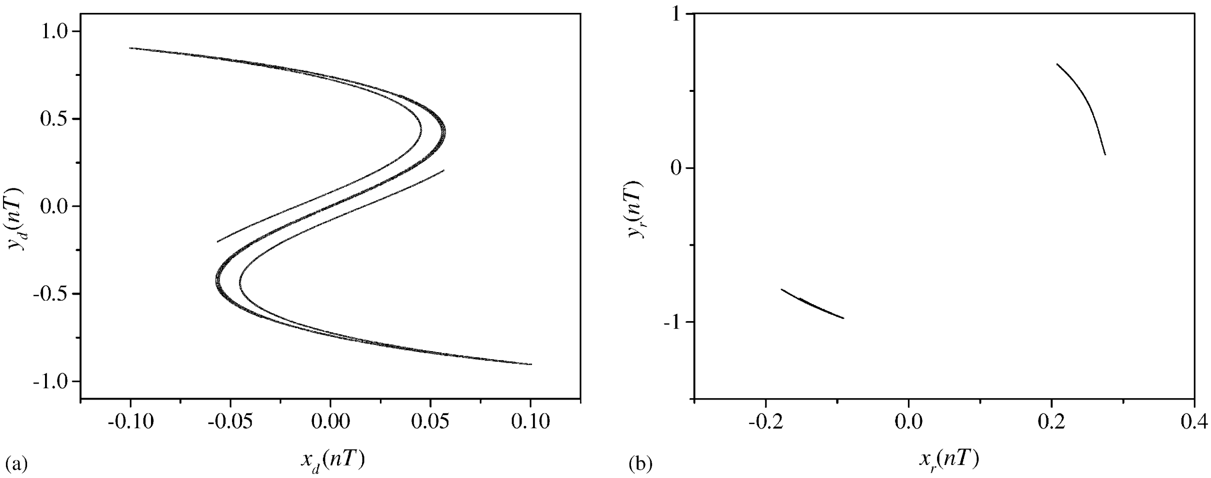

Fig. 1. The Poincare sections of the attractors. (a) $x_d(nT)$ versus $y_d(nT)$, where $\beta_d = 0.1$, $p_d = 1.28$, $\omega = 0.3$, and $T = 2\pi/\omega$; (b) $x_r(nT)$ versus $y_r(nT)$, where $\beta_r = 0.15$ and $p_r = 2$.

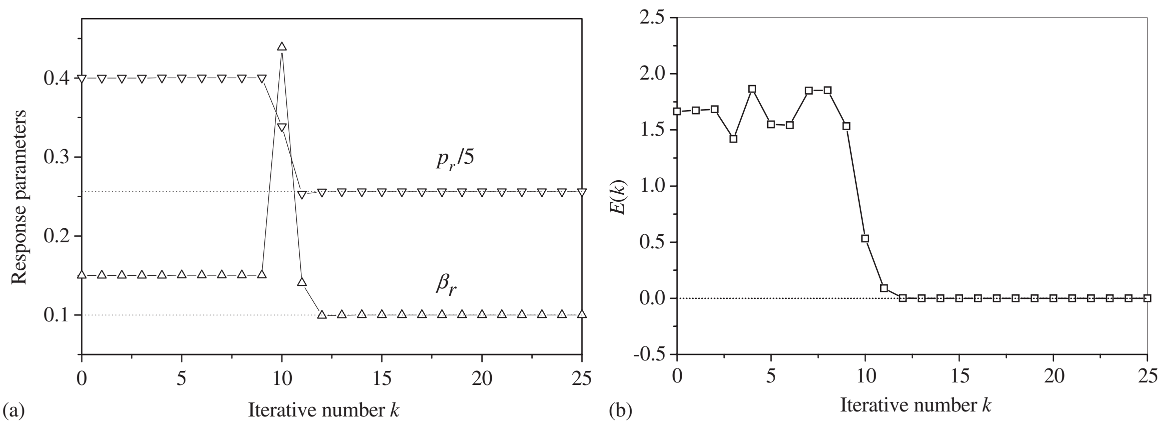

Fig. 2. (a) The convergences of the response pendulum parameters $\beta_r(k)$, $p_r(k)$ to $(0.1, 1.28)$, where $\beta_r(0) = 0.15$, $p_r(0) = 2$, $\omega = 0.3$, and feedback weight $K = 0.4$; (b) the time-averaged synchronization error $E(k)$, where $\tau = \pi/200$, $N = 10\,000$, and $\Delta = 10^{-4}$.

state differences between the drive and response pendulums are rapidly decreased. For the sufficiently large k, the response parameters converge to $\beta_r(k) = 0.1$ and $p_r(k) = 1.28$, and the corresponding time-averaged synchronization error $E = (1/N\tau)\int_0^{N\tau} \|\mathbf{x} - \mathbf{y}\|^2\,\mathrm{d}t$ approaches zero, as shown in Figs. 2(a) and (b), respectively. By applying this parameter estimation scheme to the response pendulum, two pendulums are completely synchronized and we can accurately determine the drive system parameters and dynamic states.

This parameter estimation technique enables us to exactly reproduce the drive pendulum parameters when the initial parameter values $p_r(0)$ and $\beta_r(0)$ of the response pendulum are within a certain range. In the contour planes of Fig. 3, the gray-coded values of the final parameter difference $\Delta p = \sqrt{(p_d - p_r)^2 + (\beta_d - \beta_r)^2}$, with respect to the initial parameter values $\beta_r(0)$ and $p_r(0)$, display the parameter space of the response pendulum allowing parameters estimation. Figs. 3(a) and (b) correspond to the feedback weight $K = 0.2$ and 0.4, respectively. When $p_r(0)$ and $\beta_r(0)$ are chosen within the black region I, for the sufficiently large k, $p_r(k)$ and $\beta_r(k)$ of the response parameters converge to 1.28 and 0.1 of the drive parameters, respectively, and Δp approaches 0. The drive and response pendulums are synchronous. However, when $p_r(0)$ and $\beta_r(0)$ are within the region II, the response parameters cannot be appropriately controlled to approach the drive parameters, and synchronization becomes impossible. Furthermore, the feedback weight K affects the range of the black synchronization region. In comparison to Fig. 3(a) with $K = 0.2$, Fig. 3(b) with $K = 0.4$ shows a wider synchronization region I, indicating that for x variable feedback, an increase in feedback weight will be helpful for parameter estimations. Although the initial values of the response parameters are significantly different from those of the drive parameters, the parameter estimation method yields response parameters that finally converge to the drive parameters, demonstrating its robustness to large initial parameter mismatches.

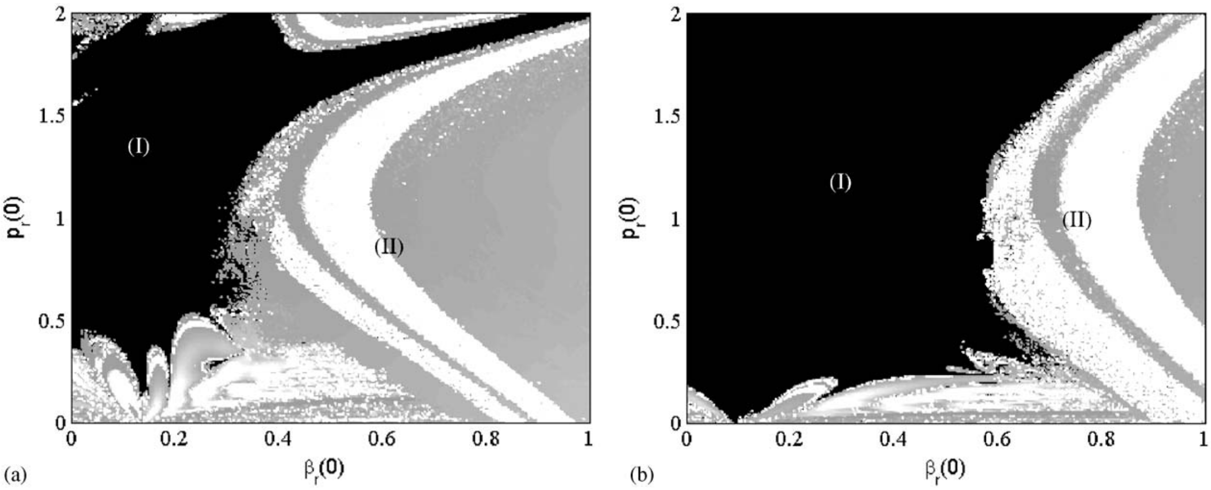

Fig. 3. The dependences of the parameter estimation result Δp on the initial response parameters $\beta_r(0)$ and $p_r(0)$. The black region I in the contour corresponds to the synchronization region where the response parameters $p_r(k)$ and $\beta_r(k)$ converge to 1.28 and 0.1 of the drive parameters, and the region II corresponds to the asynchronization region. (a) $K = 0.2$; (b) $K = 0.4$.

The feedback weight K is an important parameter for the estimation of pendulum parameters. The feedback function $\mathbf{I}$ represents another important factor affecting the parameter estimation using feedback synchronization. Figs. 4(a) and (b) show the dependences of E and Δp on the feedback weight K, respectively, where three kinds of feedback functions are applied: $\mathbf{I}_x = \begin{Bmatrix} 1 & 0 \\ 1 & 0 \end{Bmatrix}, \mathbf{I}_y = \begin{Bmatrix} 0 & 1 \\ 0 & 1 \end{Bmatrix}$, and $\mathbf{I}_{xy} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$. For the x variable feedback and the x,y variable feedback, the response parameters $p_r(k)$ and $\beta_r(k)$ converge to the drive parameters and the synchronization error approaches 0 when K is increased above a threshold value. However, for the y variable feedback, the parameter estimate can be achieved when K is within a certain range. A sufficiently small or large feedback weight of the y variable feedback will disrupt chaos synchronization of two pendulums, and then the drive parameters cannot be estimated.

Because of random perturbation and finite measurement accuracies, noise inevitably exists in practical systems. The feedback synchronization method has been found to be robust to noise and has the potential value to be applied in noisy environments [4,5,16,17]. To study the effect of noise on this parameter estimation method, we add a Gaussian noise with mean-square value σ^2 to the right side of Eq. (1). When noise amplitude is sufficiently small, external noise produces a weak perturbation to the synchronization manifold. The time-averaged synchronization error E and the parameter difference Δp linearly increases with the noise amplitude, as shown in Figs. 5(a) and (b), where the x variable feedback is applied, and the curves from the top to bottom correspond to $K = 0.2$, 0.4, and 0.8, respectively. When noise amplitude is excessively large, the noise perturbations may knock the trajectory of the response system off the synchronization manifold so that the parameter estimate cannot target its control to the drive parameters. The increase of feedback weight may be helpful for suppressing the effects of noise. In Fig. 5, for the feedback control $K = 0.2$, the drive parameters cannot be estimated when noise amplitude $\sigma = 0.01$. However, increase of feedback weight to $K = 0.4$ and 0.8 can significantly decrease the synchronization error E and the parameter difference Δp even for larger noise amplitudes

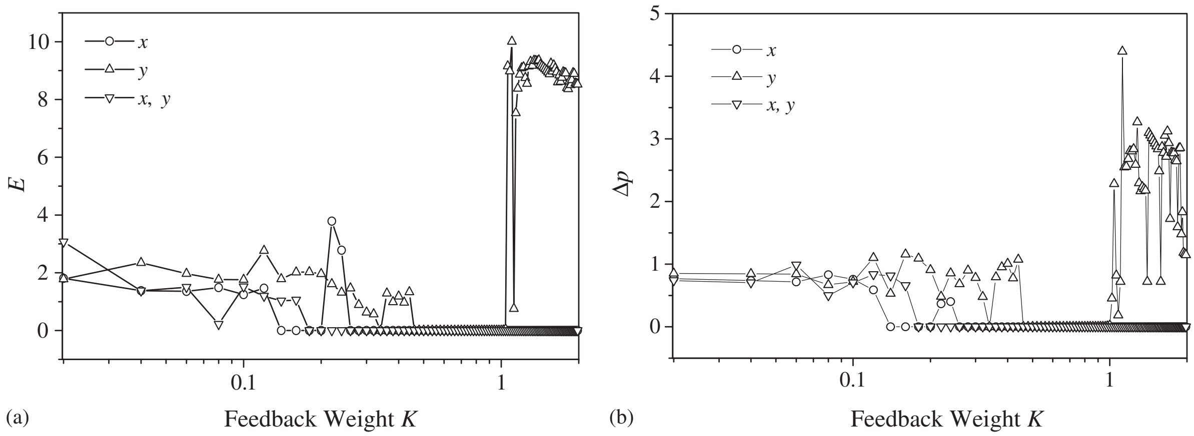

Fig. 4. The effect of feedback weight K on parameter estimate, where (a) and (b) correspond to E and Δp, respectively. —○—, x variable feedback; —△—, y variable feedback; —▽—, x and y variable feedback.

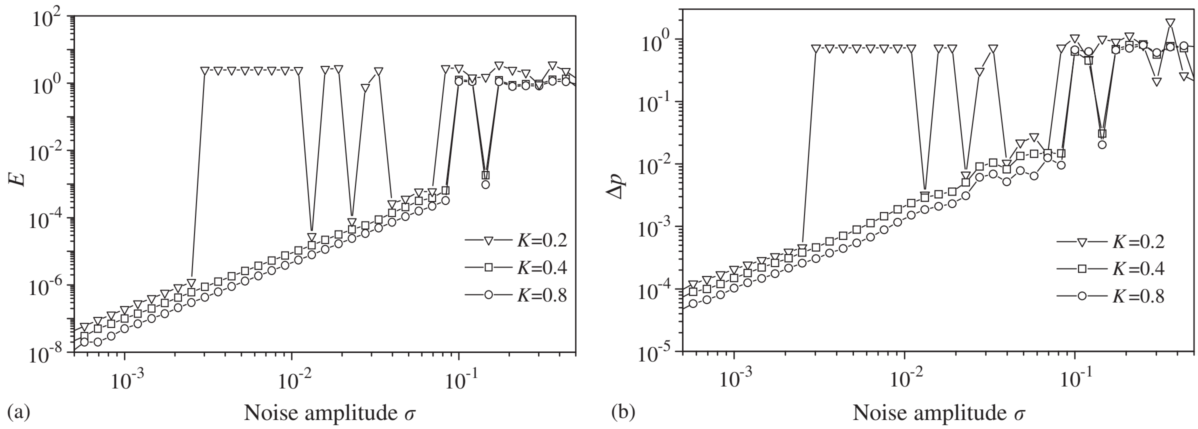

Fig. 5. The effect of noise on parameter estimates where (a) and (b) correspond to E and Δp, respectively. —∇—, $K = 0.2$; —□—, $K = 0.4$; —○—, $K = 0.8$.

($\sigma < 0.08$), demonstrating that this parameter estimation method based on feedback synchronization is robust to noise.

Finally, we apply this parameter estimation method based on feedback synchronization to reproduce the bifurcation diagram of the drive pendulum. For the system parameters $\omega = 0.3$ and $\beta_d = 0.1$, Garira and Bishop [6] found that as p_d increases within the interval (1.2, 1.28), the unsymmetrical orbit of the parametrically excited pendulum undergoes a period-doubling bifurcation to chaos, as shown in the bifurcation diagram of Fig. 6(a). Fig. 6(b) shows the bifurcation diagram of the response pendulum where the initial response parameters are $\omega = 0.3$, $\beta_r(0) = 0.3$, and $1.7 \leqslant p_r(0) \leqslant 1.78$. Although the drive and response pendulums have different system parameters, when the iterative parameter estimation procedure is imposed on the response system in Fig. 6(c), the response parameters $\beta_r(k)$ and $p_r(k)$ converge to the drive parameters β_d and p_d. The two pendulum systems are synchronized. The bifurcation diagram of the response system completely duplicates all bifurcation details of the drive pendulum, including period-doubling cascades and periodic windows.

The methods of nonlinear dynamic prediction have been previously applied to approximate model equations [10]. However, the reconstructed polynomial models in nonlinear dynamic predictions may not exactly represent the practical model, such as the sine term of the parametrically excited pendulum. Thus, the practical model parameters may not be precisely estimated using the nonlinear dynamic prediction methods. The inevitable prediction errors in nonlinear dynamic predictions may affect the applicability of these approximation methods to extremely high-dimensional systems, particularly spatio-temporal chaotic systems. Parameter estimation based on active–passive synchronization has recently been applied to exactly estimate the system parameters of the spatio-temporal chaotic systems generated by one-way coupled map lattices [15]. Two spatio-temporal chaotic parametrically excited pendulum arrays can be synchronized using the feedback method [5]. Thus, the parameter estimation method based on feedback synchronization may potentially be applied to precisely estimate the system parameters of spatio-temporal chaotic parametrically excited pendulum arrays and to synchronize two pendulum arrays with initial parameter mismatches.

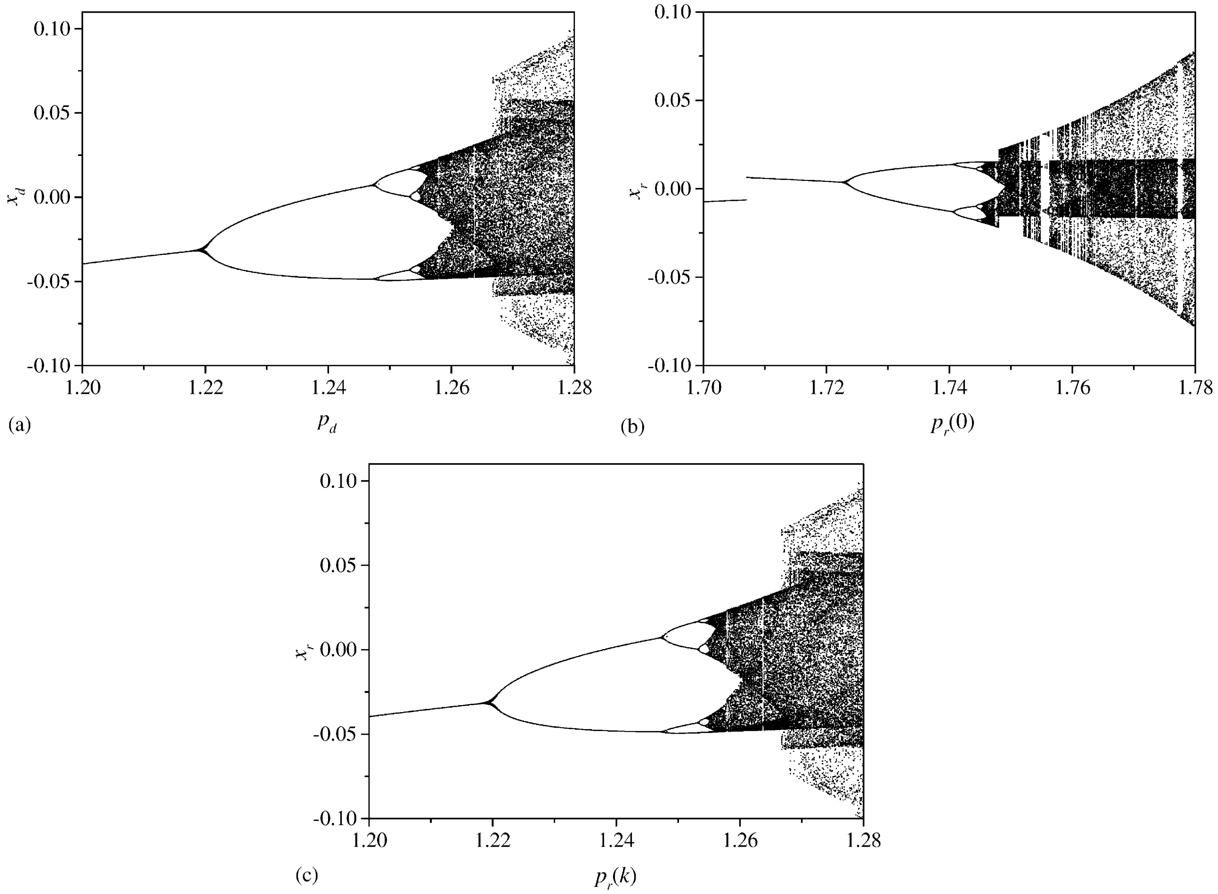

Fig. 6. The bifurcation diagrams. (a) the drive pendulum with $\omega = 0.3$ and $\beta_d = 0.1$, and $p_d \in (1.2, 1.28)$; (b) the response pendulum before parameter estimate with the initial parameter values $\beta_r(0) = 0.3$ and $p_r(0) \in (1.7, 1.78)$; (c) the response pendulum after parameter estimate where $K = 0.4$. The response parameters converge to $\beta_r(k) = \beta_d$, and $p_r(k) = p_d$ for the sufficiently large k.

4. Conclusion

In this paper, based on feedback synchronization, we investigated the parameter estimation of parametrically excited pendulums and the synchronization of two pendulum systems with initial parameter mismatches. When the initial parameter differences of the response pendulum were within a certain range, the parameter estimation method allowed the response parameters to converge to the drive parameters and synchronized the two chaotic pendulum systems. The response pendulum exactly reproduced the bifurcation diagram of the drive pendulum. Increasing the feedback weight might suppress the effects of noise and parameter mismatch. Small noise and parameter mismatches did not disrupt correct parameter estimation in the response system, suggesting that the parameter estimation scheme based on feedback synchronization might be applicable in practically noisy environments.

Acknowledgments

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