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Journal of Sound and Vibration 290 (2006) 654–675

JOURNAL OF SOUND AND VIBRATION

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Multi-objective control of vehicle active suspension systems via load-dependent controllers

Huijun Gao[a,*], James Lam[b], Changhong Wang[a]

[a]*Space Control and Inertial Technology Research Center, P.O. Box 1230, Harbin Institute of Technology, Xidazhi Street 92, Harbin, 150001, PR China*
[b]*Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong*

Received 16 August 2004; received in revised form 10 February 2005; accepted 12 April 2005
Available online 24 August 2005

Abstract

This paper presents a load-dependent controller design approach to solve the problem of multi-objective control for vehicle active suspension systems by using linear matrix inequalities. A quarter-car model with active suspension system is considered. It is assumed that the vehicle body mass resides in an interval and can be measured online. This approach of designing controllers, whose gain matrix depends on the online available information of the body mass, is based on a parameter-dependent Lyapunov function. Since the parameter-dependent idea is fully exploited, the proposed controller design approach can yield much less conservative results compared with previous approaches that design robust constant controllers in the quadratic framework. The usefulness and the advantages of the proposed controller design methodology are demonstrated via numerical simulations.

1. Introduction

Vehicle suspensions have been a hot research topic for many years due to its important role in ride comfort, vehicle safety, road damage minimization and the overall vehicle performance. To meet these requirements, many types of suspension systems, ranging from passive [1,2], semi-active [3,4], to active suspensions [5,6], are currently being employed and studied. It has been well

*Corresponding author. Fax: +86 45 186418091.
E-mail addresses: hjgao@hit.edu.cn (H. Gao), james.lam@hku.hk (J. Lam).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.04.007

recognized that active suspensions have a great potential to meet the tight performance requirements demanded by users. Therefore, in recent years more and more attention has been devoted to the development of active suspensions and various approaches have been proposed to solve the crucial problem of designing a suitable control law for these active suspension systems (see, for instance Refs. [7–12] and the references therein).

The linear quadratic regulator (LQR) has been used as one of the main control techniques for dealing with active suspension design [13]. In this framework, an optimal state feedback gain minimizing a quadratic cost function is obtained. It is noted that the model parameters are assumed to be precisely known and the optimal control strategies may collapse in the face of some uncertain parameters. However, a suspension system often contains parameters that are intrinsically uncertain, such as the sprung mass, whose value is dependent on the total load of the vehicle. Therefore, the use of robust control techniques has become a major requirement in the further development of active suspension systems, and many useful results on robust control for active suspension systems have been reported [14–17].

To achieve a compromise between several performance requirements for uncertain active suspension systems, very recently a robust multi-objective controller was designed for a quarter-car model whose system matrices are subject to parameter uncertainties characterized by a given polytope [18]. The main objective is to use a robust state-feedback controller to achieve multiple performance objectives for different controlled output signals. It is worth mentioning that the solutions are given in the quadratic framework. Although being adequate to ensure stability for systems with arbitrarily fast time-varying parameters, methods based on quadratic stability can produce conservative results since the same parameter-independent Lyapunov function must be used for the entire uncertainty domain. In addition, it is noted that the gain matrix for the designed controllers keeps constant for all the uncertain parameters. However, for active suspension systems with parameters, such as the sprung mass, that can be measured online without difficulty, the online available information of these parameters can be utilized in the realization of control strategy. This would generally allow less conservative designs to be achieved.

Motivated by the above discussion, in this paper we present a load-dependent controller design approach to solve the problem of multi-objective control for vehicle active suspension systems. A quarter-car model with active suspension system is considered and the linear matrix inequality (LMI) technique is employed to cast the controller designs into convex optimizations. It is assumed that the vehicle body mass (whose value changes with the vehicle load) resides in an interval and can be measured online. This approach of designing controllers, whose gain matrix depends on the online available information of the body mass, is based on a parameter-dependent Lyapunov function. Since the parameter-dependent idea is fully exploited, the proposed controller design approach can yield much less conservative results compared with previous approaches that design robust constant controllers in the quadratic framework. The usefulness and advantage of the proposed controller design methodology are demonstrated via numerical simulations.

The remainder of this paper is organized as follows. The problem of multi-objective load-dependent controller design for uncertain active suspension systems is formulated in Section 2. Sections 3 presents controller synthesis results by using LMI techniques. A design example illustrating the usefulness and advantage of the proposed methodology is given in Section 4 and we conclude the paper in Section 5.

Notations: The superscript T stands for matrix transposition; R^n denotes the n-dimensional Euclidean space, $R^{m\times n}$ is the set of all $m\times n$ real matrices and the notation $P>0$ means that P is symmetric and positive definite; **I** and **0** represent identity matrix and zero matrix; the notation $\|\cdot\|$ refers to the Euclidean vector norm. In addition, in symmetric block matrices or long matrix expressions, we use $*$ to represent a block in a matrix that is induced by symmetry and $\{\cdots\}$ stands for a block-diagonal matrix. For simplicity, we use sym(M) to represent $M+M^{\mathrm{T}}$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem formulation

Consider the quarter-car model shown in Fig. 1, which has been used extensively in the literature due to its simplicity while capturing many essential characteristics of a real suspension system. In this figure, m_s is the sprung mass, which represents the car chassis whose value changes with the vehicle load; m_u is the unsprung mass, which represents mass of the wheel assembly; k_s and c_s are stiffness and damping of the uncontrolled suspension, respectively; k_t serves to model the compressibility of the pneumatic tire; z_s and z_u are the displacements of the sprung and unsprung masses, respectively; z_r is the road displacement input; u is the active input of the suspension system.

The ideal dynamic equations for the sprung and unsprung masses of the quarter-car model are given by

$$\begin{aligned} &m_s\ddot{z}_s(t)+c_s[\dot{z}_s(t)-\dot{z}_u(t)]+k_s[z_s(t)-z_u(t)]=u(t),\\ &m_u\ddot{z}_u(t)+c_s[\dot{z}_u(t)-\dot{z}_s(t)]+k_s[z_u(t)-z_s(t)]+k_t[z_u(t)-z_r(t)]=-u(t). \end{aligned} \tag{1}$$

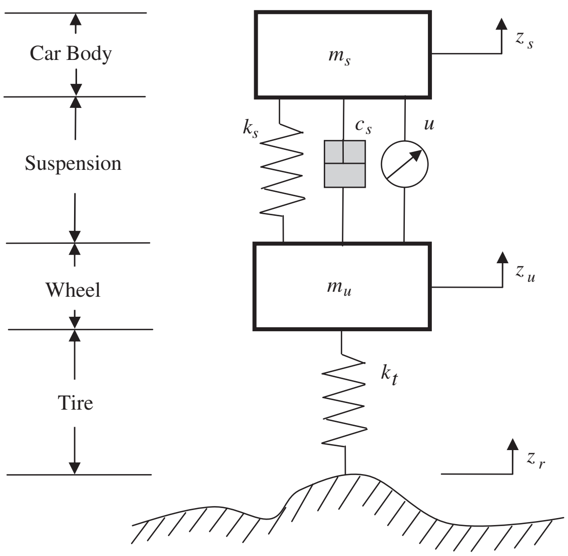

Fig. 1. Quarter-car model with an active suspension.

It can be seen from Fig. 1 that the disturbance input of the system is the road displacement z_r, which can be represented by [18,19]:

$$\dot{z}_r(t) = 2\pi q_0 \sqrt{G_0 V} w(t), \tag{2}$$

where G_0 stands for the road roughness coefficient, q_0 is the reference spatial frequency, V is the vehicle forward velocity, $w(t)$ is zero-mean white noise with identity power spectral density.

As is mentioned previously, the body mass m_s usually changes with the vehicle load. Throughout the paper, it is assumed that the vehicle body mass resides in an interval and can be measured online, that is,

$$m_1 \leqslant m_s \leqslant m_2. \tag{3}$$

Choose the following set of state variables:

$$x_1(t) = z_s(t) - z_u(t), \quad x_2(t) = z_u(t) - z_r(t), \quad x_3(t) = \dot{z}_s(t), \quad x_4(t) = \dot{z}_u(t), \tag{4}$$

where $x_1(t)$ is the suspension deflection, $x_2(t)$ is the tire deflection, $x_3(t)$ is the sprung mass speed, and $x_4(t)$ is the unsprung mass speed.

Then, by defining $x(t) \triangleq [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]^{\mathrm{T}}$, the dynamic equations in Eq. (1) can be written in the following state-space form:

$$\dot{x}(t) = A(m_s)x(t) + B(m_s)u(t) + B_w(m_s)w(t), \tag{5}$$

where

$$A(m_s) = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -k_s/m_s & 0 & -c_s/m_s & c_s/m_s \\ k_s/m_u & -k_u/m_s & c_s/m_s & -c_s/m_s \end{bmatrix},$$

$$B(m_s) = \begin{bmatrix} 0 \\ 0 \\ 1/m_s \\ -1/m_u \end{bmatrix}, \quad B_w(m_s) = \begin{bmatrix} 0 \\ -2\pi q_0 \sqrt{G_0 V} \\ 0 \\ 0 \end{bmatrix}. \tag{6}$$

It is worth mentioning that as the body mass m_s usually changes with the vehicle load, which can be measured online, we express the system matrices of the quarter-car model as functions of m_s.

In designing the control law for a suspension system, usually we need to take the following aspects into consideration:

(1) *Ride comfort*: It is well-known that ride comfort is an important performance for vehicle design, which is usually evaluated by the body acceleration in the vertical direction. Therefore, in the controller design, one of our main objectives is to minimize the vertical body acceleration $\ddot{z}_s(t)$, that is,

$$\min \ddot{z}_s(t). \tag{7}$$

(2) *Road holding ability*: In order to ensure a firm uninterrupted contact of wheels to road, the dynamic tire load should not exceed the static ones [20], that is,

$$k_u(z_u - z_r) < 9.8(m_s + m_u). \tag{8}$$

By considering Eq. (3), Eq. (8) holds if

$$k_u(z_u - z_r) < 9.8(m_1 + m_u). \tag{9}$$

(3) *Maximum suspension deflection*: Because of the constraint of mechanical structure, the maximum allowable suspension stroke has to be taken into consideration to prevent excessive suspension bottoming, which can possibly result in deterioration of ride comfort and even structural damage. The requirement is

$$|z_s(t) - z_u(t)| \leqslant z_{\max}, \tag{10}$$

where $z_{\max}$ is the maximum suspension deflection.

(4) *Saturation effect of actuator*: In view of the limited power of the hydraulic actuator, the active force for the suspension system should be confined to a certain range, that is,

$$|u(t)| \leqslant u_{\max}. \tag{11}$$

It is not difficult to see that the latter three requirements are actually constraints, while only the first one needs to be minimized. In other words, the strategy in designing control law for suspension systems is to minimize the vertical body acceleration $\ddot{z}_s(t)$ while keeping the other three requirements satisfied.

According to the above four requirements, we define the following output variables:

$$\begin{aligned}
z_1(t) &= \ddot{z}_s(t),\\
z_2(t) &= (z_s(t) - z_u(t))/z_{\max},\\
z_3(t) &= k_u(z_u(t) - z_r(t))/9.8(m_1 + m_u),\\
z_4(t) &= u(t)/u_{\max}.
\end{aligned} \tag{12}$$

Therefore, the vehicle suspension system can be described by the following state-space equation:

$$\begin{aligned}
\dot{x}(t) &= A(m_s)x(t) + B(m_s)u(t) + B_w(m_s)w(t),\\
z_l(t) &= C_l(m_s)x(t) + D_l(m_s)u(t), \quad l = 1, \ldots, 4,
\end{aligned} \tag{13}$$

where $A(m_s)$, $B(m_s)$, $B_w(m_s)$ are defined in Eq. (6), and

$$\begin{aligned}
&C_1(m_s) = [-k_s/m_s \quad 0 \quad -c_s/m_s \quad c_s/m_s], \quad D_1(m_s) = 1/m_s,\\
&C_2(m_s) = [1/z_{\max} \quad 0 \quad 0 \quad 0], \quad D_2(m_s) = 0,\\
&C_3(m_s) = [0 \quad k_u/9.8(m_1 + m_u) \quad 0 \quad 0], \quad D_3(m_s) = 0,\\
&C_4(m_s) = [0 \quad 0 \quad 0 \quad 0], \quad D_4(m_s) = 1/u_{\max}.
\end{aligned} \tag{14}$$

It is not difficult to see that the system matrices which are dependent on the body mass m_s can be expressed as

$$(A(m_s), B(m_s), B_w(m_s), C_l(m_s), D_l(m_s)) = \sum_{i=1}^{2} \alpha_i (A_i, B_i, B_{wi}, C_{li}, D_{li}),$$
$$\alpha_i \geqslant 0, \quad \alpha_1 + \alpha_2 = 1, \tag{15}$$

where

$$(A_1, B_1, B_{w1}, C_{l1}, D_{l1}) = (A(m_s), B(m_s), B_w(m_s), C_l(m_s), D_l(m_s))\big|_{m_s=m_1},$$
$$(A_2, B_2, B_{w2}, C_{l2}, D_{l2}) = (A(m_s), B(m_s), B_w(m_s), C_l(m_s), D_l(m_s))\big|_{m_s=m_2}.$$

Moreover, the relationship between the vector $\alpha \triangleq (\alpha_1, \alpha_2)$ and the online measurable body mass m_s is given by

$$\alpha_1 = \left(\frac{1}{m_s} - \frac{1}{m_2}\right) \Big/ \left(\frac{1}{m_1} - \frac{1}{m_2}\right), \quad \alpha_2 = \left(\frac{1}{m_1} - \frac{1}{m_s}\right) \Big/ \left(\frac{1}{m_1} - \frac{1}{m_2}\right). \tag{16}$$

Now we have used a two-vertex polytope to describe the load-dependent system matrices. The polytope description has been used in many references (see, for instance, [21] and [22]). As the vehicle load can be measured online easily, we can get the vector (α_1, α_2) according to the available m_s based on Eq. (16).

For the active suspension system (13), Ref. [18] designs a state-feedback control law of the following form:

$$u(t) = Kx(t), \tag{17}$$

where K is a constant feedback gain to be determined. This robust control approach to achieve multi-objective performances deserves some remarks:

(1) The controller design presented in Ref. [18] is based on the notion of quadratic stability. That is, for the entire uncertainty polytope, a fixed Lyapunov function is required to satisfy a set of LMI conditions. Such treatment has been well recognized to be conservative, and an advanced research topic in robust control is to utilize parameter-dependent Lyapunov functions.
(2) In the design example in Ref. [18], the considered uncertain parameter is the sprung mass m_s, whose value depends on the total value of the load. It is noted that the designed controller has a constant gain matrix K, which is used for all possible values of body mass m_s. In fact, the body mass m_s is usually not difficult to obtain online, thus if the controller gain can change according to the online available value of m_s, better performance may be achieved.

Based on the above points, we consider the following controller structure:

$$u(t) = K(m_s)x(t), \tag{18}$$

where $K(m_s)$ is a gain matrix function to be determined. Our purpose is to investigate the design of the controller (18) based on parameter-dependent Lyapunov functions. Since the controller design presented here does not belong to those commonly used robust control approaches, and is essentially different from those in the quadratic framework, we call Eq. (18) a *load-dependent* controller.

By applying controller (18) to the suspension system (13), we obtain the following closed-loop system:

$$\begin{aligned}\dot{x}(t) &= \bar{A}(m_s)x(t) + B_w(m_s)w(t),\\ z_l(t) &= \bar{C}_l(m_s)x(t), \quad l = 1, \ldots, 4,\end{aligned} \tag{19}$$

where

$$\begin{aligned}\bar{A}(m_s) &\triangleq A(m_s) + B(m_s)K(m_s),\\ \bar{C}_l(m_s) &\triangleq [C_l(m_s) + D_l(m_s)K(m_s)], \quad l = 1, \ldots, 4.\end{aligned} \tag{20}$$

Then, the transfer functions from the disturbance signal to the controlled outputs are given by

$$T_l(s) = \bar{C}_l(m_s)\left[sI - \bar{A}(m_s)\right]^{-1}B_w(m_s), \quad l = 1, \ldots, 4. \tag{21}$$

Similar to Ref. [18], we also introduce the H_2 and GH_2 (generalized H_2, also called L_2-L_∞ [23]) performances to evaluate the controlled outputs $z_j(t)$. Based on the aforementioned requirements for control design, the problem to be solved in this paper can be summarized as follows:

2.1. Problem load-dependent suspension control (LDSC)

Given an active suspension system (13), design a load-dependent controller in the form of Eq. (18) via the following minimization problem:

$$\min \gamma_1 \quad \text{s.t.} \quad \begin{cases} \left\|T_1(s)\right\|_2 \leqslant \gamma_1, \\ \left\|T_l(s)\right\|_G \leqslant \gamma_l, \quad l = 2, 3, 4, \\ OC, \end{cases}$$

where γ_l is a given constant,

$$\begin{aligned}\left\|T_1(s)\right\|_2 &\triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\ \ T_1^*(\mathrm{j}\omega)T_1(\mathrm{j}\omega)\,\mathrm{d}\omega},\\ \left\|T_l(s)\right\|_G &\triangleq \sup\left\{\left\|z_l(t)\right\| : \quad x(0) = 0, \quad t \geqslant 0, \quad \int_0^t \left\|\omega(\tau)\right\|\,\mathrm{d}\tau \leqslant 1\right\}\end{aligned}$$

and OC represents other constraints, such as pole constraints of the closed-loop system. The proposed control system diagram is given in Fig. 2.

Remark 1. It is worth mentioning that the control strategy proposed above is much different from the standard PID control, as seen in the following two aspects:

(1) In the standard PID controller design, it is often difficult to take the variation of the body mass m_s into consideration, which is dependent on the vehicle load. However, in the control strategy proposed above, the body mass m_s is assumed to reside in an interval, which characterizes the real situation more precisely.

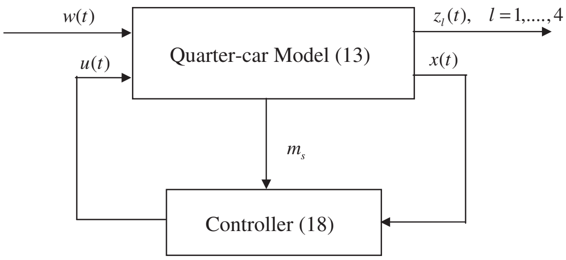

Fig. 2. Control system diagram.

(2) In the control strategy proposed above, the multiple requirements (including ride comfort, road holding ability, suspension deflection limit and saturation effect of actuator) are formulated in a unified framework, based on which the controller design is cast into a multiple-objective minimization problem. However, in the standard PID controller design, we are usually difficult to take these factors into account simultaneously.

3. Load-dependent controller design

In this section, we will investigate the problem of multi-objective control through load-dependent controllers formulated in the above section. First, according to Ref. [24], the closed-loop system in Eq. (19) is asymptotically stable with $\|T_1(s)\|_2 \leqslant \gamma_1$ and $\|T_j(s)\|_G \leqslant \gamma_l$ $(l = 2, 3, 4)$ if and only if there exist matrix functions $P(m_s) > 0$ and $S(m_s) > 0$ satisfying

$$\mathrm{Tr}(S(m_s)) < \gamma_1^2, \tag{22}$$

$$\begin{bmatrix} \bar{A}^{\mathrm{T}}(m_s)P(m_s) + P(m_s)\bar{A}(m_s) & P(m_s)B_w(m_s) \\ * & -I \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} -S(m_s) & \bar{C}_1(m_s) \\ * & -P(m_s) \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} -\gamma_l^2 I & \bar{C}_l(m_s) \\ * & -P(m_s) \end{bmatrix} < 0, \quad l = 2, 3, 4. \tag{25}$$

In addition, in order to obtain desired dynamics of the closed-loop systems, usually some pole placement constraints need to be imposed. In this paper, we consider the following two kinds of regional pole constraints [25]:

(1) *Disk region*: Let $\mho(\eta, \rho)$ denotes any disk region centered in η with radius ρ in the complex plane ($\eta, \rho \in \mathbb{R}$ and $\rho > 0$). Then, all the eigenvalues of $\bar{A}(m_s)$ in Eq. (19) lie in the region

$\mho(\eta,\rho)$ if and only if there exists a matrix function $P(m_s)>0$ satisfying

$$\begin{bmatrix} -P(m_s) & P(m_s)\big(\bar{A}(m_s)-\eta I\big) \\ * & -\rho^2 P(m_s) \end{bmatrix} < 0. \tag{26}$$

Vertical strip: Let $\wp(v,\mu)$ denotes a vertical strip lying within the bounds v and μ ($v<\mu$, v, $\mu \in \mathbb{R}$). Then, all the eigenvalues of $\bar{A}(m_s)$ in Eq. (19) lie in the region $\wp(v,\mu)$ if and only if there exists a matrix function $P(m_s)>0$ satisfying

$$\big(\bar{A}(m_s)-\mu I\big)^{\mathrm{T}} P(m_s) + P(m_s)\big(\bar{A}(m_s)-\mu I\big) < 0, \tag{27}$$

$$-\big(\bar{A}(m_s)-vI\big)^{\mathrm{T}} P(m_s) - P(m_s)\big(\bar{A}(m_s)-vI\big) < 0. \tag{28}$$

In the multi-objective synthesis, in order to cast the controller design into convex optimization problems, we usually need to set a common Lyapunov matrix for different performance objectives. Thus, the closed-loop system (19) is asymptotically stable with $\|T_1(s)\|_2 \leqslant \gamma_1$, $\|T_l(s)\|_G \leqslant \gamma_l$, $l=2,3,4$, and all the eigenvalues of $\bar{A}(m_s)$ lie in the region $\mho(\eta,\rho)$ (or $\wp(v,\mu)$) if there exist matrix functions $P(m_s)>0$ and $S(m_s)>0$ satisfying Eqs. (22)–(26) (or Eqs. (22)–(25), (27) and (28)). Ref. [18] presents a robust controller design by setting $P(m_s)\equiv P$ for the entire uncertainty domain. In the following, we will present a new approach based on parameter-dependent Lyapunov functions.

First define the following invertible matrix functions:

$$J_1 \triangleq \mathrm{diag}\{P^{-1}(m_s), I\},\;\; J_2 \triangleq \mathrm{diag}\{I, P^{-1}(m_s)\},\; J_3 \triangleq \mathrm{diag}\{P^{-1}(m_s), P^{-1}(m_s)\}. \tag{29}$$

By performing congruence transformations to Eqs. (23)–(28) by J_1, J_2, J_2, J_3, $P^{-1}(m_s)$, $P^{-1}(m_s)$ respectively, and by changing the matrix variables with

$$\bar{P}(m_s) \triangleq P^{-1}(m_s), \quad \bar{K}(m_s) \triangleq K(m_s)P^{-1}(m_s) \tag{30}$$

we obtain

$$\begin{bmatrix} \bar{P}(m_s)A^{\mathrm{T}}(m_s) + \bar{K}^{\mathrm{T}}(m_s)B^{\mathrm{T}}(m_s) + A(m_s)\bar{P}(m_s) + B(m_s)\bar{K}(m_s) & B_w(m_s) \\ * & -I \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} -S(m_s) & C_1(m_s)\bar{P}(m_s) + D_1(m_s)\bar{K}(m_s) \\ * & -\bar{P}(m_s) \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} -\gamma_l I & C_l(m_s)\bar{P}(m_s) + D_l(m_s)\bar{K}(m_s) \\ * & -\bar{P}(m_s) \end{bmatrix} < 0, \quad l=2,3,4, \tag{33}$$

$$\begin{bmatrix} -\bar{P}(m_s) & (A(m_s)-\eta I)\bar{P}(m_s) + B(m_s)\bar{K}(m_s) \\ * & -\rho^2 \bar{P}(m_s) \end{bmatrix} < 0, \tag{34}$$

$$\bar{P}(m_s)(A(m_s)-\mu I)^{\mathrm{T}} + (A(m_s)-\mu I)\bar{P}(m_s) + \bar{K}^{\mathrm{T}}(m_s)B^{\mathrm{T}}(m_s) + B(m_s)\bar{K}(m_s) < 0, \tag{35}$$

$$-\bar{P}(m_s)(A(m_s) - vI)^{\mathrm{T}} - (A(m_s) - vI)\bar{P}(m_s) - \bar{K}^{\mathrm{T}}(m_s)B^{\mathrm{T}}(m_s) - B(m_s)\bar{K}(m_s) < 0. \tag{36}$$

Eqs. (22), (31)–(34) are the conditions for the existence of admissible controllers with disk pole constraint, and Eqs. (22), (31)–(33), (35), (36) are the conditions for the existence of admissible controllers with vertical strip pole constraint. From Eq. (30) we know that if there exist matrix functions $\bar{P}(m_s)$, $S(m_s)$ and $\bar{K}(m_s)$ satisfying the above required conditions, the gain matrix function for an admissible controller in the form of Eq. (18) can be given by

$$K(m_s) \triangleq \bar{K}(m_s)\bar{P}^{-1}(m_s). \tag{37}$$

It is noted that for fixed m_s, conditions (22), (31)–(36) are LMIs, which can be readily solved via standard numerical software. However, these conditions cannot be implemented due to their infinite-dimensional nature in the parameter m_s. Our purpose hereafter is to transform these conditions into tractable LMI-based conditions.

According to the inner property of the polytopic uncertain systems, we assume the matrix functions $\bar{P}(m_s)$, $S(m_s)$ and $\bar{K}(m_s)$ in Eqs. (22), (31)–(36) to be of the following form:

$$\bar{P}(m_s) = \sum_{i=1}^{2} \alpha_i \bar{P}_i, \quad S(m_s) = \sum_{i=1}^{2} \alpha_i S_i, \quad \bar{K}(m_s) = \sum_{i=1}^{2} \alpha_i \bar{K}_i. \tag{38}$$

Then, Eq. (22) holds if

$$\mathrm{Tr}(S_i) < \gamma_1^2, \quad i = 1, 2. \tag{39}$$

In addition, it is not difficult to rewrite Eq. (31) in the following form:

$$\begin{aligned} \mathrm{X}(m_s) &\triangleq \begin{bmatrix} \mathrm{sym}\big(A(m_s)\bar{P}(m_s) + B(m_s)\bar{K}(m_s)\big) & B_w(m_s) \\ * & -I \end{bmatrix} \\ &= \sum_{j=1}^{2}\sum_{i=1}^{2} \alpha_i\alpha_j \mathrm{X}_{ij} = \sum_{j=1}^{2} \alpha_i^2 \mathrm{X}_i + \alpha_1\alpha_2 \mathrm{X}_{12}, \end{aligned}$$

where

$$\mathrm{X}_{ij} \triangleq \begin{bmatrix} \bar{P}_i A_j^{\mathrm{T}} + \bar{K}_i^{\mathrm{T}} B_j^{\mathrm{T}} + A_j \bar{P}_i + B_j \bar{K}_i & B_{wj} \\ * & -I \end{bmatrix}.$$

Therefore, Eq. (31) holds if

$$\begin{bmatrix} \bar{P}_i A_i^{\mathrm{T}} + \bar{K}_i^{\mathrm{T}} B_i^{\mathrm{T}} + A_i \bar{P}_i + B_i \bar{K}_i - \tilde{A}_{ii} & B_{wi} - \tilde{B}_{ii} \\ * & -I - \tilde{D}_{ii} \end{bmatrix} < 0, \quad i = 1, 2, \tag{40}$$

$$\begin{bmatrix} \mathrm{sym}\big(A_2\bar{P}_1 + B_2\bar{K}_1 + A_1\bar{P}_2 + B_1\bar{K}_2 - \tilde{A}_{12}\big) & B_{w2} + B_{w1} - \tilde{B}_{12} - \tilde{C}_{12}^{\mathrm{T}} \\ * & -2I - \tilde{D}_{12} - \tilde{D}_{12}^{\mathrm{T}} \end{bmatrix} \leqslant 0, \tag{41}$$

$$\begin{bmatrix} \begin{bmatrix} \tilde{A}_{11} & \tilde{B}_{11} \\ * & \tilde{D}_{11} \end{bmatrix} & \begin{bmatrix} \tilde{A}_{12} & \tilde{B}_{12} \\ \tilde{C}_{12} & \tilde{D}_{12} \end{bmatrix} \\ * & \begin{bmatrix} \tilde{A}_{22} & \tilde{B}_{22} \\ * & \tilde{D}_{22} \end{bmatrix} \end{bmatrix} < 0. \tag{42}$$

By using similar techniques, it can be established that Eq. (32) holds if

$$\begin{bmatrix} -S_i - \tilde{E}_{ii} & C_{1i}\bar{P}_i + D_{1i}\bar{K}_i - \tilde{F}_{ii} \\ * & -\bar{P}_i - \tilde{H}_{ii} \end{bmatrix} < 0, \quad i = 1, 2, \tag{43}$$

$$\begin{bmatrix} -S_1 - S_2 - \tilde{E}_{12} - \tilde{E}_{12}^{\mathrm{T}} & C_{12}\bar{P}_1 + D_{12}\bar{K}_1 + C_{11}\bar{P}_2 + D_{11}\bar{K}_2 - \tilde{F}_{12} - \tilde{G}_{12}^{\mathrm{T}} \\ * & -\bar{P}_1 - \bar{P}_2 - \tilde{H}_{12} - \tilde{H}_{12}^{\mathrm{T}} \end{bmatrix} \leqslant 0, \tag{44}$$

$$\begin{bmatrix} \begin{bmatrix} \tilde{E}_{11} & \tilde{F}_{11} \\ * & \tilde{H}_{11} \end{bmatrix} & \begin{bmatrix} \tilde{E}_{12} & \tilde{F}_{12} \\ \tilde{G}_{12} & \tilde{H}_{12} \end{bmatrix} \\ * & \begin{bmatrix} \tilde{E}_{22} & \tilde{F}_{22} \\ * & \tilde{H}_{22} \end{bmatrix} \end{bmatrix} < 0. \tag{45}$$

Eq. (33) holds if

$$\begin{bmatrix} -\gamma_l^2 I - \tilde{I}_{lii} & C_{li}\bar{P}_i + D_{li}\bar{K}_i - \tilde{J}_{lii} \\ * & -\bar{P}_i - \tilde{L}_{lii} \end{bmatrix} < 0, \quad i = 1, 2, \quad l = 2, 3, 4, \tag{46}$$

$$\begin{bmatrix} -2\gamma_l^2 I - \tilde{I}_{l12} - \tilde{I}_{l12}^{T} & C_{l2}\bar{P}_1 + D_{l2}\bar{K}_1 + C_{l1}\bar{P}_2 + D_{l1}\bar{K}_2 - \tilde{J}_{l12} - \tilde{K}_{l12}^{T} \\ * & -\bar{P}_1 - \bar{P}_2 - \tilde{L}_{l12} - \tilde{L}_{l12}^{T} \end{bmatrix} \leqslant 0, \quad l = 2, 3, 4, \tag{47}$$

$$\begin{bmatrix} \begin{bmatrix} \tilde{I}_{l11} & \tilde{J}_{l11} \\ * & \tilde{L}_{l11} \end{bmatrix} & \begin{bmatrix} \tilde{I}_{l12} & \tilde{J}_{l12} \\ \tilde{K}_{l12} & \tilde{L}_{l12} \end{bmatrix} \\ * & \begin{bmatrix} \tilde{I}_{l22} & \tilde{J}_{l22} \\ * & \tilde{L}_{l22} \end{bmatrix} \end{bmatrix} < 0, \quad l = 2, 3, 4. \tag{48}$$

Eq. (34) holds if

$$\begin{bmatrix} -\bar{P}_i - \tilde{M}_{ii} & (A_i - \eta I)\bar{P}_i + B_i\bar{K}_i - \tilde{N}_{ii} \\ * & -\rho^2\bar{P}_i - \tilde{P}_{ii} \end{bmatrix} < 0, \quad i = 1, 2, \tag{49}$$

$$\begin{bmatrix} -\bar{P}_1 - \bar{P}_2 - \tilde{M}_{12} - \tilde{M}_{12}^{\mathrm{T}} & (A_2 - \eta I)\bar{P}_1 + B_2\bar{K}_1 + (A_1 - \eta I)\bar{P}_2 + B_1\bar{K}_2 - \tilde{N}_{12} - \tilde{O}_{12}^{\mathrm{T}} \\ * & -\rho^2\bar{P}_1 - \rho^2\bar{P}_2 - \tilde{P}_{12} - \tilde{P}_{12}^{\mathrm{T}} \end{bmatrix} \leqslant 0, \tag{50}$$

$$\begin{bmatrix} \begin{bmatrix} \tilde{M}_{11} & \tilde{N}_{11} \\ * & \tilde{P}_{11} \end{bmatrix} & \begin{bmatrix} \tilde{M}_{12} & \tilde{N}_{12} \\ \tilde{O}_{12} & \tilde{P}_{12} \end{bmatrix} \\ * & \begin{bmatrix} \tilde{M}_{22} & \tilde{N}_{22} \\ * & \tilde{P}_{22} \end{bmatrix} \end{bmatrix} < 0. \tag{51}$$

Eq. (35) holds if

$$\bar{P}_i(A_i - \mu I)^{\mathrm{T}} + (A_i - \mu I)\bar{P}_i + \bar{K}_i^{\mathrm{T}} B_i^{\mathrm{T}} + B_i\bar{K}_i - \tilde{Q}_{ii} < 0, \quad i = 1, 2, \tag{52}$$

$$\mathrm{sym}\big((A_2 - \mu I)\bar{P}_1 + B_2\bar{K}_1 + (A_1 - \mu I)\bar{P}_2 + B_1\bar{K}_2 - \tilde{Q}_{12}\big) \leqslant 0, \tag{53}$$

$$\begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ * & \tilde{Q}_{22} \end{bmatrix} < 0. \tag{54}$$

Eq. (36) holds if

$$-\bar{P}_i(A_i - vI)^{\mathrm{T}} - (A_i - vI)\bar{P}_i - \bar{K}_i^{\mathrm{T}} B_i^{\mathrm{T}} - B_i\bar{K}_i - \tilde{R}_{ii} < 0, \quad i = 1, 2, \tag{55}$$

$$\mathrm{sym}\big(-(A_2 - vI)\bar{P}_1 - B_2\bar{K}_1 - (A_1 - vI)\bar{P}_2 - B_1\bar{K}_2 - \tilde{R}_{12}\big) \leqslant 0, \tag{56}$$

$$\begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ * & \tilde{R}_{22} \end{bmatrix} < 0. \tag{57}$$

Now, we have transformed conditions (22), (31)–(36) into a set of LMI conditions. Based on these conditions, the multi-objective load-dependent controller design in Problem LDSC can be solved via the following convex optimization problem:

$$\min \gamma_1 \quad \text{s.t. (39)}-\text{(48) and OC}, \tag{58}$$

where OC refers to the pole placement constraints (49)–(51) (disk region) or (52)–(57) (vertical strip region). If the optimization problem (58) has a set of feasible solutions, by substituting the matrix functions (38) into (37), the feedback gain matrix function for controller (18) can be given by

$$K(m_s) = \left(\sum_{i=1}^{2} \alpha_i \bar{K}_i\right)\left(\sum_{i=1}^{2} \alpha_i \bar{P}_i\right)^{-1}. \tag{59}$$

Remark 2. The obtained controller gain matrix function in Eq. (59) based on the convex optimization problem (58) is nonlinearly dependent on the vector α (consequently nonlinearly dependent on m_s), which constitutes the essential difference from previous constant gain controller design.

Remark 3. As can be seen in the above derivation process, the Lyapunov matrices for any given body mass m_s can be given by

$$P(m_s) = \left(\sum_{i=1}^{2} \alpha_i \bar{P}_i \right)^{-1}$$

which is also dependent on the parameter m_s.

4. A design example

In this section, we use an example to illustrate the usefulness and advantage of the load-dependent controller design method proposed in the above sections. Model parameters are borrowed from Ref. [20] and listed in Table 1. The values listed in Table 1 are for the nominal system. We assume that the sprung mass m_s changes with the vehicle load, which is expressed as

$$m_s = (320 + \lambda)\,\mathrm{kg},$$

where λ is a parameter satisfying $|\lambda| \leqslant \bar{\lambda}$. In this case, the state-space model (13) can be represented by a two-vertex polytope.

First, assume that $\bar{\lambda} = 64\,\mathrm{kg}$ (that is, the sprung mass m_s fluctuates around its nominal value by 20%). In addition, assume the maximum allowable suspension stroke $z_{\max} = 0.08\,\mathrm{m}$, the maximum force output $u_{\max} = 1000\,\mathrm{N}$, the road roughness coefficient $G_0 = 512 \times 10^{-6}\,\mathrm{m}^3$, the reference spatial frequency $q_0 = 0.1\,\mathrm{m}^{-1}$ and the vehicle forward speed $V = 30\,\mathrm{m/s}$. Our purpose is to design a load-dependent controller in the form of Eq. (18), such that the closed-loop system (19) satisfies

(1) $\|T_1(s)\|_2 \leqslant \gamma_1$,
(2) $\|T_l(s)\|_G \leqslant 1, \quad l = 2, 3, 4$,
(3) All the eigenvalues of $\bar{A}(\lambda)$ lie in the region $\wp\,(-38,-2)$.

By solving the convex optimization problem (58) in the MATLAB environment [26], we have $\gamma_1^* = \min \|T_1(s)\|_2 = 2.7256\,\mathrm{m/s}^2$, and the associated matrices are as follows (for brevity, here we only list the matrices that are necessary for the construction of the admissible controllers):

$$\bar{P}_1 = \begin{bmatrix} 0.0030 & -0.0002 & -0.0095 & -0.0034 \\ -0.0002 & 0.0002 & 0.0006 & -0.0030 \\ -0.0095 & 0.0006 & 0.0311 & 0.0038 \\ -0.0034 & -0.0030 & 0.0038 & 1.0277 \end{bmatrix},$$

Table 1
Parameters of the quarter-car model

m_s	k_s	c_s	k_u	m_u
320 kg	18,000 N/m	1000 Ns/m	200,000 N/m	40 kg

$$\bar{P}_2 = \begin{bmatrix} 0.0046 & -0.0001 & -0.0141 & -0.0026 \\ -0.0001 & 0.0002 & 0.0008 & -0.0032 \\ -0.0141 & 0.0008 & 0.1038 & 0.0027 \\ -0.0026 & -0.0032 & 0.0027 & 0.9664 \end{bmatrix}$$

$$\bar{K}_1 = \begin{bmatrix} 51.2114 & -1.5133 & -160.3566 & -393.0366 \end{bmatrix},$$

$$\bar{K}_2 = \begin{bmatrix} 58.1277 & -0.6991 & -243.2434 & -263.5746 \end{bmatrix}.$$

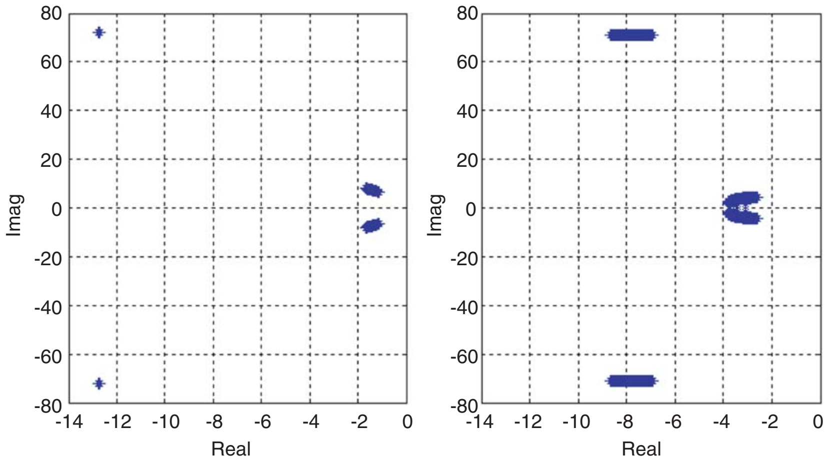

Fig. 3. Poles of open- and closed-loop systems.

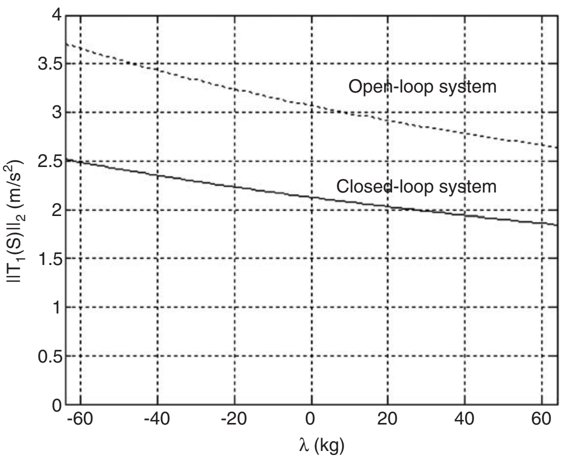

Fig. 4. $\|T_1(s)\|_2$ of open- and closed-loop systems versus parameter λ.

Therefore, the gain matrix function for an admissible load-dependent controller is given by

$$K(m_s) = \left(\sum_{i=1}^{2} \alpha_i \bar{K}_i\right)\left(\sum_{i=1}^{2} \alpha_i \bar{P}_i\right)^{-1}, \tag{60}$$

where

$$\begin{aligned} \alpha_1 &= \left(\frac{1}{m_s} - \frac{1}{320+\bar{\lambda}}\right) \Big/ \left(\frac{1}{320-\bar{\lambda}} - \frac{1}{320+\bar{\lambda}}\right), \\ \alpha_2 &= \left(\frac{1}{320-\bar{\lambda}} - \frac{1}{m_s}\right) \Big/ \left(\frac{1}{320-\bar{\lambda}} - \frac{1}{320+\bar{\lambda}}\right). \end{aligned} \tag{61}$$

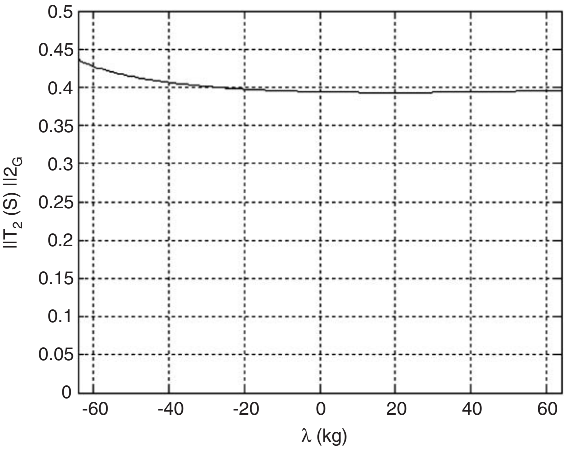

Fig. 5. $\left\|T_2(s)\right\|_G$ of closed-loop system versus parameter λ.

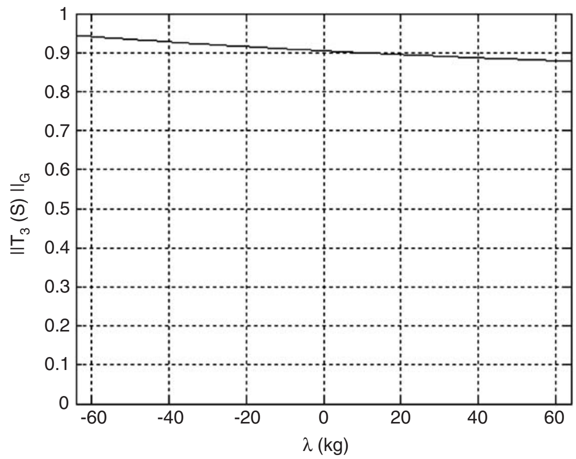

Fig. 6. $\left\|T_3(s)\right\|_G$ of closed-loop system versus parameter λ.

Fig. 3 depicts the eigenvalues of the open- and closed-loop systems in the complex plane, from which we can see that the designed controller renders the poles of the closed-loop system to lie inside the expected region. The H_2 norms of the transfer function $T_1(s)$ for different λ in the admissible interval $|\lambda| \leq \bar{\lambda}$ are shown in Fig. 4. It can be seen from this figure that for all admissible parameter λ, we have $\|T_1(s)\|_2 < \gamma_1^* = 2.7256\,\mathrm{m/s^2}$. In addition, $\|T_l(s)\|_G$, $l = 2, 3, 4$ for different admissible λ are also presented in Figs. 5–7, which clearly show $\|T_l(s)\|_G < 1$.

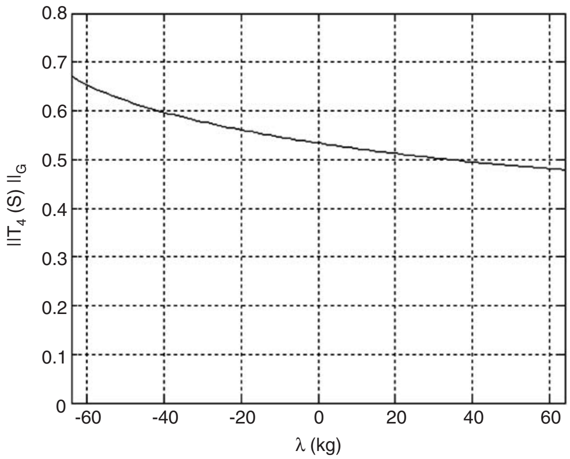

Fig. 7. $\|T_4(s)\|_G$ of closed-loop system versus parameter λ.

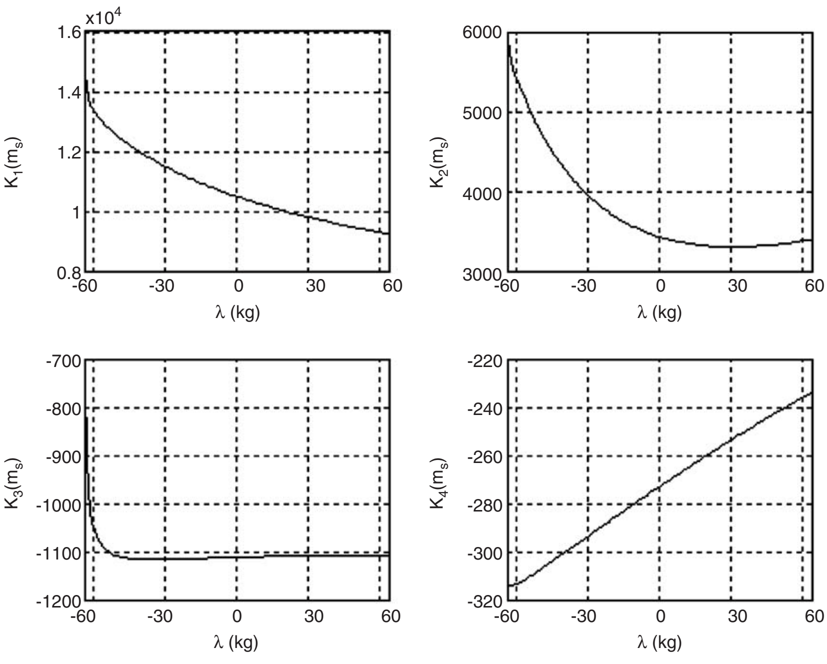

Fig. 8. Nonlinear behavior of controller gains versus parameter λ.

As is mentioned in the above section, the controller gain matrix function (60) is in fact a nonlinear function in terms of the parameter λ. In order to see clear the relationship between $K(m_s)$ and λ, Fig. 8 depicts the four components of $K(m_s)$ for different λ.

Fig. 9 shows the open- and closed-loop frequency responses from the ground vertical velocity $\dot{z}_r(t)$ to the body acceleration $\ddot{z}_s(t)$. From this figure we can see that the closed-loop system has a significant reduction in amplitude when compared with the open-loop system, especially for the frequency band (4–8 Hz), in which the human body is more sensitive to vertical vibration. Therefore, the ride comfort has been improved significantly under the designed load-dependent controller.

Now assume the disturbance input from the ground $\omega(t)$ to be zero-mean white noise with identity power spectral density (shown in Fig. 10). Fig. 11 shows the body accelerations of the

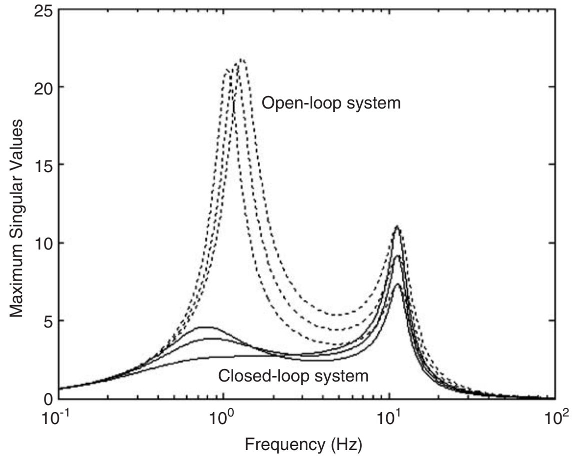

Fig. 9. Frequency response of open- and closed-loop systems from ground velocity $\dot{z}_r(t)$ to body acceleration $\ddot{z}_s(t)$ ($\lambda = -64, 0, 64$).

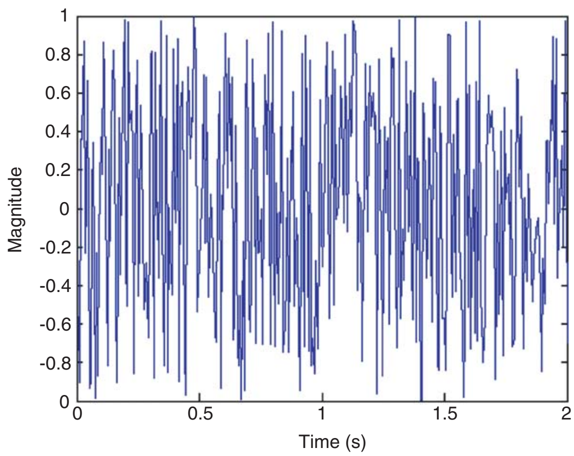

Fig. 10. Disturbance input from the ground $\omega(t)$.

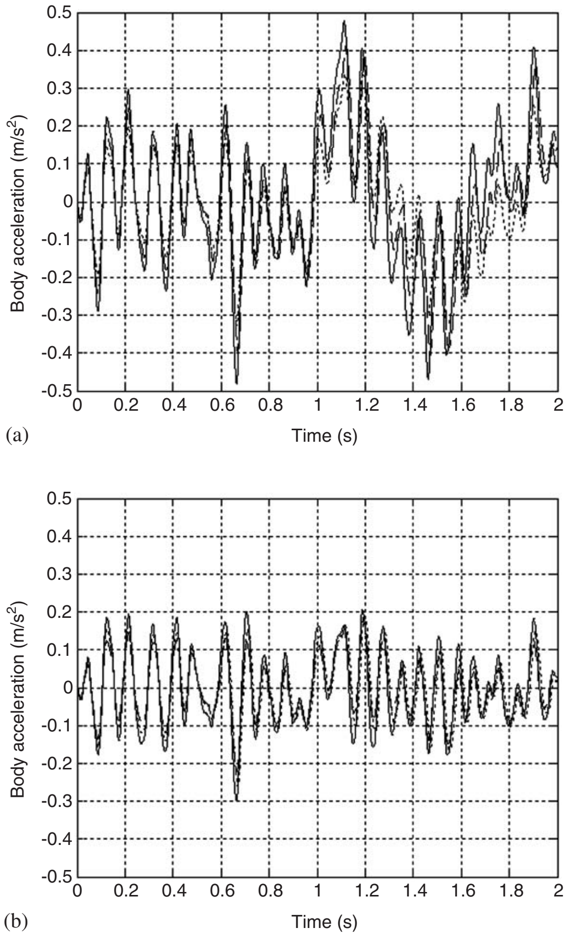

Fig. 11. Vertical accelerations of (a) open- and (b) closed-loop systems. ————, $\lambda = -64$; ------, $\lambda = 0$; ......, $\lambda = 64$.

Table 2
Obtained minimum $\|T_1(s)\|_2$ (comparing results with Ref. [18])

$\bar{\lambda}$ (kg)	32	64	96	128	144
Our method (m/s^2)	2.2608	2.7256	3.4386	4.5863	5.4659
Ref. [18] (m/s^2)	2.5739	3.2689	4.2967	5.9526	Infeasible

open- and closed-loop systems, from which we can see the effectiveness of the designed load-dependent controller (in this figure, the solid line, dashed line and dotted line represent the case $\lambda = -64$, $\lambda = 0$ and $\lambda = 64$, respectively).

Finally, a comparison between the load-dependent controller design and the constant controller design presented in Ref. [18] is carried out. Table 2 lists the obtained minimum H_2 norm of $T_1(s)$ for different $\bar{\lambda}$. It can be seen that under the same conditions, the load-dependent controller approach can yield much less conservative designs than the constant gain approach. Notably for $\bar{\lambda} = 144$ where the constant controller method fails to find feasible solutions, our load-dependent approach is still able to provide desired controllers. To highlight the benefit of the load-dependent controller design, in the following we will present some computer simulations. To this end, we still assume the disturbance input from the ground $\omega(t)$ to be zero-mean white noise with identity power spectral density (shown in Fig. 10). For $\bar{\lambda} = 32$, Fig. 12 presents the body accelerations of the closed-loop systems by the load-dependent controller and the constant controller respectively; for $\bar{\lambda} = 64$. Fig. 13 presents the body accelerations of the closed-loop systems by the load-dependent controller and the constant controller, respectively. From these figures, we can see that

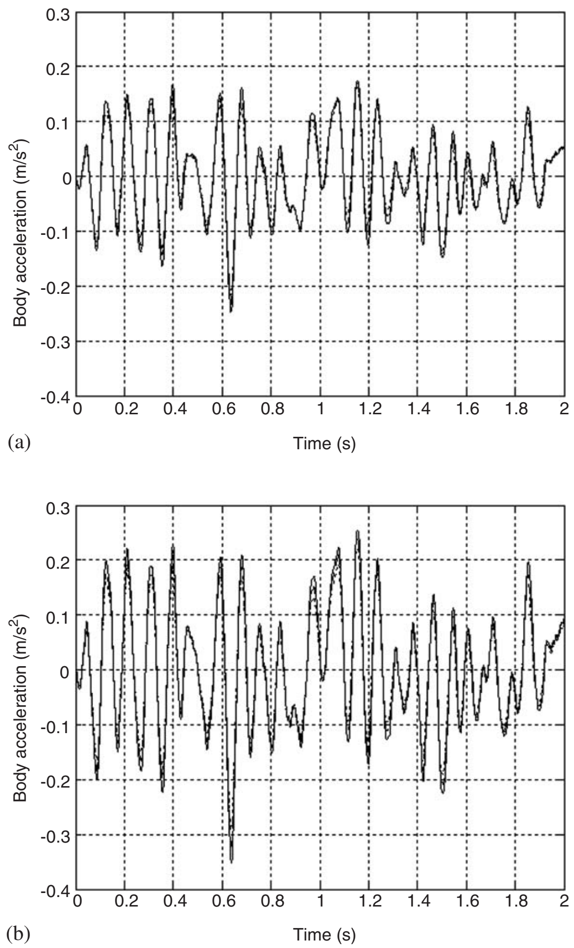

Fig. 12. Vertical accelerations of closed-loop systems by different controllers for $\bar{\lambda} = 32$: (a) load-dependent controller; (b) constant controller. ————, $\lambda = -32$; - - - - - -, $\lambda = 0$; . . . . . . , $\lambda = 32$.

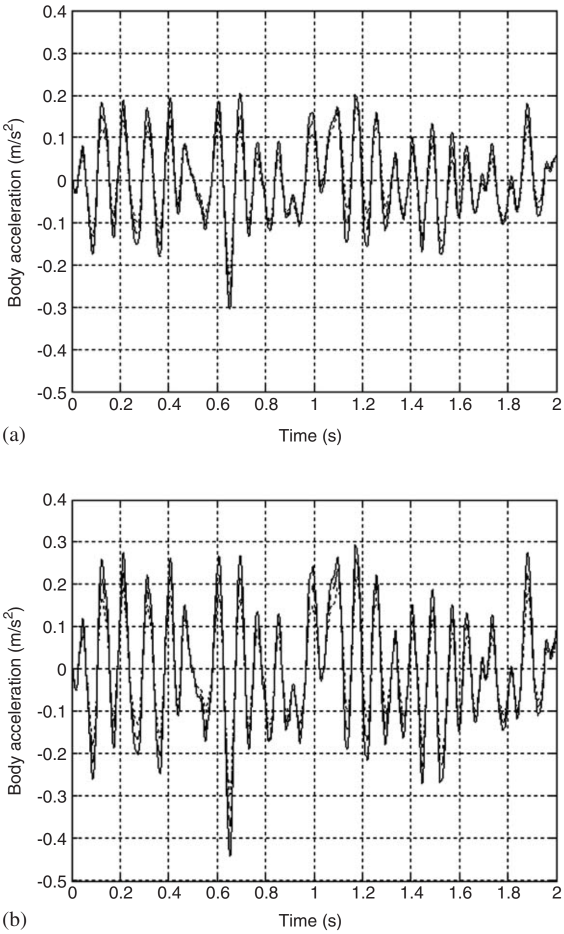

Fig. 13. Vertical accelerations of closed-loop systems by different controllers for $\bar{\lambda} = 64$: (a) load-dependent controller; (b) constant controller. ————, $\lambda = -64$; - - - - - -, $\lambda = 0$; . . . . . . , $\lambda = 64$.

the load-dependent controller design yields better controllers than the constant controller design approach.

5. Concluding remarks

A load-dependent controller design approach has been proposed to solve the problem of multi-objective control of active suspension systems with uncertain parameters. This approach designs controllers whose gain matrix depends on the online available information of the body mass based on parameter-dependent Lyapunov functions. Compared with previous approaches that design robust constant controllers, the proposed load-dependent approach can yield much less conservative results. The usefulness and the advantages of the proposed controller design methodology are illustrated via simulations. Finally, it is worth mentioning that as only state-feedback case is considered in this paper, future research effort can be directed at solving the problem of output-feedback controller design (such as that considered in Ref. [12]), which is more suitable for the case when some of the state variables are not measurable.

Acknowledgements

The authors are grateful to the anonymous reviewers for their valuable comments and suggestions that helped them improve the presentation of the paper. This work was partly supported by RGC HKU 7103/01P.

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