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Stochastic stabilization of uncontrolled and controlled Duffing–van der Pol systems under Gaussian white-noise excitation

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Abstract

In this paper, the stochastic stability of uncontrolled and controlled Duffing–van der Pol systems under Gaussian white-noise excitation is investigated. On the one hand, Lyapunov exponent as a measure is used to estimate the local stability with probability one for the trivial solution of uncontrolled and controlled systems. The difference in Lyapunov exponents between these two kinds of systems is given. On the other hand, the boundary classification of Hamiltonian as a criterion is chosen to judge the global stability of coupled Duffing–van der Pol systems. And the Hamiltonians associated with controlled and uncontrolled systems are also expressed.

1. Introduction

In the last century, the theory of stochastic optimal control for systems under random excitations has developed rapidly with wide applications in many scientific fields, especially in economics and physics [1–4]. For the problem of stochastic stabilization control, the purpose is mainly to design a control law to make unstable random dynamic systems become stable, or to enhance the stability balance of a stable random dynamic systems. Besides, the dynamic stability

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and control of stochastic systems under parametrical and external excitations were paid attention to by several researchers [5–8], mainly focusing on linear systems.

In recent years, based on the stochastic averaging method for quasi-Hamiltonian systems [9–11], Zhu and his co-workers proposed a stochastic dynamical stability strategy [12–14], which can be applied either to linear or nonlinear systems under random excitations. By applying the stochastic averaging method, the system can be described in averaged Ito stochastic differential equations, whose solutions are Markov processes. Generally, the Lyapunov exponent is a measure to judge the stability of a random system [15]. However, the ergodic control based on the Lyapunov exponent can only estimate local stability with probability one for a trivial solution of the system. For a random controlled system governed by one-dimensional diffusion process, the classification of boundaries for the governing FPK equation can be applied to judge the global stability of the system. Recently Zhu and Huang [16] discussed the stability problems of two linearly and nonlinearly coupled van der Pol oscillators. In the present paper, we apply the same strategy to a two-dof coupled Duffing–van der Pol quasi-non-intergrable Hamiltonian system with nonlinear damping to achieve stochastic stabilization and stochastic stability control, which to our knowledge has not been studied yet.

The paper is arranged as follows. Firstly, the local and global stability for coupled Duffing–van der Pol systems subjected to parametric random excitations are analyzed in detail by means of the Lyapunov exponent and the boundary classification. Secondly, the stochastic stability of unbounded control within semi-infinite time interval for coupled Duffing–van der Pol systems is investigated in a similar way. Finally, the numerical results of stochastic stability for uncontrolled and controlled Duffing–van der Pol systems are compared through illustrative figures.

2. Stochastic average for Duffing–van der Pol systems

The Duffing model of an electro-magnetized vibrating beam and the van der Pol model of an electrical circuit with a triode value whose resistance changes with current are two of the most common examples in nonlinear oscillation texts and research articles. In recent years, the coupled Duffing–van der Pol systems has attracted the attention of researches because of their special nonlinear characters in dynamical theories [17]. Moreover, some of the modified Duffing–van der Pol systems are being used to analyze the vibration behavior in electronic oscillator in practice [18,19]. Thus we consider a two-dof Duffing–van der Pol systems subject to random parametric excitation in this article to explore the stability and control of it, of which the Lagrange motion equation is given here as

$$
\begin{aligned}
&\ddot{X}_1 + \beta_1(1 - \lambda_1 X_1^2 - \lambda_1 X_2^2)\dot{X}_1 + \alpha\omega_1^2 X_1(\omega_1^2 X_1^2 + \omega_2^2 X_2^2) + \omega_1^2 X_1 = b_1 X_1 \xi_1(t),\\
&\ddot{X}_2 + \beta_2(1 - \lambda_2 X_1^2 - \lambda_2 X_2^2)\dot{X}_2 + \alpha\omega_2^2 X_2(\omega_1^2 X_1^2 + \omega_2^2 X_2^2) + \omega_2^2 X_2 = b_2 X_2 \xi_2(t),
\end{aligned} \tag{1}
$$

where parameters β_i, λ_i, b_i^2 $(i = 1, 2)$ are positive parameters and small enough of order ε, α is the parameter for strongly nonlinear stiffness term, and $\xi_i(t)$ $(i = 1, 2)$ are mutually independent Gaussian white-noise with zero mean and noisy intensity $2D_i$, D_i are also positive and small parameters. So Eq. (1) defines a quasi-Hamiltonian system with light damping forces and strong nonlinear restoring forces.

The canonical Hamiltonian equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{2}$$

lead to the following stochastic differential equations in the sense of Stratonovich with zero Wong–Zakai correction terms

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= \left(-\frac{\partial H}{\partial q_i} - c_{ii}\frac{\partial H}{\partial p_i}\right) + f_{ii}\xi_i(t) \end{aligned} \quad (i = 1, 2) \tag{3}$$

with

$$c_{ii} = \beta_i(1 - \lambda_i q_1^2 - \lambda_i q_2^2), \quad f_{ii} = b_i q_i, \tag{4}$$

where q_i and $p_i\,(i = 1, 2)$ express generalized displacement and generalized momentum respectively, the Hamiltonian $H(t)$ represents the total energy of the uncontrolled systems. By letting $\varepsilon = 0$ and ignoring small terms, the Hamiltonian can be expressed as follows:

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + U(q_1, q_2),\ U(q_1, q_2) = \tfrac{1}{2}\omega_1^2 q_1^2 + \tfrac{1}{2}\omega_2^2 q_2^2 + \frac{\alpha}{4}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2)^2. \tag{5}$$

The Hamiltonian system governed by Eq. (1) is non-integrable, since $U(q_1, q_2)$ is non-separable when $\alpha \neq 0$. The damping is light and random excitations are weak, so Eq. (1) describes a quasi-non-integrable Hamiltonian system [5]. By applying the stochastic averaging method of quasi-non-integrable-Hamiltonian system [9], the Hamiltonian $H(t)$ converges in probability to a one-dimensional diffusion process, governed by

$$\mathrm{d}H = m(H)\,\mathrm{d}t + \sigma(H)\,\mathrm{d}B(t). \tag{6}$$

It is seen that $H = 0$ is a trivial solution of the systems, and what we care about most is the stability of this trivial solution. Therefore, we pay special attention to the neighbor of this solution to keep the analysis simple. Note that the drift coefficient $m(H)$ and diffusion coefficient $\sigma(H)$ can be computed approximately by integrating in the state–space [5]. They are

$$m(H) = m_1 H + m_2 H^2 + m_3 H^3, \quad H \to 0, \tag{7}$$

$$\sigma^2(H) = \sigma_1^2 H^2 + \sigma_2^2 H^3, \quad H \to 0, \tag{8}$$

where

$$m_1 = \frac{1}{2}\left[\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2} - \beta_1 - \beta_2\right], \tag{9}$$

$$m_2 = \frac{1}{6}\left(\frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_1^2} + \frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_2^2}\right) + \frac{\alpha}{3}(\beta_1 + \beta_2), \tag{10}$$

$$m_3 = -\frac{\alpha}{4}\left(\frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_1^2} + \frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_2^2}\right), \tag{11}$$

$$\sigma_1^2 = \frac{1}{3}\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right), \quad \sigma_2^2 = -\frac{\alpha}{2}\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right). \tag{12}$$

3. Lyapunov exponent and stochastic stability

In the theory of stochastic stabilization, the Lyapunov exponent can be a measure to judge the stability of a random system. Letting $\mathbf{Z} = [\mathbf{q}^{\mathrm{T}}, \mathbf{p}^{\mathrm{T}}]^{\mathrm{T}}$, one can see from Eq. (1) that $\mathbf{Z} = 0$ is a trivial solution of the original systems. The definition for the local stability and the asymptotic stability of the trivial solution are given as follows.

For any small $\varepsilon > 0$, the trivial solution $\mathbf{Z} = 0$ is known to be stable with probability one if

$$\lim_{\|\mathbf{z}_0\| \to 0} P\left\{\sup_{t \geqslant 0} \left\|\mathbf{Z}(t, \mathbf{z}_0)\right\| < \varepsilon\right\} = 1$$

and the trivial solution $\mathbf{Z} = 0$ is known to be asymptotic stable with probability one if

$$\lim_{\|\mathbf{z}_0\| \to 0} P\left\{\lim_{t \to \infty} \left\|\mathbf{Z}(t, \mathbf{z}_0)\right\| = 0\right\} = 1,$$

where $\mathbf{z}_0 = \mathbf{Z}(0)$ is the deterministic initial state and $\|\mathbf{Z}\|$ denotes the norm of $\mathbf{Z}$ which is usually Euclidean norm, i.e. $\|\mathbf{Z}\| = (Z_i Z_i)^{1/2}$. On the basis of Oseledec Multiplicative ergodic theorem, the Lyapunov exponent of the linearized equations is defined as

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln\left\|\mathbf{Z}(t, \mathbf{z}_0)\right\|$$

and the most important character is that the sign of largest Lyapunov exponent determines the stability behavior of random system [15]. And the necessary and sufficient condition for the asymptotic stability with probability one of the system's trivial solution is that the largest Lyapunov exponent should be less than zero.

However, for the two-dof Duffing–van der Pol systems, the linearized equations are 2×2 dimensions which are very difficult to obtain the analytical expression of the largest Lyapunov exponent. To overcome this difficulty, Zhu and Huang [20] proposed a new norm $H^{1/2}$ to replace the Euclidean norm and gave the explanation for the rationality of this norm.

Firstly, we apply Lyapunov exponent method to estimate the stability of coupled Duffing–van der Pol systems. The Lyapunov exponent can be evaluated by introducing the transformation $Y(t) = H^{1/2}(t)$, and the diffusion process governing $Y(t)$ can be obtained by the Ito differential rule on the basis of averaged Ito differential equation (6), that is

$$\mathrm{d}Y(t) = a(Y)\,\mathrm{d}t + b(Y)\,\mathrm{d}B(t). \tag{13}$$

The drift term $a(Y)$ and diffusion term $b(Y)$ satisfy the following equations:

$$a(Y) = \tfrac{1}{2} Y^{-1} m(Y) - \tfrac{1}{8} Y^{-3} \sigma^2(Y)|_{H=Y^2}, \quad b^2(Y) = \tfrac{1}{4} Y^{-2} \sigma^2(Y)|_{H=Y^2}. \tag{14}$$

Substituting Eqs. (9)–(12) into Eq. (14), we can rewrite the drift coefficient $a(Y)$ and diffusion coefficient $b(Y)$ as

$$a(Y) = \tfrac{1}{2}(m_1 - \tfrac{1}{4}\sigma_1^2)Y + \tfrac{1}{2}(m_2 - \tfrac{1}{4}\sigma_2^2)Y^3 + \tfrac{1}{2}m_3 Y^5, \tag{15}$$

$$b^2(Y) = \tfrac{1}{4}\sigma_1^2 Y^2 + \tfrac{1}{4}\sigma_2^2 Y^4. \tag{16}$$

The linearized Ito equation at $Y = 0$ is of the form

$$\mathrm{d}Y = a'(0)H\,\mathrm{d}t + b'(0)H\,\mathrm{d}B(t), \tag{17}$$

whose solution is

$$Y(t) = Y(0)\exp\left[\int_0^t \left(a'(0) - \frac{1}{2}b'^2(0)\right)\right]\mathrm{d}s + \int_0^t b'(0)\,\mathrm{d}B(s). \tag{18}$$

Then the Lyapunov exponent corresponding to the new norm is

$$\lambda_u = \lim_{t\to\infty}\frac{1}{t}\ln Y(t) = a'(0) - \frac{1}{2}b'(0) = \frac{1}{2}m_1 - \frac{1}{4}\sigma_1^2. \tag{19}$$

According to the necessary and sufficient condition for asymptotic stability with probability one of the trivial solution, when $\lambda_u < 0$, namely

$$\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right) < \frac{3}{2}(\beta_1 + \beta_2),$$

$H = 0$ is locally asymptotic stable.

Secondly, we discriminate the stability for coupled Duffing–van der Pol systems by the boundary classification. The ergodic control based on the Lyapunov exponent is effective to estimate local asymptotic stability with probability one for a trivial solution of the systems, but incapable of global stability. For this reason, the boundary classification of Hamiltonian governed by averaged Ito equation is applied to judge the global stability for randomly controlled systems.

Note that the asymptotic expressions for $a(Y)$ and $b^2(Y)$ as $Y \to 0$ are

$$a(Y) = \tfrac{1}{2}(m_1 - \tfrac{1}{4}\sigma_1^2)Y + o(Y), \quad Y \to 0, \tag{20}$$

$$b^2(Y) = \tfrac{1}{4}\sigma_1^2 Y^2 + o(Y^2), \quad Y \to 0. \tag{21}$$

The left boundary $H = 0$ is a singular boundary of the first kind. The diffusion exponent, drift exponent and character value are, respectively,

$$x_1 = 2, \quad y_1 = 1, \quad c_1 = \frac{4m_1 - \sigma_1^2}{\sigma_1^2} = 5 - 6(\beta_1 + \beta_2)/\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right). \tag{22}$$

On the basis of the classification for boundary in Table 4.5.2 in [21], $Y = 0$ is repulsively natural if $c_1 > 1$, strictly natural if $c_1 = 1$, and attractively natural if $c_1 < 1$.

The right boundary $H \to \infty$ is the singular boundary of the second kind, and the asymptotic expressions for $a(Y)$ and $b^2(Y)$ as $Y \to \infty$ are

$$a(Y) = \frac{1}{2}m_3 Y^5 + o(Y^5) = -\frac{\alpha}{8}\left(\frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_1^2} + \frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_2^2}\right)Y^5 + o(Y^5), \quad Y \to \infty, \tag{23}$$

$$b^2(Y) = \frac{1}{4}\sigma_2^2 Y^4 + o(Y^4) = -\frac{\alpha}{8}\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right) Y^4 + o(Y^4), \quad Y \to \infty. \tag{24}$$

The diffusion exponent, drift exponent and character value are, respectively,

$$x_2 = 4, \quad y_2 = 5, \quad c_2 = \frac{4m_3}{\sigma_2^2} = 2\left(\frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_1^2} + \frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_2^2}\right) \Big/ \left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right). \tag{25}$$

Note that $y_2 > x_2 - 1$, $a(+\infty) < 0$ and $y_2 > 1$. Refer to Table 4.5.3 in [21], it is concluded that the right boundary $H \to \infty$ is the entrance. In the case where the left boundary being attractively natural and the right boundary being entrance are the necessary conditions of global asymptotic stability for the trivial solution of the systems, to summarize the constrained condition, the trivial solution $H = 0$ is globally asymptotically stable only if $c_1 < 1$, that is

$$\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right) < \frac{3}{2}(\beta_1 + \beta_2). \tag{26}$$

This coincides with the result derived by Lyapunov exponent. Hence, the trivial solution $H = 0$ is not stable if

$$\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right) > \frac{3}{2}(\beta_1 + \beta_2). \tag{27}$$

4. Stochastic stabilization control

After obtaining the unstable condition (27) for uncontrolled system, the key problem now turns to search an effective controlled strategy to realize the stability control from unstable to stable. So we impose the controlled terms on the coupled Duffing–van der Pol oscillators, whose motion equations can be expressed as follows:

$$\begin{aligned} &\ddot{X}_1 + \beta_1(1 - \lambda_1 X_1^2 - \lambda_1 X_2^2)\dot{X}_1 + \alpha\omega_1^2 X_1(\omega_1^2 X_1^2 + \omega_2^2 X_2^2) + \omega_1^2 X_1 = b_1 X_1 \xi_1(t) + u_1, \\ &\ddot{X}_2 + \beta_2(1 - \lambda_2 X_1^2 - \lambda_2 X_2^2)\dot{X}_2 + \alpha\omega_2^2 X_2(\omega_1^2 X_1^2 + \omega_2^2 X_2^2) + \omega_2^2 X_2 = b_2 X_2 \xi_2(t) + u_2. \end{aligned} \tag{28}$$

The Hamiltonian of these systems converges to a partially averaged Ito differential equation

$$\mathrm{d}H = \bar{m}(H)\,\mathrm{d}t + \sigma(H)\,\mathrm{d}B(t), \tag{29}$$

where

$$\bar{m}(H) = m(H) + \left\langle \frac{\partial H}{\partial p_1} u_1 + \frac{\partial H}{\partial p_2} u_2 \right\rangle \tag{30}$$

and $m(H)$ is in accordance with Eq. (7). Now we design a control strategy to change the systems from unstable to stable. Consider the unbounded control within semi-infinite time interval $[0, \infty]$. Select the performance index

$$J(\mathbf{u}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [f(H) + \langle \mathbf{u}^{\mathrm{T}} \mathbf{N} \mathbf{u} \rangle]\,\mathrm{d}t, \tag{31}$$

where matrix $\mathbf{N}$ is positive definite and diagonal. For convenience, suppose $\mathbf{N}$ has the form

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \text{ with } N_i > 0 \quad (i = 1, 2).$$

The dynamical programming equation corresponding to this problem is

$$\min_{\mathbf{u}} \left\{ \frac{1}{2}\sigma^2(H)\frac{\mathrm{d}^2 V}{\mathrm{d}H^2} + \langle \mathbf{u}^{\mathrm{T}}\mathbf{N}\mathbf{u} \rangle + \frac{\mathrm{d}V}{\mathrm{d}H}\left[m(H) + \left\langle \frac{\partial H}{\partial p_i} u_i \right\rangle \right] + f(H) \right\} = \chi, \tag{32}$$

where

$$\chi = \lim_{T\to\infty} \frac{1}{T}\int_0^T [f(H) + \mathbf{u}^{*\mathrm{T}}\mathbf{N}\mathbf{u}^*]\,\mathrm{d}t$$

is the optimal average cost function, $\mathbf{u}^*$ is the optimal control. By the necessary conditions of Eq. (32), the optimal control terms can be derived

$$u_i^* = -\frac{1}{2N_i}\frac{\mathrm{d}V}{\mathrm{d}H} p_i. \tag{33}$$

Replacing u_i in equality (32) by u_i^*, we obtain the final dynamical programming equation:

$$\frac{1}{2}\sigma^2(H)\frac{\mathrm{d}^2 V}{\mathrm{d}H^2} + m(H)\frac{\mathrm{d}V}{\mathrm{d}H} - \frac{1}{4N_i}\left\langle \left(\frac{\partial H}{\partial p_i}\right)^2 \right\rangle \left(\frac{\mathrm{d}V}{\mathrm{d}H}\right)^2 + f(H) = \chi \tag{34}$$

and the controlled terms can be rewritten as

$$\langle u_1^* p_1 + u_2^* p_2 \rangle = -\left\langle \frac{p_1^2}{2N_1} + \frac{p_2^2}{2N_2} \right\rangle \frac{\mathrm{d}V}{\mathrm{d}H}. \tag{35}$$

Now, we apply Lypunov exponent method to check the stability of controlled systems for trivial solution. Taking the average for formula (35) in the state–space, we obtain

$$\langle p_1^2 \rangle = \langle p_2^2 \rangle = \tfrac{1}{2}H + o(H), \quad H \to 0. \tag{36}$$

It is easy to find the drift coefficient and diffusion coefficient approach the following equations as $H \to 0$:

$$m(H) = \frac{1}{2}\left[\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2} - (\beta_1 + \beta_2) \right] H + o(H) = m_1 H + o(H), \quad H \to 0, \tag{37}$$

$$\sigma^2(H) = \frac{1}{3}\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2} \right) H^2 + o(H^2) = \sigma_1^2 H^2 + o(H^2), \quad H \to 0. \tag{38}$$

For satisfying the dynamical programming equation, it is necessary to assume that

$$f_1(H) - \chi = f_0 H + o(H), \quad H \to 0. \tag{39}$$

The solution of final dynamical programming equation can be obtained as

$$\frac{\mathrm{d}V}{\mathrm{d}H} = k + o(H^0), \quad H \to 0, \tag{40}$$

where

$$k = 4[m_1 + (m_1^2 + f_0 n_0/2)^{1/2}]/n_0, \quad n_0 = \left(\frac{1}{N_1} + \frac{1}{N_2}\right). \tag{41}$$

Therefore, the fully averaged Ito differential equation for Eq. (29) can be derived by taking

$$\bar{m}(H) = m_1 H - \tfrac{1}{4} n_0 k H + o(H), \quad H \to 0. \tag{42}$$

The largest Lyapunov exponent for trivial solution of controlled systems can be expressed as

$$\lambda_c = \tfrac{1}{2}[\bar{m}'(0) - \tfrac{1}{2}(\sigma'(0))^2] = \tfrac{1}{2} m_1 - \tfrac{1}{8} n_0 k - \tfrac{1}{4}\sigma_1^2 \tag{43}$$

and the trivial solution is local asymptotic stable, if $\lambda_c < 0$.

Comparing the two largest Lyapunov exponents derived by the uncontrolled and controlled Duffing–van der Pol systems, the difference between them is

$$\lambda_u - \lambda_c = \tfrac{1}{8} n_0 k. \tag{44}$$

Then the key is how to select f_0 and N_i to realize the optimal stability control.

Now we apply the boundary classification method to check the stability of systems. For the left boundary $H = 0$, the drift coefficient and diffusion for controlled Ito differential equation approach the following equations as $H \to 0$.

$$\bar{m}(H) = m_1 H - \tfrac{1}{4} n_0 k H + o(H), \quad H \to 0, \tag{45}$$

$$\sigma^2(H) = \frac{1}{3}\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right) H^2 + o(H^2) = \sigma_1^2 H^2 + o(H^2). \tag{46}$$

The diffusion exponent, drift exponent and character value are, respectively,

$$x_3 = 2, \quad y_3 = 1, \quad c_3 = (2m_1 - \tfrac{1}{4} n_0 k)/\sigma_1^2. \tag{47}$$

According to the classification for the boundary of Hamiltonian [21], the left boundary $H = 0$ is an attractively natural boundary if $c_3 < 1$, and this requires that

$$(2m_1 - \tfrac{1}{4} n_0 k) < \sigma_1^2.$$

For the right boundary $H \to \infty$, we note that the drift coefficient and the diffusion one for the controlled Ito differential equation approach the following equations as $H \to \infty$

$$m(H) = m_3 H^3 + o(H^3) = -\frac{\alpha}{4}\left(\frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_1^2} + \frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\omega_2^2}\right) + o(H^3), \quad H \to \infty, \tag{48}$$

$$\sigma^2(H) = \sigma_2^2 H^3 + o(H^3) = -\frac{\alpha}{2}\left(\frac{D_1 b_1^2}{\omega_1^2} + \frac{D_2 b_2^2}{\omega_2^2}\right) + o(H^3), \quad H \to \infty. \tag{49}$$

The diffusion exponent, drift exponent and character value are, respectively,

$$x_4 = 3, \quad y_4 = 3, \quad c_4 = -\frac{2m_3}{\sigma_2^2}. \tag{50}$$

By the hypothesis for the parameters, α is a positive constant, which leads to the result that $m(+\infty) = -\infty$. On the other hand, if $y_4 > x_4 - 1 > 1$, on the basis of classification for singular boundary $H \to \infty$ of second kind, then the right boundary $H \to \infty$ is an entrance boundary.

The requirement for asymptotic stability in probability one of the trivial solution $H = 0$ is that the left boundary $H = 0$ is attractively natural and the right boundary $H \to \infty$ is an entrance or repulsively natural. Therefore, $c_3 < 1$ is the only restraint for asymptotic stability for trivial solution. By Eq. (47), it should be

$$(2m_1 - \tfrac{1}{4}n_0 k) < \sigma_1^2. \tag{51}$$

5. Numerical results

The results derived above are shown in the following figures. Fig. 1 displays the Lyapunov exponents of uncontrolled Duffing–van der Pol systems (1) and controlled systems (28) under Gaussian white-noise excitations. Corresponding to the same set of parametric values, the Lyapunov exponent of uncontrolled systems are always positive and the trivial solution is locally unstable. However, by selecting the exact controlled values for parameters n_0 and k according to the condition (43), the Lyapunov exponent becomes negative, which means the trivial solution turns to be locally stable. The energy functions of uncontrolled systems (1) and its averaged system (6) are given in Fig. 2. Note that the parametric value satisfies the restrictive condition (27). The energies are explosive and have an increasing tendency in the neighborhood of the trivial solution with the time. This implies that this solution of uncontrolled systems (1) and its averaged system (6) are globally unstable. Comparatively, the energy functions $H(t)$ of both controlled systems (28) and (29) are given in Fig. 3. By the review inequality (51), we can find the parametric values selected fulfill the conditions (27) and (51). In this case, the energy functions tend to the

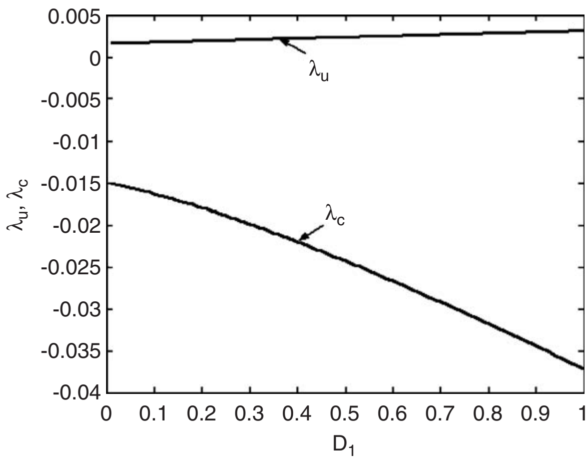

Fig. 1. The Lyapunov exponent of uncontrolled and controlled systems with Gaussian excitation. The parameters are as follows: $\omega_1 = 1,\ \omega_2 = 2,\ \alpha = 2,\ \beta_1 = 0.001,\ \beta_2 = 0.002,\ \lambda_1 = 0.4,\ \lambda_2 = 0.6,\ b_1 = 0.3,\ b_2 = 0.5,\ D_2 = 0.2,\ N_1 = 1,$ $N_2 = 2, f_0 = 0.001$.

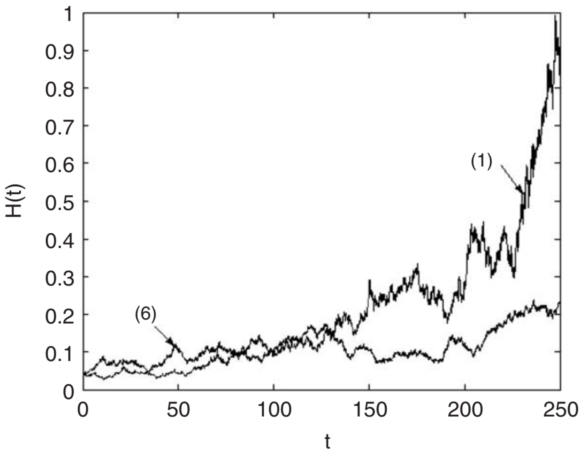

Fig. 2. The energy functions $H(t)$ of uncontrolled systems (1) and its averaged systems (6) with initial values $q_1(0) = 0.15, p_1(0) = 0.12, q_2(0) = 0.09, p_2(0) = 0.08$. The parameters are as follows: $\omega_1 = 1$, $\omega_2 = 2$, $\alpha = 2$, $\beta_1 = 0.001$, $\beta_2 = 0.002$, $\lambda_1 = 0.4$, $\lambda_2 = 0.6$, $b_1 = 0.3$, $b_2 = 0.5$, $D_1 = 0.1$, $D_2 = 0.2$.

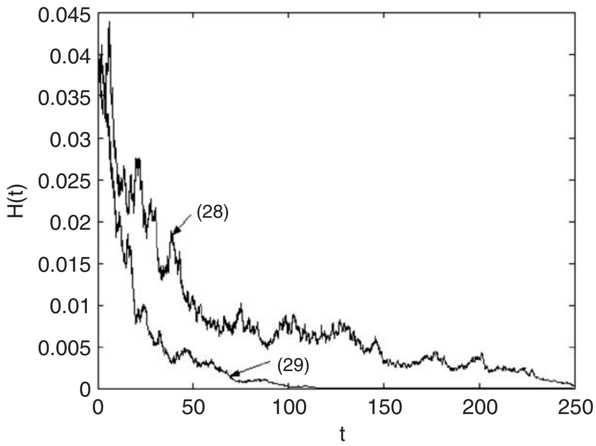

Fig. 3. The energy functions $H(t)$ of controlled systems (28) and its averaged systems (29). $N_1 = 1, N_2 = 2, f_0 = 0.001$. The rest parameters and the initial values are the same with Fig. 2.

trivial solution $H = 0$ consistently with the time. This in return verifies that condition (51) does play a significant role in conversion from unstable to stable globally. Figs. 4 and 5 show the Hamiltonian functions of both uncontrolled systems and controlled systems with a different parameter $f_0 = 0.02$ from those of in Figs. 2 and 3. It is seen that the variations of Hamiltonian functions of either for uncontrolled systems or for controlled systems are almost synchronous with the time. It illustrates that the parametric value f_0 can improve the effectiveness of the stochastic averaging method.

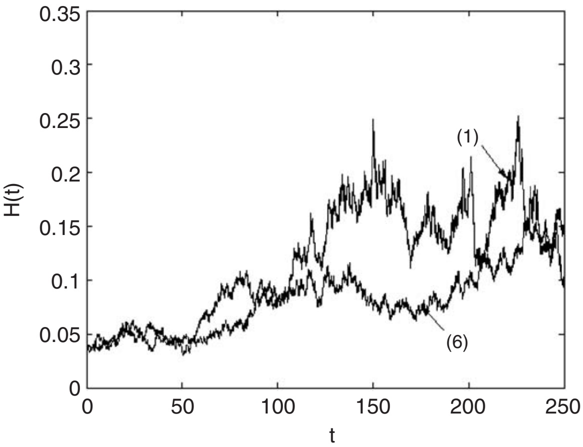

Fig. 4. The energy functions $H(t)$ of uncontrolled systems (1) and its averaged systems (6). $f_0 = 0.02$, the rest parameters and the initial values are the same with Fig. 2.

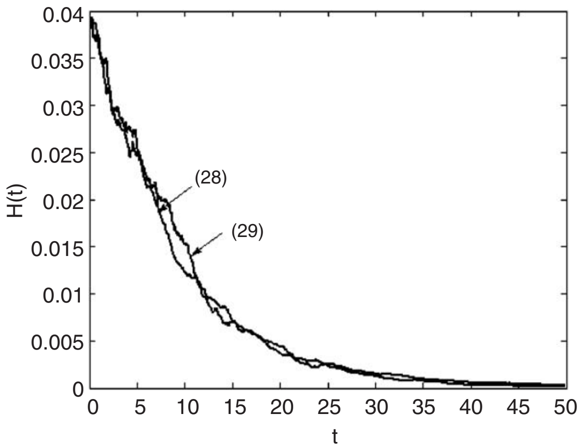

Fig. 5. The energy functions $H(t)$ of controlled systems (28) and its averaged systems (29). The parameters and initial value are the same with Fig. 4.

6. Conclusions

In the present paper, the stability and control of two-dof coupled Duffing–van der Pol systems under stochastic Gaussian excitations are investigated in detail. It has been shown that the considered systems can be reduced into a one-dimensional diffusion process. In this way, two different approaches have been followed to identify the stability property of systems. The method of Lyapunov exponent based on ergodic theorem can estimate the local stability and boundary classification of Hamiltonian can examine the global stability for uncontrolled systems. By using these two methods, the conditions restricting local stability and global stability are obtained. For the unstable uncontrolled systems, the dynamical behavior changes from unstable to stable by applying the strategy of dynamical programming rule and by selecting the controlled parametric value exactly. The numerical results illustrate the effectiveness of controlled terms.

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