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Journal of Sound and Vibration 291 (2006) 19–47

JOURNAL OF SOUND AND VIBRATION

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The transient responses of a special non-homogeneous magneto-electro-elastic hollow cylinder for axisymmetric plane strain problem

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Received 14 June 2004; received in revised form 6 May 2005; accepted 23 May 2005
Available online 19 August 2005

Abstract

By virtue of the introduction of new dependent variable and the separation of variables technique, the transient responses of a special non-homogeneous magneto-electro-elastic hollow cylinder are transformed to two Volterra integral equations of the second kind of about two functions with respect to time. These integral equations can be solved successfully by means of the interpolation method. Then, the complete solutions of displacements, stresses, electric potential, electric displacements, magnetic potential and magnetic inductions are obtained. The present method is suitable for a magneto-electro-elastic hollow cylinder with an arbitrary thickness subjected to arbitrary axisymmetric mechanical and electromagnetic loads. Numerical results are finally presented.

1. Introduction

The analysis for dynamic problems of elastic bodies is an important and interesting research field for engineers and scientists. Being of the common structural form, the hollow cylinders are studied extensively. For pure elastic media, Cinelli [1] obtained the theoretical solutions of

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.05.022

dynamics of cylindrical and spherical shells. The dynamic responses of cylindrical and spherical shells were studied by Chou and Koenig [2] and Rose et al. [3]. Wang and Gong [4] studied the stress responses of isotropic cylindrical shells shocked at the inner surface. For piezoelectric media, Adelman and Stavsky [5,6] studied the axisymmetric free vibrations of radially and axially polarized piezoelectric ceramic hollow cylinders. Shul'ga et al. [7] and Paul and Venkatesan [8] investigated the axisymmetric and three-dimensional electroelastic waves in a hollow piezoelectric ceramic cylinder. The free vibrations of piezoelectric, empty and also compressible fluid filled cylindrical shells for three-dimensional problems were studied by Ding et al. [9,10] in recent years.

There are also many works that have been done for non-homogeneous materials. Among them, the special case where Young's modulus has a power law dependence on the radial coordinate, while the linear thermal expansion coefficient and Poisson's ratio are constants, has been considered by many scientists and engineers. For instance, Shaffer [11] obtained the general solutions for a non-homogeneous orthotropic annular disk in plane stress subjected to uniform pressures at the internal and external surfaces. The rotation problem of a non-homogeneous orthotropic composite cylinder was considered by El-Naggar et al. [12]. Horgan and Chan [13] investigated the pressured functionally graded isotropic hollow cylinder and disk problems. The transient thermal stresses in a rotating non-homogeneous cylindrically orthotropic composite tube and in a non-homogeneous spherically orthotropic elastic medium with spherical cavity were studied by Abd-Alla et al. [14,15]. Tarn [16] obtained the exact solutions of functionally graded anisotropic cylinders subjected to thermal and mechanical loads for steady-state problem. Ding et al. [17] gave the solution of a non-homogeneous orthotropic cylindrical shell for axisymmetric plane strain dynamic thermoelastic problems. The torsional oscillations of a finite non-homogeneous piezoelectric cylindrical shell were also investigated by Sarma [18]. In the above studies, the variation of material density is often assumed to be the same as that of Young's.

More recent advances are the intelligent composites made of piezoelectric/piezomagnetic materials. This material not only has the ability of converting energy from one form to the other (among magnetic, electric and mechanical energies), but also exhibits a magnetoelectric effect that is not present in single-phase piezoelectric or piezomagnetic materials [19–21]. Most works for magneto-electro-elastic composites are focused on the optimization of material properties [19–32], especially the magneto-electro effect. For static problems, Wang and Shen [33] obtained the general solution of three-dimensional problems in transversely isotropic magneto-electro-elastic media and further derived the fundamental solution for dislocation and Green's functions in half-space. In addition, Wang and Shen [34] studied the two-dimensional problem of inclusions of arbitrary shape in magneto-electro-elastic composites. Liu et al. [35] obtained the Green's functions for an infinite two-dimensional anisotropic magneto-electro-elastic medium containing an elliptical cavity. Pan [36] derived the exact solutions for three-dimensional, anisotropic, magneto-electro-elastic, simply supported and multilayered rectangular plates under static loadings. For dynamic problems, the authors only found that Pan and Heyliger [37] studied the free vibrations of simply supported and multilayered rectangular plates and derived the analytical solutions. Ding et al. [38], Hou et al. [39] obtained the analytical solution for the axisymmetric plane strain electroelastic dynamics of a non-homogeneous piezoelectric hollow cylinder. Hou and Leung [40] further study the corresponding problem of magneto-electro-elastic hollow cylinders.

In this paper, the transient responses of a special non-homogeneous magneto-electro-elastic hollow cylinder subjected to arbitrary mechanical and electromagnetic loads are searched. Firstly, a new dependant variable is introduced to rewrite the governing equations, the mechanical boundary conditions and the initial conditions. Secondly, a special function is introduced to transform the inhomogeneous mechanical boundary conditions into the homogeneous ones. Thirdly, by virtue of the separation of variables technique, and utilizing the initial conditions and electromagnetic boundary conditions, two second kind of Volterra integral equations about two functions with respect to time are derived, which can be solved by means of the interpolation method. Thus, the complete transient responses of the displacements, stresses, electric potential, electric displacements, magnetic potential and magnetic inductions are obtained. At last, the transient responses of the magneto-electro-elastic hollow cylinder with different non-homogeneous state are investigated for the sudden constant load and dynamic combined load, respectively.

2. Basic equations

As suggested by Pan [37], when the body force, free charge density and current density are absent, the basic equations for the dynamics of magneto-electro-elasticity can be expressed as

$$\sigma_{ij,j} = \rho\,\frac{\partial^2 u_i}{\partial t^2}, \quad D_{j,j} = 0, \quad B_{j,j} = 0, \tag{1a−c}$$

$$\sigma_{ij} = C_{ijkl}\bar{\varepsilon}_{kl} - e_{kij}E_k - d_{kij}H_k, \tag{2a}$$

$$D_i = e_{ikl}\bar{\varepsilon}_{kl} + \varepsilon_{ik}E_k + g_{ik}H_k, \tag{2b}$$

$$B_i = d_{ikl}\bar{\varepsilon}_{kl} + g_{ik}E_k + \mu_{ik}H_k, \tag{2c}$$

$$\bar{\varepsilon}_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i = -\Phi_{,i}, \quad H_i = -\Psi_{,i}, \tag{3a−c}$$

where σ_{ij}, $\bar{\varepsilon}_{ij}$, u_i, E_i, D_i, H_i and B_i are the components of stress, strain, displacement, electric field, electric displacement, magnetic field and magnetic induction, respectively; Φ and Ψ are electric potential and magnetic potential, respectively; c_{ijkl}, e_{kij}, d_{kij}, ε_{ij}, g_{ij} and μ_{ij} are elastic, piezoelectric, piezomagnetic, dielectric, electromagnetic and magnetic coefficients, respectively; and ρ is the mass density of the material.

Here, we consider orthotropic and radially polarized magneto-electro-elastic media with the non-homogeneous case that all physical coefficients have a power law dependence on the radial coordinate. Thus, the corresponding basic equations in a cylindrical coordinate system (r, ϕ, z) can be obtained from Eqs. (1–3) as follows:

$$\begin{gathered}
\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{r\phi}}{r\partial\phi} + \frac{\partial\sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = \rho\left(\frac{r}{b}\right)^{2N}\frac{\partial^2 u_r}{\partial t^2},\\
\frac{\partial\sigma_{r\phi}}{\partial r} + \frac{\partial\sigma_{\phi\phi}}{r\partial\phi} + \frac{\partial\sigma_{\phi z}}{\partial z} + \frac{2\partial\sigma_{r\phi}}{r} = \rho\left(\frac{r}{b}\right)^{2N}\frac{\partial^2 u_\phi}{\partial t^2},\\
\frac{\partial\sigma_{zr}}{\partial r} + \frac{\partial\sigma_{\phi z}}{r\partial\phi} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} = \rho\left(\frac{r}{b}\right)^{2N}\frac{\partial^2 u_z}{\partial t^2},
\end{gathered} \tag{4a}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r)+\frac{1}{r}\frac{\partial D_\phi}{\partial \phi}+\frac{\partial D_z}{\partial z}=0, \tag{4b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r)+\frac{1}{r}\frac{\partial B_\phi}{\partial \phi}+\frac{\partial B_z}{\partial z}=0, \tag{4c}$$

$$\begin{aligned}
\sigma_{\phi\phi}&=(c_{11}\gamma_{\phi\phi}+c_{12}\gamma_{zz}+c_{13}\gamma_{rr}-e_{31}E_r-d_{31}H_r)(r/b)^{2N},\\
\sigma_{zz}&=(c_{12}\gamma_{\phi\phi}+c_{22}\gamma_{zz}+c_{23}\gamma_{rr}-e_{32}E_r-d_{32}H_r)(r/b)^{2N},\\
\sigma_{rr}&=(c_{13}\gamma_{\phi\phi}+c_{23}\gamma_{zz}+c_{33}\gamma_{rr}-e_{33}E_r-d_{33}H_r)(r/b)^{2N},\\
\sigma_{zr}&=(2c_{44}\gamma_{rz}-e_{24}E_z-d_{24}H_z)(r/b)^{2N},\\
\sigma_{r\phi}&=(2c_{55}\gamma_{r\phi}-e_{15}E_\phi-d_{15}H_\phi)(r/b)^{2N},\\
\sigma_{\phi z}&=2c_{66}\gamma_{\phi z}(r/b)^{2N},
\end{aligned} \tag{5a}$$

$$\begin{aligned}
D_\phi&=(2e_{15}\gamma_{r\phi}+\varepsilon_{11}E_\phi+g_{11}H_\phi)(r/b)^{2N},\\
D_z&=(2e_{24}\gamma_{rz}+\varepsilon_{22}E_z+g_{22}H_z)(r/b)^{2N},\\
D_r&=(e_{31}\gamma_{\phi\phi}+e_{32}\gamma_{zz}+e_{33}\gamma_{rr}+\varepsilon_{33}E_r+g_{33}H_r)(r/b)^{2N},
\end{aligned} \tag{5b}$$

$$\begin{aligned}
B_\phi&=(2d_{15}\gamma_{r\phi}+g_{11}E_\phi+\mu_{11}H_\phi)(r/b)^{2N},\\
B_z&=(2d_{24}\gamma_{rz}+g_{22}E_z+\mu_{22}H_z)(r/b)^{2N},\\
B_r&=(d_{31}\gamma_{\phi\phi}+d_{32}\gamma_{zz}+d_{33}\gamma_{rr}+g_{33}E_r+\mu_{33}H_r)(r/b)^{2N},
\end{aligned} \tag{5c}$$

$$\begin{aligned}
&\gamma_{rr}=\frac{\partial u_r}{\partial r},\quad \gamma_{\phi\phi}=\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r},\quad \gamma_{zz}=\frac{\partial u_z}{\partial z},\\
&\gamma_{zr}=\frac{1}{2}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right),\quad \gamma_{r\phi}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right),\\
&\gamma_{\phi z}=\frac{1}{2}\left(\frac{\partial u_\phi}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \phi}\right),
\end{aligned} \tag{6a}$$

$$E_r=-\frac{\partial \Phi}{\partial r},\quad E_\phi=-\frac{1}{r}\frac{\partial \Phi}{\partial \phi},\quad E_z=-\frac{\partial \Phi}{\partial z}, \tag{6b}$$

$$H_r=-\frac{\partial \Psi}{\partial r},\quad H_\phi=-\frac{1}{r}\frac{\partial \Psi}{\partial \phi},\quad H_z=-\frac{\partial \Psi}{\partial z}, \tag{6c}$$

where b and N are determined constants. There are altogether 28 independent constants in Eqs. (5), which include 9 elastic constants, 5 piezoelectric constants, 5 piezomagetic constants, 3 dielectric constants, 3 electromagnetic constants and 3 magnetic constants. When $N=0$, Eqs. (4–6) degenerated to the basic equations of homogeneous media.

3. Problem description

Consider an orthotropic, radially polarized and non-homogeneous magneto-electro-elastic hollow cylinder with inner and outer radii of a and b (Fig. 1). There are axisymmetric mechanical and electromagnetic loads acting on its boundary.

From the view point of three dimensions, this is an axisymmetric problem with $u_r = u_r(r, z, t)$, $u_\phi = 0$, $u_z = u_z(r, z, t)$, $\Phi = \Phi(r, z, t)$ and $\Psi = \Psi(r, z, t)$. In this paper, a plane strain condition for this problem is considered. Thus,

$$u_r = u_r(r, t), \quad u_\phi = u_z = 0, \quad \Phi = \Phi(r, t), \quad \Psi = \Psi(r, t). \tag{7}$$

Substitution of Eq. (7) into Eq. (6) yields

$$\gamma_{rr} = \frac{\partial u_r}{\partial r} = u_r', \quad \gamma_{\phi\phi} = \frac{u_r}{r}, \quad \gamma_{zz} = \gamma_{zr} = \gamma_{r\phi} = \gamma_{\phi z} = 0, \tag{8a}$$

$$E_r = -\frac{\partial \Phi}{\partial r} = -\Phi', \quad E_\phi = E_z = 0, \tag{8b}$$

$$H_r = -\frac{\partial \Psi}{\partial r} = -\Psi', \quad H_\phi = H_z = 0. \tag{8c}$$

Substitution of Eq. (8) into Eq. (5) yields

$$\begin{aligned}
\sigma_{\phi\phi} &= \left(c_{11}\frac{u_r}{r} + c_{13}\frac{\partial u_r}{\partial r} + e_{31}\frac{\partial \Phi}{\partial r} + d_{31}\frac{\partial \Psi}{\partial r}\right)\left(\frac{r}{b}\right)^{2N}, \\
\sigma_{zz} &= \left(c_{12}\frac{u_r}{r} + c_{23}\frac{\partial u_r}{\partial r} + e_{32}\frac{\partial \Phi}{\partial r} + d_{32}\frac{\partial \Psi}{\partial r}\right)\left(\frac{r}{b}\right)^{2N}, \\
\sigma_{rr} &= \left(c_{13}\frac{u_r}{r} + c_{33}\frac{\partial u_r}{\partial r} + e_{33}\frac{\partial \Phi}{\partial r} + d_{33}\frac{\partial \Psi}{\partial r}\right)\left(\frac{r}{b}\right)^{2N}, \\
\sigma_{zr} &= \sigma_{r\phi} = \sigma_{\phi z} = 0,
\end{aligned} \tag{9a}$$

$$\begin{aligned}
D_r &= \left(e_{31}\frac{u_r}{r} + e_{33}\frac{\partial u_r}{\partial r} - \varepsilon_{33}\frac{\partial \Phi}{\partial r} - g_{33}\frac{\partial \Psi}{\partial r}\right)\left(\frac{r}{b}\right)^{2N}, \\
D_\phi &= D_z = 0,
\end{aligned} \tag{9b}$$

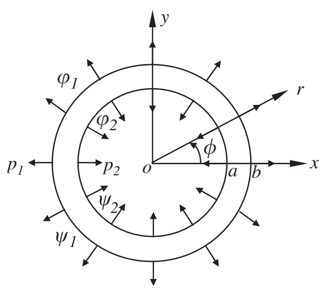

Fig. 1. Non-homogeneous magneto-electro-elastic hollow cylinder under coupling loads.

$$B_r = \left(d_{31}\,\frac{u_r}{r} + d_{33}\,\frac{\partial u_r}{\partial r} - g_{33}\,\frac{\partial \Phi}{\partial r} - \mu_{33}\,\frac{\partial \Psi}{\partial r}\right)\left(\frac{r}{b}\right)^{2N},$$
$$B_\phi = B_z = 0. \tag{9c}$$

Then, by virtue of Eq. (9), Eq. (4) can be simplified as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = \rho\left(\frac{r}{b}\right)^{2N}\frac{\partial^2 u_r}{\partial t^2}, \tag{10a}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) = 0, \qquad \frac{1}{r}\frac{\partial}{\partial r}(rB_r) = 0. \tag{10b,c}$$

The non-dimensional parameters, coordinates and variables are introduced as

$$\begin{aligned}
&c_i = \frac{c_{1i}}{c_{33}}, \quad c_4 = \frac{c_{23}}{c_{33}}, \quad e_i = \frac{e_{3i}}{\sqrt{c_{33}\varepsilon_{33}}}, \quad d_i = \frac{d_{3i}}{\sqrt{c_{33}\mu_{33}}} \quad (i = 1, 2, 3),\\
&g = \frac{g_{33}}{\sqrt{\varepsilon_{33}\mu_{33}}}, \quad c = \sqrt{\frac{c_{33}}{\rho}}, \quad s = \frac{a}{b}, \quad \xi = \frac{r}{b}, \quad \tau = \frac{ct}{b},\\
&u = \frac{u_r}{b}, \quad \sigma_r = \frac{\sigma_{rr}}{c_{33}}, \quad \sigma_\phi = \frac{\sigma_{\phi\phi}}{c_{33}}, \quad \sigma_z = \frac{\sigma_{zz}}{c_{33}},\\
&\varphi = \frac{\Phi}{b}\sqrt{\frac{\varepsilon_{33}}{c_{33}}}, \quad D = \frac{D_r}{\sqrt{c_{33}\varepsilon_{33}}}, \quad \psi = \frac{\Psi}{b}\sqrt{\frac{\mu_{33}}{c_{33}}}, \quad B = \frac{B_r}{\sqrt{c_{33}\mu_{33}}}.
\end{aligned} \tag{11}$$

Thus, we have

$$\begin{aligned}
&u_r' = \frac{\partial u_r}{\partial r} = \frac{\partial u}{\partial \xi} = u', \qquad \frac{u_r}{r} = \frac{u}{\xi},\\
&\Phi' = \frac{\partial \Phi}{\partial r} = \sqrt{\frac{c_{33}}{\varepsilon_{33}}}\frac{\partial \varphi}{\partial \xi} = \sqrt{\frac{c_{33}}{\varepsilon_{33}}}\varphi',\\
&\Psi' = \frac{\partial \Psi}{\partial r} = \sqrt{\frac{c_{33}}{\mu_{33}}}\frac{\partial \psi}{\partial \xi} = \sqrt{\frac{c_{33}}{\mu_{33}}}\psi'.
\end{aligned} \tag{12}$$

Then, based on Eqs. (11,12), Eqs. (9,10) can be translated into following non-dimensional forms:

$$\begin{aligned}
&\sigma_\phi = \left(c_1\,\frac{u}{\xi} + c_3\,\frac{\partial u}{\partial \xi} + e_1\,\frac{\partial \varphi}{\partial \xi} + d_1\,\frac{\partial \psi}{\partial \xi}\right)\xi^{2N},\\
&\sigma_z = \left(c_2\,\frac{u}{\xi} + c_4\,\frac{\partial u}{\partial \xi} + e_2\,\frac{\partial \varphi}{\partial \xi} + d_2\,\frac{\partial \psi}{\partial \xi}\right)\xi^{2N},\\
&\sigma_r = \left(c_3\,\frac{u}{\xi} + \frac{\partial u}{\partial \xi} + e_3\,\frac{\partial \varphi}{\partial \xi} + d_3\,\frac{\partial \psi}{\partial \xi}\right)\xi^{2N},
\end{aligned} \tag{13a}$$

$$D = \left(e_1\,\frac{u}{\xi} + e_3\,\frac{\partial u}{\partial \xi} - \frac{\partial \varphi}{\partial \xi} - g\,\frac{\partial \psi}{\partial \xi}\right)\xi^{2N}, \tag{13b}$$

$$B = \left(d_1\,\frac{u}{\xi} + d_3\,\frac{\partial u}{\partial \xi} - g\,\frac{\partial \varphi}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\right)\xi^{2N}, \tag{13c}$$

$$\frac{\partial \sigma_r}{\partial \xi} + \frac{\sigma_r - \sigma_\phi}{\xi} = \xi^{2N} \frac{\partial^2 u}{\partial \tau^2}, \tag{14a}$$

$$\frac{\partial}{\partial \xi}(\xi D) = 0, \qquad \frac{\partial}{\partial \xi}(\xi B) = 0. \tag{14b,c}$$

The corresponding non-dimensional boundary conditions and initial conditions are

$$\sigma_r(1,\tau) = p_1(\tau), \qquad \sigma_r(s,\tau) = p_2(\tau), \tag{15a}$$

$$\varphi(1,\tau) = \varphi_1(\tau), \qquad \varphi(s,\tau) = \varphi_2(\tau), \tag{15b}$$

$$\psi(1,\tau) = \psi_1(\tau), \qquad \psi(s,\tau) = \psi_2(\tau), \tag{15c}$$

$$u(\xi,0) = g_1(\xi), \qquad \dot{u}(\xi,0) = \left[\frac{\partial u(\xi,\tau)}{\partial \tau}\right]_{\tau=0} = g_2(\xi). \tag{16}$$

Every dot over a variable in this paper denotes a partial derivative with respect to initial conditions time τ.

Thus, the complete governing equations for this problem are obtained and consisted of Eqs. (13–16).

4. Solving technology

4.1. Transform for the governing equations

At first, Eqs. (13b,c) are rewritten as

$$\frac{\partial \varphi}{\partial \xi} = \alpha_3 \frac{u}{\xi} + \beta_3 \frac{\partial u}{\partial \xi} - \frac{\delta}{\xi^{2N}}(D - gB), \tag{17a}$$

$$\frac{\partial \psi}{\partial \xi} = \alpha_4 \frac{u}{\xi} + \beta_4 \frac{\partial u}{\partial \xi} - \frac{\delta}{\xi^{2N}}(B - gD). \tag{17b}$$

Then, substituting Eq. (17) into Eq. (13a) yields

$$\sigma_\phi = \left(\alpha_1 \frac{u}{\xi} + \alpha_2 \frac{\partial u}{\partial \xi}\right)\xi^{2N} - \alpha_3 D - \alpha_4 B, \tag{18a}$$

$$\sigma_z = \left(\gamma_1 \frac{u}{\xi} + \gamma_2 \frac{\partial u}{\partial \xi}\right)\xi^{2N} - \gamma_3 D - \gamma_4 B, \tag{18b}$$

$$\sigma_r = \left(\beta_1 \frac{u}{\xi} + \beta_2 \frac{\partial u}{\partial \xi}\right)\xi^{2N} - \beta_3 D - \beta_4 B, \tag{18c}$$

where

$$\alpha_1 = c_1 + e_1\alpha_3 + d_1\alpha_4, \quad \alpha_2 = c_3 + e_1\beta_3 + d_1\beta_4, \quad \alpha_3 = \delta(e_1 - gd_1), \quad \alpha_4 = \delta(d_1 - ge_1),$$

$$\beta_1 = \alpha_2, \quad \beta_2 = 1 + e_3\beta_3 + d_3\beta_4, \quad \beta_3 = \delta(e_3 - gd_3), \quad \beta_4 = \delta(d_3 - ge_3), \quad \delta = 1/(1 - g^2),$$
$$\gamma_1 = c_2 + e_2\alpha_3 + d_2\alpha_4, \quad \gamma_2 = c_4 + e_2\beta_3 + d_2\beta_4, \quad \gamma_3 = \delta(e_2 - gd_2), \quad \gamma_4 = \delta(d_2 - ge_2). \tag{19}$$

Based on Eqs. (14b,c), D and B can be derived as

$$D(\xi, \tau) = \frac{1}{\xi} d(\tau), \qquad B(\xi, \tau) = \frac{1}{\xi} b(\tau), \tag{20a,b}$$

where $d(\tau)$ and $b(\tau)$ are undetermined functions with respect to non-dimensional time τ.

Substituting Eqs. (18a,c) into Eq. (14a) and utilizing Eq. (20) yield

$$\frac{\partial^2 u}{\partial \xi^2} + (2N+1)\frac{1}{\xi}\frac{\partial u}{\partial \xi} - \lambda\frac{u}{\xi^2} - \frac{1}{\beta_2}\frac{\partial^2 u}{\partial \tau^2} = -\frac{\alpha_3}{\beta_2}\frac{d(\tau)}{\xi^{2(N+1)}} - \frac{\alpha_4}{\beta_2}\frac{b(\tau)}{\xi^{2(N+1)}}, \tag{21}$$

where

$$\lambda = \frac{\alpha_1 - 2N\beta_1}{\beta_2}. \tag{22}$$

Secondly, a new dependent variable $v(\xi, \tau)$ is introduced as

$$v(\xi, \tau) = \xi^N u(\xi, \tau). \tag{23}$$

Thus, Eq. (21) can be rewritten as

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial v}{\partial \xi} - n^2\frac{v}{\xi^2} - \frac{1}{\beta_2}\frac{\partial^2 v}{\partial \tau^2} = -\frac{\alpha_3}{\beta_2}\frac{d(\tau)}{\xi^{N+2}} - \frac{\alpha_4}{\beta_2}\frac{b(\tau)}{\xi^{N+2}}, \tag{24}$$

where

$$n = \sqrt{N^2 + \lambda}. \tag{25}$$

By virtue of Eqs. (18c,20,23), mechanical boundary condition (15a) can be rewritten into following forms:

$$\left[\beta_2\frac{\partial v}{\partial \xi} + \beta_5\frac{v}{\xi}\right]_{\xi=1} = P_1(\tau), \tag{26a}$$

$$\left[\beta_2\frac{\partial v}{\partial \xi} + \beta_5\frac{v}{\xi}\right]_{\xi=s} = P_2(\tau), \tag{26b}$$

where

$$P_1(\tau) = p_1(\tau) + \beta_3 d(\tau) + \beta_4 b(\tau),$$
$$P_2(\tau) = \frac{1}{s^N}\left[p_2(\tau) + \frac{\beta_3}{s} d(\tau) + \frac{\beta_4}{s} b(\tau)\right],$$
$$\beta_5 = \beta_1 - N\beta_2. \tag{27}$$

Thirdly, the inhomogeneous boundary conditions (26) will be further transformed into the homogeneous ones by taking

$$v(\xi, \tau) = w(\xi, \tau) + A_0(\xi - s)^m P_1(\tau) + B_0(\xi - 1)^m P_2(\tau), \tag{28}$$

where $w(\xi,\tau)$ is undetermined function and

$$A_0=\frac{1}{\beta_2 m(1-s)^{m-1}+\beta_5(1-s)^m},\qquad B_0=\frac{1}{\beta_2 m(s-1)^{m-1}+\beta_5 s^{-1}(s-1)^m}. \tag{29}$$

Generally, one can take $m=2$ if both the denominators of A_0 and B_0 are non-zero; otherwise, $m=3,4,5,\ldots$ can be adopted.

Substituting Eq. (28) into Eqs. (24,26) yields

$$\frac{\partial^2 w}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial w}{\partial\xi}-n^2\frac{w}{\xi^2}-\frac{1}{\beta_2}\frac{\partial^2 w}{\partial\tau^2}=Q(\xi,\tau), \tag{30}$$

$$\left[\beta_2\frac{\partial w}{\partial\xi}+\beta_5\frac{w}{\xi}\right]_{\xi=s\text{ and }\xi=1}=0, \tag{31}$$

where

$$Q(\xi,\tau)=F(\xi,\tau)+I_1(\xi)d(\tau)+I_2(\xi)\ddot{d}(\tau)+L_1(\xi)b(\tau)+L_2(\xi)\ddot{b}(\tau) \tag{32}$$

and

$$F(\xi,\tau)=-A_0 f(\xi,s)p_1(\tau)-\frac{B_0}{s^N}f(\xi,1)p_2(\tau)+\frac{A_0}{\beta_2}(\xi-s)^m\ddot{p}_1(\tau)+\frac{B_0}{\beta_2 s^N}(\xi-1)^m\ddot{p}_2(\tau),$$

$$\begin{aligned}
I_1(\xi)&=-\frac{\alpha_3}{\beta_2}\frac{1}{\xi^{N+2}}-\beta_3\left[A_0 f(\xi,s)+\frac{B_0}{s^{N+1}}f(\xi,1)\right],\\
I_2(\xi)&=\frac{\beta_3}{\beta_2}\left[A_0(\xi-s)^m+\frac{B_0}{s^{N+1}}(\xi-1)^m\right],\\
L_1(\xi)&=-\frac{\alpha_4}{\beta_2}\frac{1}{\xi^{N+2}}-\beta_4\left[A_0 f(\xi,s)+\frac{B_0}{s^{N+1}}f(\xi,1)\right],\\
L_2(\xi)&=\frac{\beta_4}{\beta_2}\left[A_0(\xi-s)^m+\frac{B_0}{s^{N+1}}(\xi-1)^m\right],\\
f(\xi,\eta)&=m(m-1)(\xi-\eta)^{m-2}+m\frac{(\xi-\eta)^{m-1}}{\xi}-n^2\frac{(\xi-\eta)^m}{\xi^2}.
\end{aligned} \tag{33}$$

Based on Eqs. (23,28), initial condition (16) can be transformed into

$$\begin{aligned}
w(\xi,0)&=h_1(\xi)+h_2(\xi)d(0)+h_3(\xi)b(0),\\
\dot{w}(\xi,0)&=h_4(\xi)+h_2(\xi)\dot{d}(0)+h_3(\xi)\dot{b}(0),
\end{aligned} \tag{34}$$

where

$$h_1(\xi)=\xi^N g_1(\xi)-A_0(\xi-s)^m p_1(0)-\frac{B_0}{s^N}(\xi-1)^m p_2(0),$$

$$h_2(\xi)=-\beta_3\left[A_0(\xi-s)^m+\frac{B_0}{s^{N+1}}(\xi-1)^m\right],$$

$$h_3(\xi) = -\beta_4\left[A_0(\xi - s)^m + \frac{B_0}{s^{N+1}}(\xi - 1)^m\right],$$

$$h_4(\xi) = \xi^N g_2(\xi) - A_0(\xi - s)^m \dot{p}_1(0) - \frac{B_0}{s^N}(\xi - 1)^m \dot{p}_2(0). \tag{35}$$

Thus, some governing equations are transformed into Eqs. (30,31,34).

4.2. Solution

By virtue of the method of variable separation, the solution of Eq. (30) can be assumed in the following form:

$$w(\xi, \tau) = \sum_{i=1}^{\infty} R_i(\xi) T_i(\tau), \tag{36}$$

where $T_i(\tau)$ is undetermined functions of τ and

$$R_i(\xi) = Y(n, k_i, 1) J_n(k_i\xi) - J(n, k_i, 1) Y_n(k_i\xi), \tag{37}$$

here $J_n(k_i\xi)$ and $Y_n(k_i\xi)$ are, respectively, the first and second kind of Bessel functions with order of n. k_i is an incremental series of positive roots of the following equation:

$$J(n, k_i, s) Y(n, k_i, 1) - J(n, k_i, 1) Y(n, k_i, s) = 0, \tag{38}$$

where

$$J(n, k_i, \xi) = \frac{k_i}{2n}[(\beta_5 + n\beta_2) J_{n-1}(k_i\xi) + (\beta_5 - n\beta_2) J_{n+1}(k_i\xi)],$$

$$Y(n, k_i, \xi) = \frac{k_i}{2n}[(\beta_5 + n\beta_2) Y_{n-1}(k_i\xi) + (\beta_5 - n\beta_2) Y_{n+1}(k_i\xi)], \tag{39}$$

and n, β_2, β_5 are constants defined in Eqs. (19,25,27).

Thus, we can find that $R_i(\xi)$ satisfies

$$\left[\beta_2 \frac{\mathrm{d}R}{\mathrm{d}\xi} + \beta_5 \frac{R}{\xi}\right]_{\xi=s \text{ and } \xi=1} = 0. \tag{40}$$

So $w(\xi, \tau)$ expressed in Eq. (36) satisfies the homogeneous boundary conditions (31).

In addition, by virtue of the orthogonal property of Bessel functions, it is easy to find that $R_i(\xi)$ also satisfies

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi \frac{\mathrm{d}R_i}{\mathrm{d}\xi}\right) + \left(k_i^2 - \frac{n^2}{\xi^2}\right) R_i = 0, \tag{41}$$

$$\int_s^1 \xi R_i R_j \,\mathrm{d}\xi = \delta_{ij} N_i, \tag{42}$$

where δ_{ij} is the Kronecker delta, and

$$N_i = \frac{1}{2k_i^2}\left[\xi^2\left(\frac{\mathrm{d}R_i}{\mathrm{d}\xi}\right)^2 + (k_i^2\xi^2 - n^2)R_i^2(\xi)\right]_{\xi=s}^{\xi=1}. \tag{43}$$

Substituting solution (36) into Eq. (30) and using Eq. (41) yield

$$\sum_{i=1}^{\infty} R_i\left[k_i^2 T_i(\tau) + \frac{1}{\beta_2}\ddot{T}_i(\tau)\right] = -Q(\xi,\tau). \tag{44}$$

Then, by virtue of the orthogonal property (42), the equation to determine $T_i(\tau)$ can be derived from Eq. (44) as follows:

$$\ddot{T}_i(\tau) + \omega_i^2 T_i(\tau) = q_i(\tau), \tag{45}$$

where

$$\omega_i^2 = \beta_2 k_i^2, \qquad q_i(\tau) = -\frac{\beta_2}{N_i}\int_s^1 Q(\xi,\tau)R_i\xi\,\mathrm{d}\xi. \tag{46}$$

The solution of Eq. (45) is

$$T_i(\tau) = T_i(0)\cos\omega_i\tau + \frac{1}{\omega_i}\dot{T}_i(0)\sin\omega_i\tau + \frac{1}{\omega_i}\int_0^\tau q_i(\eta)\sin\omega_i(\tau-\eta)\,\mathrm{d}\eta, \tag{47}$$

where $T_i(0)$ and $\dot{T}_i(0)$ can be determined by substituting solution (36) into initial conditions (34) and with the aid of orthogonal property (42) as

$$T_i(0) = A_{1i} + A_{2i}d(0) + A_{3i}b(0),$$

$$\dot{T}_i(0) = A_{4i} + A_{2i}\dot{d}(0) + A_{3i}\dot{b}(0) \tag{48}$$

and

$$A_{ji} = \frac{1}{N_i}\int_s^1 h_j(\xi)R_i\xi\,\mathrm{d}\xi \quad (j = 1,2,3,4; i = 1,2,3,\ldots). \tag{49}$$

It can be seen that the solution is still not obtained for the undetermined functions $d(\tau)$ and $b(\tau)$ in expression (32) for $Q(\xi,\tau)$, which is used in Eqs. (46,47) to determine $q_i(\tau)$ and $T_i(\tau)$. For this object, the electromagnetic boundary conditions (15b,c) will be used as follows.

At first, by virtue of Eqs. (23,28,36), the displacement $u(\xi,\tau)$ can be expressed as

$$u(\xi,\tau) = \frac{1}{\xi^N}\left[\sum_{i=1}^{\infty} R_i(\xi)T_i(\tau) + A_0(\xi-s)^m P_1(\tau) + B_0(\xi-1)^m P_2(\tau)\right]. \tag{50}$$

Substituting Eq. (20) into Eq. (17) and integrating the result using electromagnetic boundary conditions (15b,c) yield

$$\varphi(\xi,\tau) = \alpha_3 \int_s^{\xi} \frac{1}{\eta}\, u(\eta,\tau)\,\mathrm{d}\eta + \beta_3[u(\xi,\tau) - u(s,\tau)] + \varphi_2(\tau) + \begin{cases} -\delta \ln\left(\dfrac{\xi}{s}\right)[d(\tau) - gb(\tau)], & N = 0, \\ \dfrac{\delta}{2N}\left(\dfrac{1}{\xi^{2N}} - \dfrac{1}{s^{2N}}\right)[d(\tau) - gb(\tau)], & N \neq 0, \end{cases} \tag{51a}$$

$$\psi(\xi,\tau) = \alpha_4 \int_s^{\xi} \frac{1}{\eta}\, u(\eta,\tau)\,\mathrm{d}\eta + \beta_4[u(\xi,\tau) - u(s,\tau)] + \psi_2(\tau) + \begin{cases} -\delta \ln\left(\dfrac{\xi}{s}\right)[b(\tau) - gd(\tau)], & N = 0, \\ \dfrac{\delta}{2N}\left(\dfrac{1}{\xi^{2N}} - \dfrac{1}{s^{2N}}\right)[b(\tau) - gd(\tau)], & N \neq 0. \end{cases} \tag{51b}$$

Substitution of Eq. (50) into Eq. (51) and subsequently into the electromagnetic boundary conditions (15b,c) yields

$$\begin{aligned} \varphi_1(\tau) = {} & \alpha_3 \left[\sum_{i=1}^{\infty} T_i(\tau) \int_s^1 \frac{1}{\eta^{N+1}}\, R_i(\eta)\,\mathrm{d}\eta + A_0 P_1(\tau) \int_s^1 \frac{1}{\eta^{N+1}}\,(\eta - s)^m\,\mathrm{d}\eta \right. \\ & \left. + B_0 P_2(\tau) \int_s^1 \frac{1}{\eta^{N+1}}\,(\eta - 1)^m\,\mathrm{d}\eta \right] \\ & + \beta_3 \left[\sum_{i=1}^{\infty} R_i(1) T_i(\tau) + A_0 (1-s)^m P_1(\tau) \right] + \varphi_2(\tau) - \beta_3 u(s,\tau) \\ & + \begin{cases} \delta \ln s [d(\tau) - gb(\tau)], & N = 0, \\ \dfrac{\delta}{2N}\left(1 - \dfrac{1}{s^{2N}}\right)[d(\tau) - gb(\tau)], & N \neq 0, \end{cases} \end{aligned} \tag{52a}$$

$$\begin{aligned} \psi_1(\tau) = {} & \alpha_4 \left[\sum_{i=1}^{\infty} T_i(\tau) \int_s^1 \frac{1}{\eta^{N+1}}\, R_i(\eta)\,\mathrm{d}\eta + A_0 P_1(\tau) \int_s^1 \frac{1}{\eta^{N+1}}\,(\eta - s)^m\,\mathrm{d}\eta \right. \\ & \left. + B_0 P_2(\tau) \int_s^1 \frac{1}{\eta^{N+1}}\,(\eta - 1)^m\,\mathrm{d}\eta \right] \\ & + \beta_4 \left[\sum_{i=1}^{\infty} R_i(1) T_i(\tau) + A_0 (1-s)^m P_1(\tau) \right] + \psi_2(\tau) - \beta_4 u(s,\tau) \end{aligned}$$

$$+\begin{cases} \delta \ln s[b(\tau) - gd(\tau)], & N = 0, \\ \dfrac{\delta}{2N}\left(1 - \dfrac{1}{s^{2N}}\right)[b(\tau) - gd(\tau)], & N \neq 0. \end{cases} \tag{52b}$$

Moving the positions of $\varphi_2(\tau)$ and $\psi_2(\tau)$ from right hand side to left hand side of Eq. (52) yields

$$\varphi_1(\tau) - \varphi_2(\tau) = \sum_{i=1}^{\infty} A_i T_i(\tau) + G_1 P_1(\tau) + G_2 P_2(\tau) + \begin{cases} \delta \ln s[d(\tau) - gb(\tau)], & N = 0, \\ \dfrac{\delta}{2N}\left(1 - \dfrac{1}{s^{2N}}\right)[d(\tau) - gb(\tau)], & N \neq 0, \end{cases} \tag{53a}$$

$$\psi_1(\tau) - \psi_2(\tau) = \sum_{i=1}^{\infty} B_i T_i(\tau) + H_1 P_1(\tau) + H_2 P_2(\tau) + \begin{cases} \delta \ln s[b(\tau) - gd(\tau)], & N = 0, \\ \dfrac{\delta}{2N}\left(1 - \dfrac{1}{s^{2N}}\right)[b(\tau) - gd(\tau)], & N \neq 0, \end{cases} \tag{53b}$$

where G_1, G_2, A_i, H_1, H_2, B_i are constants defined in (A.1,A.2).

Substitution of Eq. (32) into the second equation of Eq. (46) yields

$$q_i(\tau) = X_i(\tau) + Y_{1i} d(\tau) + Y_{2i} \ddot{d}(\tau) + Z_{1i} b(\tau) + Z_{2i} \ddot{b}(\tau), \tag{54}$$

where Y_{1i}, Y_{2i}, Z_{1i}, Z_{2i} are constants defined in Eq. (A.3), and

$$X_i(\tau) = -\frac{\beta_2}{N_i} \int_s^1 F(\xi, \tau) R_i \xi \, \mathrm{d}\xi. \tag{55}$$

Substitution of Eqs. (48,54) into Eq. (47) and the result of $T_i(\tau)$ and $P_1(\tau)$, $P_2(\tau)$ in Eq. (27) into Eq. (53), using the integration-by-parts formula, yields

$$\varphi_1(\tau) - \varphi_2(\tau) = U_1(\tau) + V_{d1} d(\tau) + V_{b1} b(\tau) + \sum_{i=1}^{\infty} W_{d1i} \int_0^{\tau} d(\eta) \sin \omega_i(\tau - \eta) \, \mathrm{d}\eta + \sum_{i=1}^{\infty} W_{b1i} \int_0^{\tau} b(\eta) \sin \omega_i(\tau - \eta) \, \mathrm{d}\eta, \tag{56a}$$

$$\psi_1(\tau) - \psi_2(\tau) = U_2(\tau) + V_{d2} d(\tau) + V_{b2} b(\tau) + \sum_{i=1}^{\infty} W_{d2i} \int_0^{\tau} d(\eta) \sin \omega_i(\tau - \eta) \, \mathrm{d}\eta + \sum_{i=1}^{\infty} W_{b2i} \int_0^{\tau} b(\eta) \sin \omega_i(\tau - \eta) \, \mathrm{d}\eta, \tag{56b}$$

where V_{d1}, V_{b1}, W_{d1i}, W_{b1i}, V_{d2}, V_{b2}, W_{d2i}, W_{b2i} are constants defined in Eqs. (A.4) and (A.5), and

$$\begin{aligned} U_1(\tau) = G_1 p_1(\tau) + \frac{G_2}{s^N} p_2(\tau) + \sum_{i=1}^{\infty} A_i \Big\{ & [A_{1i} + (A_{2i} - Y_{2i}) d(0) + (A_{3i} - Z_{2i}) b(0)] \cos \omega_i \tau \\ & + \frac{1}{\omega_i} [A_{4i} + (A_{2i} - Y_{2i}) \dot{d}(0) + (A_{3i} - Z_{2i}) \dot{b}(0)] \sin \omega_i \tau \\ & + \frac{1}{\omega_i} \int_0^\tau X_i(\eta) \sin \omega_i (\tau - \eta) \, d\eta \Big\}, \end{aligned} \tag{57a}$$

$$\begin{aligned} U_2(\tau) = H_1 p_1(\tau) + \frac{H_2}{s^N} p_2(\tau) + \sum_{i=1}^{\infty} B_i \Big\{ & [A_{1i} + (A_{2i} - Y_{2i}) d(0) + (A_{3i} - Z_{2i}) b(0)] \cos \omega_i \tau \\ & + \frac{1}{\omega_i} [A_{4i} + (A_{2i} - Y_{2i}) \dot{d}(0) + (A_{3i} - Z_{2i}) \dot{b}(0)] \sin \omega_i \tau \\ & + \frac{1}{\omega_i} \int_0^\tau X_i(\eta) \sin \omega_i (\tau - \eta) d\eta \Big\}. \end{aligned} \tag{57b}$$

Let $\tau = 0$, one can obtain from Eq. (56)

$$a_{11} d(0) + a_{12} b(0) = b_1, \qquad a_{21} d(0) + a_{22} b(0) = b_2, \tag{58}$$

where a_{ij}, b_i $(i,j = 1,2)$ are constants defined in (A.6).

From Eq. (56), one can derive

$$\begin{aligned} \dot{\varphi}_1(\tau) - \dot{\varphi}_2(\tau) = \dot{U}_1(\tau) + V_{d1} \dot{d}(\tau) + V_{b1} \dot{b}(\tau) + \sum_{i=1}^{\infty} \omega_i W_{d1i} \int_0^\tau d(\eta) \cos \omega_i (\tau - \eta) \, d\eta \\ + \sum_{i=1}^{\infty} \omega_i W_{b1i} \int_0^\tau b(\eta) \cos \omega_i (\tau - \eta) \, d\eta, \end{aligned} \tag{59a}$$

$$\begin{aligned} \dot{\psi}_1(\tau) - \dot{\psi}_2(\tau) = \dot{U}_2(\tau) + V_{d2} \dot{d}(\tau) + V_{b2} \dot{b}(\tau) + \sum_{i=1}^{\infty} \omega_i W_{d2i} \int_0^\tau d(\eta) \cos \omega_i (\tau - \eta) \, d\eta \\ + \sum_{i=1}^{\infty} \omega_i W_{b2i} \int_0^\tau b(\eta) \cos \omega_i (\tau - \eta) \, d\eta. \end{aligned} \tag{59b}$$

Similarly, substituting $\tau = 0$ into Eq. (59) yields

$$a_{11} \dot{d}(0) + a_{12} \dot{b}(0) = d_1, \qquad a_{21} \dot{d}(0) + a_{22} \dot{b}(0) = d_2, \tag{60}$$

where a_{ij}, d_i $(i,j = 1,2)$ are constants defined in Eq. (A.6). Thus, $d(0)$, $b(0)$, $\dot{d}(0)$ and $\dot{b}(0)$ can be obtained from Eqs. (58,60).

Substituting the obtained $d(0)$, $\dot{d}(0)$, $b(0)$ and $\dot{b}(0)$ into Eqs. (57), one can see that $U_1(\tau)$ and $U_2(\tau)$ are two determined functions, so that Eq. (56) becomes two Volterra integral equations of the second kind [41]. It is known that they have unique solution at all times. For some cases, the analytical solution can be obtained. While for general cases, numerical methods are needed. In this paper, the recursion formula are constructed by making use of the linear interpolation function. In practice, the numerical result can be obtained efficiently by the present method. In order to show the method of solving the integral equation, the time interval $[0, \tau]$ is divided into n subintervals. The discrete time points are $\tau_0 = 0, \tau_1, \tau_2, \ldots, \tau_n$. Then the interpolation function at the time interval $[\tau_{j-1}, \tau_j]$ is

$$d(\tau) = \zeta_j(\tau)d(\tau_{j-1}) + \eta_j(\tau)d(\tau_j), \quad b(\tau) = \zeta_j(\tau)b(\tau_{j-1}) + \eta_j(\tau)b(\tau_j) \ \ (j = 1, 2, \ldots, n), \tag{61}$$

where

$$\zeta_j(\tau) = \frac{\tau - \tau_j}{\tau_{j-1} - \tau_j}, \quad \eta_j(\tau) = \frac{\tau - \tau_{j-1}}{\tau_j - \tau_{j-1}} \ \ (j = 1, 2, \ldots, n). \tag{62}$$

Substituting Eq. (61) into Eq. (56) yields

$$\begin{aligned}\varphi_1(\tau_j) - \varphi_2(\tau_j) = {} & U_1(\tau_j) + V_{d1}d(\tau_j) + V_{b1}b(\tau_j) + \sum_{i=1}^{\infty} W_{d1i} \sum_{k=1}^{j} [K_{ijk}d(\tau_{k-1}) + M_{ijk}d(\tau_k)] \\ & + \sum_{i=1}^{\infty} W_{b1i} \sum_{k=1}^{j} [K_{ijk}b(\tau_{k-1}) + M_{ijk}b(\tau_k)],\end{aligned} \tag{63a}$$

$$\begin{aligned}\psi_1(\tau_j) - \psi_2(\tau_j) = {} & U_2(\tau_j) + V_{d2}d(\tau_j) + V_{b2}b(\tau_j) + \sum_{i=1}^{\infty} W_{d2i} \sum_{k=1}^{j} [K_{ijk}d(\tau_{k-1}) + M_{ijk}d(\tau_k)] \\ & + \sum_{i=1}^{\infty} W_{b2i} \sum_{k=1}^{j} [K_{ijk}b(\tau_{k-1}) + M_{ijk}b(\tau_k)],\end{aligned} \tag{63b}$$

where

$$\begin{aligned} K_{ijk} &= \textstyle\int_{\tau_{k-1}}^{\tau_k} \zeta_k(p) \sin \omega_i(\tau_j - p)\,\mathrm{d}p \\ M_{ijk} &= \textstyle\int_{\tau_{k-1}}^{\tau_k} \eta_k(p) \sin \omega_i(\tau_j - p)\,\mathrm{d}p \end{aligned} \quad (k = 1, 2, \ldots, j, \ j = 1, 2, \ldots, n). \tag{64}$$

Then one can derive the following formula from Eq. (63):

$$e_{11}d(\tau_j) + e_{12}b(\tau_j) = f_1, \quad e_{21}d(\tau_j) + e_{22}b(\tau_j) = f_2 \quad (j = 1, 2, \ldots, n), \tag{65}$$

where

$$e_{11} = V_{d1} + \sum_{i=1}^{\infty} W_{d1i}M_{ijj}, \qquad e_{12} = V_{b1} + \sum_{i=1}^{\infty} W_{b1i}M_{ijj},$$

$$e_{21} = V_{d2} + \sum_{i=1}^{\infty} W_{d2i}M_{ijj}, \qquad e_{22} = V_{b2} + \sum_{i=1}^{\infty} W_{b2i}M_{ijj},$$

$$\begin{aligned} f_1 = {} & \varphi_1(\tau_j) - \varphi_2(\tau_j) - U_1(\tau_j) - \sum_{i=1}^{\infty} W_{d1i}\left(K_{ijj}d(\tau_{j-1}) + \sum_{k=1}^{j-1} [K_{ijk}d(\tau_{k-1}) + M_{ijk}d(\tau_k)]\right) \\ & - \sum_{i=1}^{\infty} W_{b1i}\left(K_{ijj}b(\tau_{j-1}) + \sum_{k=1}^{j-1} [K_{ijk}b(\tau_{k-1}) + M_{ijk}b(\tau_k)]\right),\end{aligned}$$

$$f_2 = \psi_1(\tau_j) - \psi_2(\tau_j) - U_2(\tau_j) - \sum_{i=1}^{\infty} W_{d2i}\left(K_{ijj}d(\tau_{j-1}) + \sum_{k=1}^{j-1}[K_{ijk}d(\tau_{k-1}) + M_{ijk}d(\tau_k)]\right)$$
$$- \sum_{i=1}^{\infty} W_{b2i}\left(K_{ijj}b(\tau_{j-1}) + \sum_{k=1}^{j-1}[K_{ijk}b(\tau_{k-1}) + M_{ijk}b(\tau_k)]\right). \tag{66}$$

Thus, once $d(0)$ and $b(0)$ are obtained by Eq. (58), one can obtain $d(\tau_j)$ and $b(\tau_j)$ ($j = 1, 2, \ldots, n$) step by step by solving Eqs. (65,66). Thus $d(\tau)$ and $b(\tau)$ are then determined. Based on these, the complete transient responses including $u(\xi,\tau)$, $\phi(\xi,\tau)$, $\psi(\xi,\tau)$, $D(\xi,\tau)$, $B(\xi,\tau)$, $\sigma_r(\xi,\tau)$, $\sigma_\phi(\xi,\tau)$ and $\sigma_z(\xi,\tau)$ can be finally determined.

It is noted that only one side of inner and outer surfaces can be prescribed by electric displacement and magnetic induction, the other side must be prescribed by electric potential and magnetic potential. For example, the inner surface ($\xi = s$) are prescribed by

$$D(s,\tau) = D_2(\tau), \qquad B(s,\tau) = B_2(\tau). \tag{67}$$

Substituting Eq. (20) into Eq. (67), one can obtain

$$d(\tau) = sD_2(\tau), \qquad b(\tau) = sB_2(\tau). \tag{68}$$

Substitution of Eq. (68) into Eq. (20) yields

$$D(\xi,\tau) = \frac{s}{\xi} D_2(\tau), \qquad B(\xi,\tau) = \frac{s}{\xi} D_2(\tau). \tag{69}$$

Thus the electric displacement and magnetic induction on the outer surface have been determined by Eq. (69) as

$$D(1,\tau) = sD_2(\tau), \qquad B(1,\tau) = sD_2(\tau). \tag{70}$$

So the outer side must be prescribed by electric potential and magnetic potential as Eq. (15).

In this case, the displacement solution had been obtained until Eq. (49), and the procedure of solving the Volterra integral equation to determine $d(\tau)$ and $b(\tau)$, which had been obtained by Eq. (68), is avoided. In addition, Eq. (51), which is used to determine the solutions of electric potential and magnetic potential, contains $\varphi_2(\tau)$ and $\psi_2(\tau)$, so if the boundary conditions of electric potential $\varphi_1(\tau)$ and magnetic potential $\psi_1(\tau)$ are prescribed in outer surface, Eq. (53) should be used to obtain $\varphi_2(\tau)$ and $\psi_2(\tau)$.

All above solutions can be degenerated into corresponding non-homogeneous piezoelectric, piezomagnetic and purely elastic solutions as its special cases. For purely elastic problem (with the coupling physical coefficients e_{kij}, d_{kij} and g_{ij} in Eq. (2) being set to zero), the solution had been obtained until Eq. (49), because the undetermined functions $d(\tau)$ and $b(\tau)$ in expression (32) of $Q(\xi,\tau)$ vanish. For piezoelectric or piezomagnetic problem (with the coupling of physical constants d_{kij}, g_{ij} and e_{kij}, g_{ij} in Eq. (2) being set to zero, respectively), one of $d(\tau)$ and $b(\tau)$ vanishes and the solutions can be obtained similarly by solving one Volterra integral equation instead of two. These can be found in Ref. [38] for homogeneous piezoelectric hollow cylinder and in Ref. [39] for non-homogeneous hollow cylinder. In addition, the transient response of homogeneous magneto-electro-elastic hollow cylinder [40] can be easily obtained by let non-homogeneous parameter $N = 0$. All these conclusions had been proved out by numerical results. For example, when $N = 0$, the transient responses at $\xi = 0.75$ (the middle surface) in Figs. 2 and 5 of next section agree with corresponding transient responses at $\xi = 0.75$ in Figs. 3 and 6 of Ref. [40].

5. Numerical results and discussions

Having derived the exact solutions, some numerical results will be presented for the transient responses of a non-homogeneous magneto-electro-elastic hollow cylinder under zero initial conditions $g_1(\xi) = g_2(\xi) = 0$ and subjected to following loads.

(a) *Sudden constant pressure load*:

$$p_1 = 0, \quad p_2(\tau) = -\sigma_0 H(\tau), \quad \varphi_1(\tau) = \varphi_2(\tau) = 0, \quad \psi_1(\tau) = \psi_2(\tau) = 0. \tag{71}$$

(b) *Dynamic combined loads*:

$$\begin{aligned} p_1(\tau) &= \sigma_0 \sin(\tau), & p_2(\tau) &= \sigma_0 \sin(2\tau), \\ \varphi_1(\tau) &= \varphi_0 \cos(3\tau), & \varphi_2(\tau) &= \varphi_0 \sin(4\tau), \\ \psi_1(\tau) &= \psi_0 \sin(5\tau), & \psi_2(\tau) &= \psi_0 \cos(6\tau), \end{aligned} \tag{72}$$

where σ_0, φ_0, ψ_0 are prescribed constant stress, constant electric potential and constant magnetic potential and $\sigma_0 = 1.0$, $\varphi_0 = 1.0$, $\psi_0 = 1.0$ are adopted. In addition, $H(\tau)$ is the Heaviside function.

In this course, $s = 0.5$, $m = 2$, $\tau_n = \tau_{300} = 12$ and the first 40 terms in Eq. (36) are taken. The physical constants of the magneto-electro-elastic material are listed in Table 1.

5.1. Transient responses in non-homogeneous magneto-electro-elastic hollow cylinders under sudden constant pressure load

In cases of $N = -1$, 0 and 1, the transient responses of all dimensional components at $\xi = 0.75$ (the middle surface) in non-homogeneous magneto-electro-elastic hollow cylinder subjected to loads (71) are compared in Fig. 2. In addition, the distributions of all dimensional components along ξ at two determined time $\tau = 4.0$ and 8.0 are plotted in Fig. 3.

Table 1
Physical constants of magneto-electro-elastic hollow cylinder [24]

c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	g_{11}
2.86×10^{11}	1.73×10^{11}	1.70×10^{11}	2.695×10^{11}	4.53×10^{10}	5.0×10^{-12}
e_{15}	e_{31}	e_{33}	ε_{11}	ε_{33}	g_{33}
11.6	−4.4	18.6	8.0×10^{-11}	9.3×10^{-11}	3.0×10^{-12}
d_{15}	d_{31}	d_{33}	μ_{11}	μ_{33}	
550	580.3	699.7	-5.90×10^{-4}	1.57×10^{-4}	

Units: elastic constants: $\mathrm{N\,m^{-2}}$; piezoelectric constants: $\mathrm{C\,m^{-2}}$; piezomagnetic constants: $\mathrm{N\,A^{-1}\,m^{-1}}$; dielectric constants: $\mathrm{C^2\,N^{-1}\,m^{-2}}$; electromagnetic constants: $\mathrm{N\,s\,V^{-1}\,C^{-1}}$; magnetic constants: $\mathrm{N\,s^2\,C^{-2}}$.

From Figs. 2 and 3, some information can be obtained as follows:

(1) The peak values of all components increase quickly with the increase of N (Fig. 2).
(2) The peak value of σ_ϕ is larger than those of σ_r and σ_z, and becomes the primary stress in any case of N (Figs. 2b–d).

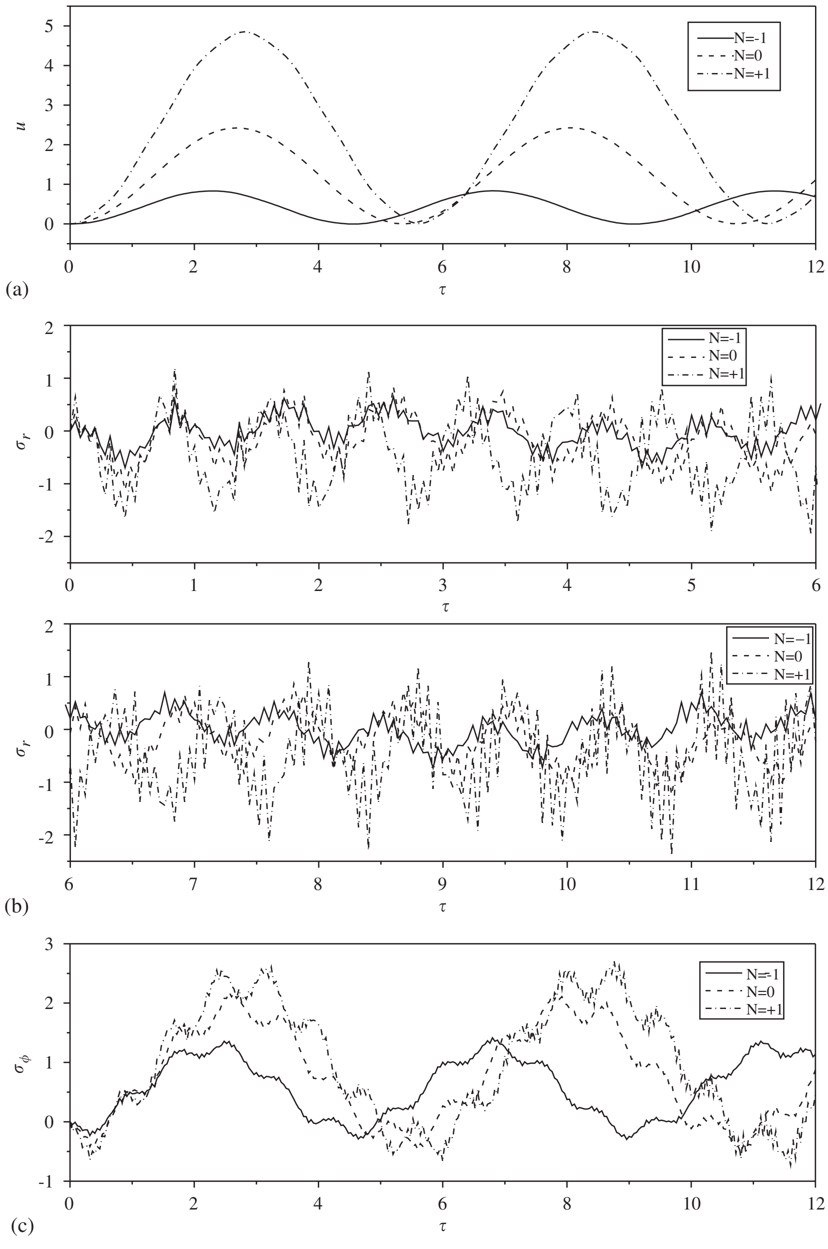

Fig. 2. Transient responses of all non-dimensional components at $\xi = 0.75$ (the middle surface) for sudden constant pressure.

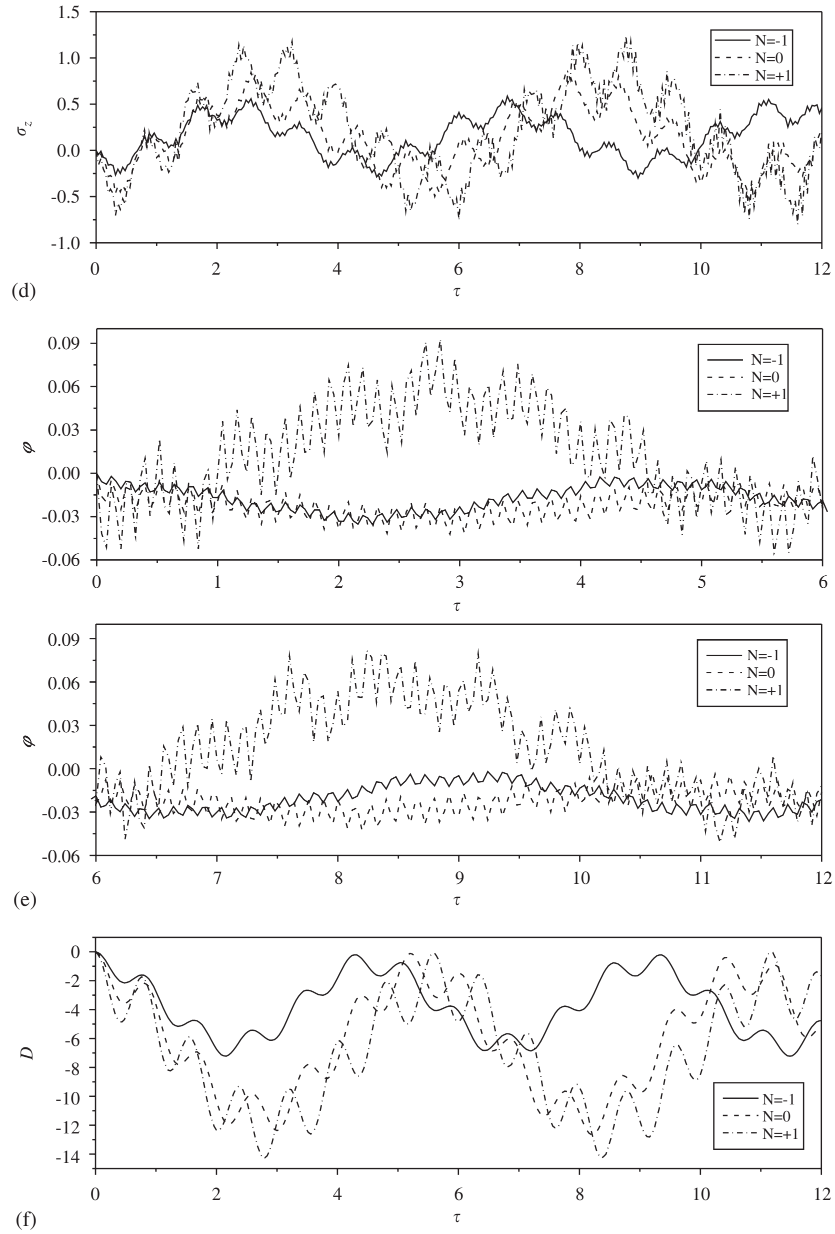

Fig. 2. (*Continued*)

(3) The transient responses of stress σ_r and electric potential φ change intensely along with τ (Figs. 2b and e), i.e. they are sensitive to loadings and can be used as a suitable feedback in smart system. So we can regard them as the smartest one of all components.

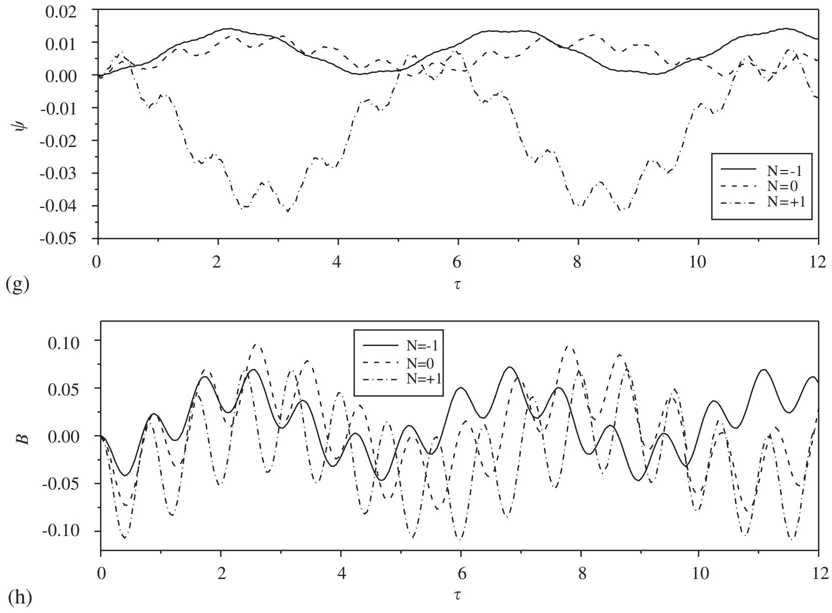

Fig. 2. (*Continued*)

(4) The distribution of stress σ_r changes dramatically along ξ and show very complicated characteristics compared to those of other components (Fig. 3b).

5.2. *Transient responses in non-homogeneous magneto-electro-elastic hollow cylinders under dynamic combined loads*

In cases of $N = -1$, 0 and 1, the parallel numerical results for dynamic combined loads (72) are plotted in Figs. 4 and 5.

From Figs. 4 and 5, some information can be obtained as follows:

(1) The relation between the peak values of all components with N shows some new characteristics. Except u, which is same as that for constant pressure and increase with the increase of N (Fig. 4a), σ_r, σ_ϕ, σ_z, D and B are contrary to those for constant pressure and decrease with the increase of N (Figs. 4b–h). In addition, the peak values of φ and ψ show an undetermined relation with N (Figs. 4e and g).
(2) Contrary to constant pressure, σ_ϕ is not still the primary stress, it has the same importance with σ_r and σ_z for their same level of peak values (Figs. 4b–d).
(3) The transient responses of stress σ_r for combined dynamic loads is still the smartest one of all components (Fig. 4b), while the electric potential φ is not in smart form (Figs. 4b and e);
(4) Similar to constant pressure, the distribution of stress σ_r along ξ still shows complicated characteristics than those of other components (Fig. 5b).

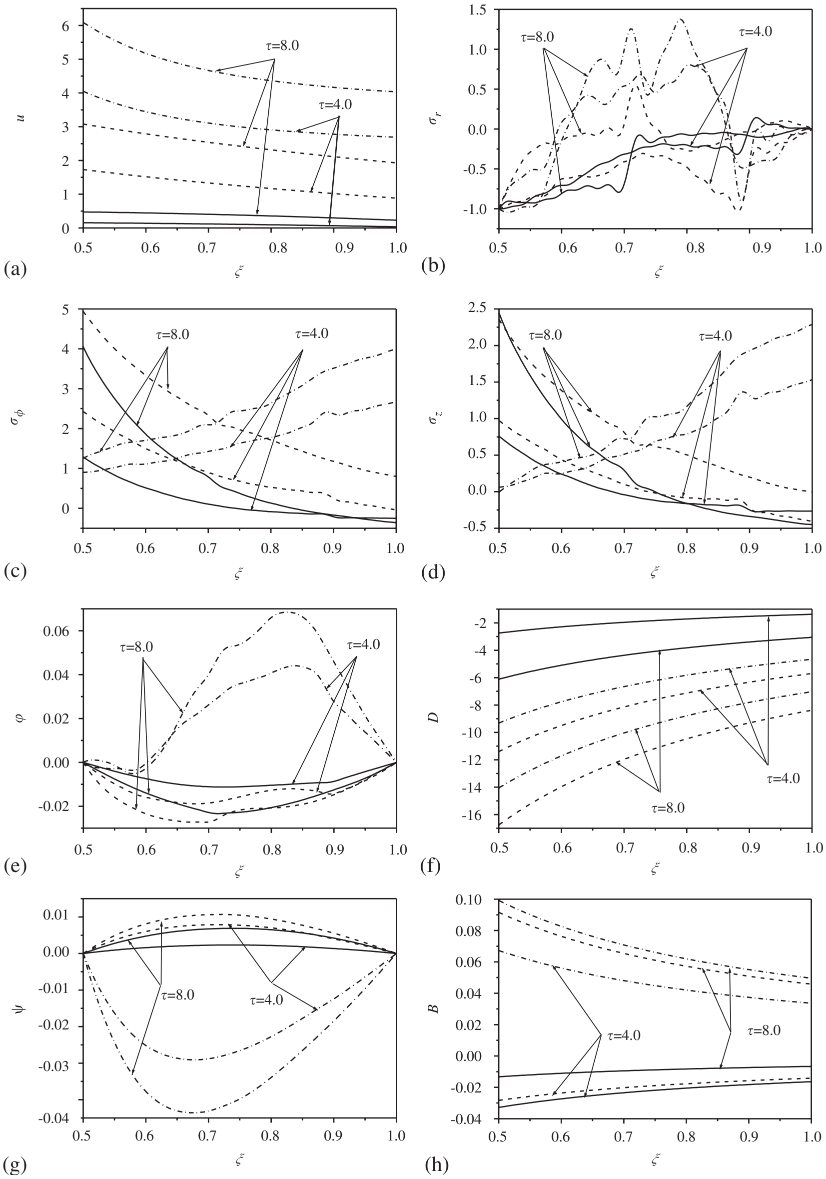

Fig. 3. Distributions of all non-dimensional components along ξ at two determined time $\tau = 4.0$ and 8.0 for sudden constant pressure. Solid line (——) for case of $N = -1$, dash line (– –) for case of $N = 0$, dash dot line (–‐ ‐ –‐) for case of $N = 1$.

In addition, the calculation also shows us some information as follows:

(1) The numerical results on the boundaries of $\xi = 0.5$ and 1 satisfy the prescribed boundary conditions and the precision level is up to 10^{-5} at least. This also can be seen from Figs. 3b,e,g and 5b,e,g.

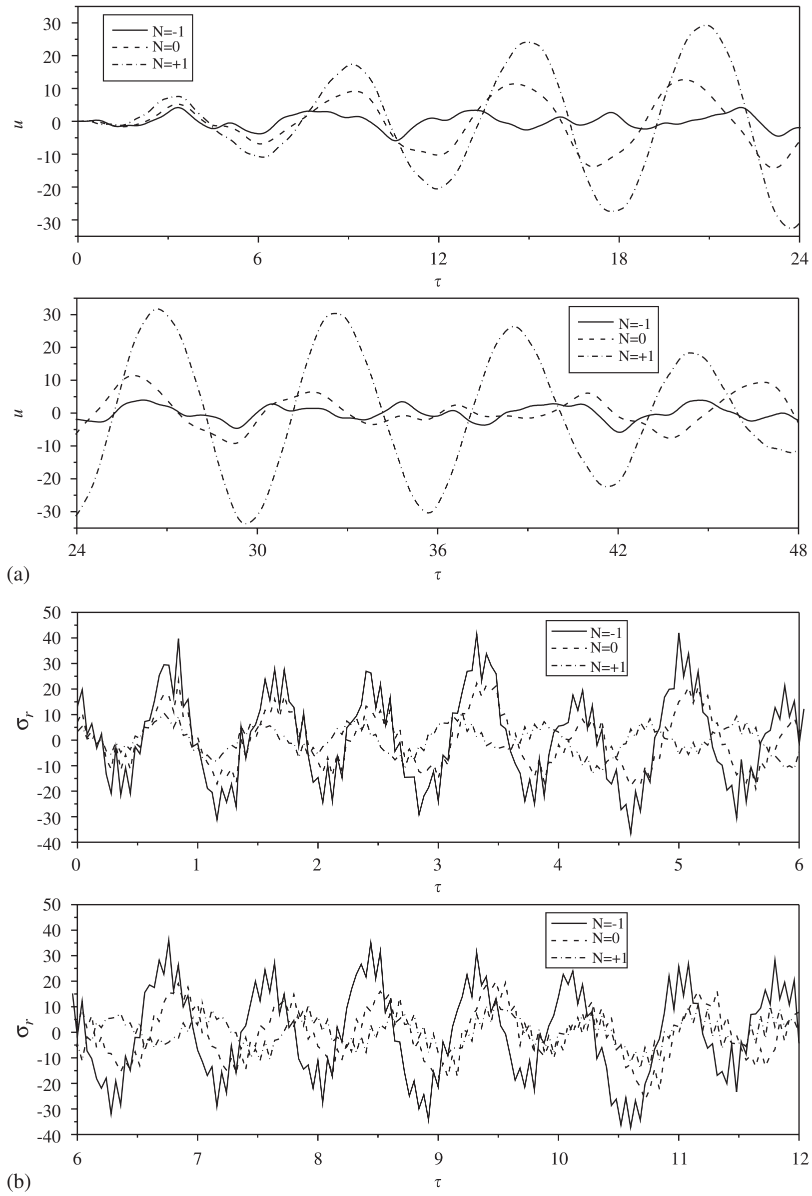

Fig. 4. Transient responses of all non-dimensional components at $\xi = 0.75$ (the middle surface) for dynamic combined loads.

(2) By using linear interpolation functions or high-order interpolation functions, the accurate results can be obtained effectively. It is noted here that the recursion formula becomes very simple when linear interpolation functions are used. Particularly, the

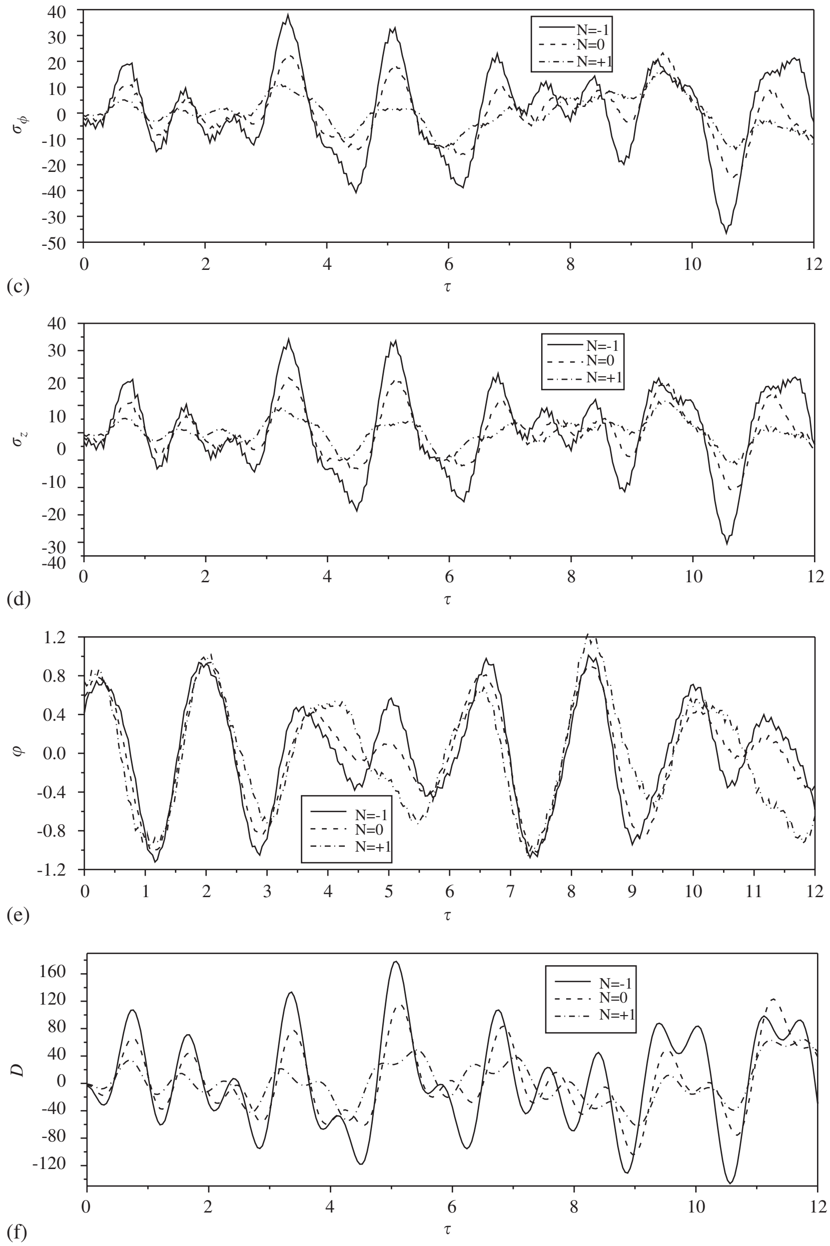

Fig. 4. (*Continued*)

simplest recursion formula will be obtained when equal time steps are used. Based on many kinds of tests, we conclude that the satisfactory numerical results can be obtained when $\Delta\tau \leqslant 0.05$.

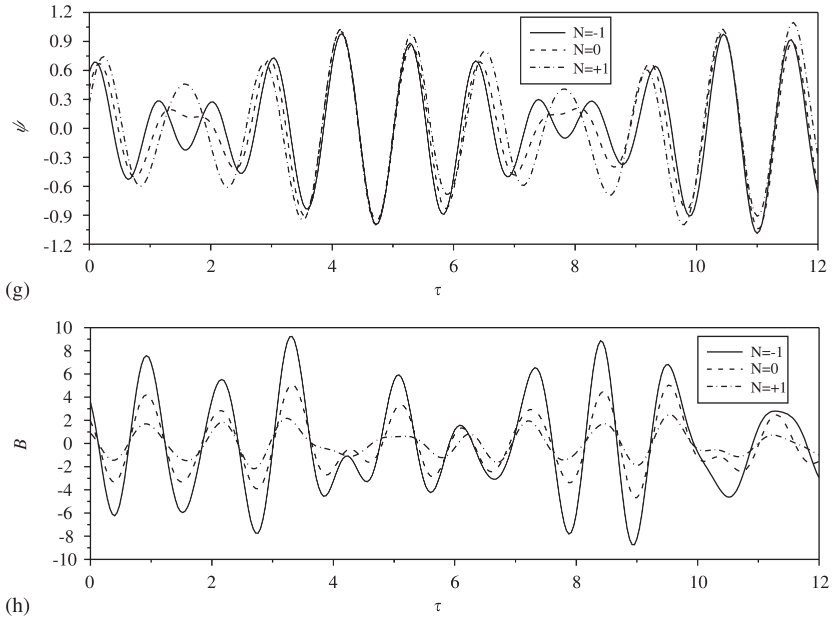

Fig. 4. (*Continued*)

(3) Calculations for the sudden constant electric potential and magnetic potential on the inner surface of non-homogeneous magneto-electro-elastic cylinder show that they have a similar form of transient responses with those for sudden constant pressure.

6. Conclusions

In this paper, we have derived an analytical solution for the transient responses of a special non-homogeneous magneto-electro-elastic hollow cylinder with arbitrary thickness subjected to arbitrary mechanical and electromagnetic loads. The present solution can be degenerated to those of corresponding purely elastic, piezomagnetic and piezoelectric problems as its special cases. They can not only provide benchmarks for numerical methods, such as the finite element and boundary element methods, but also can offer a simple and accurate tool for the prediction, identification and study of the complex dynamic characteristics of coupling mechanical and electromagnetic fields in the working magneto-electro-elastic components, such as sensors and actuators in active structures.

In three non-homogeneous cases, typical numerical examples are presented for magneto-electro-elastic hollow cylinders, which are subjected to sudden constant pressure load and dynamic combined loads, respectively. It can be concluded that the transient responses of coupled fields show complicated characteristics, including the relations between the peak values of all components with non-homogeneous parameter N, the determination of the smartest component and primary stress etc. All these should be determined

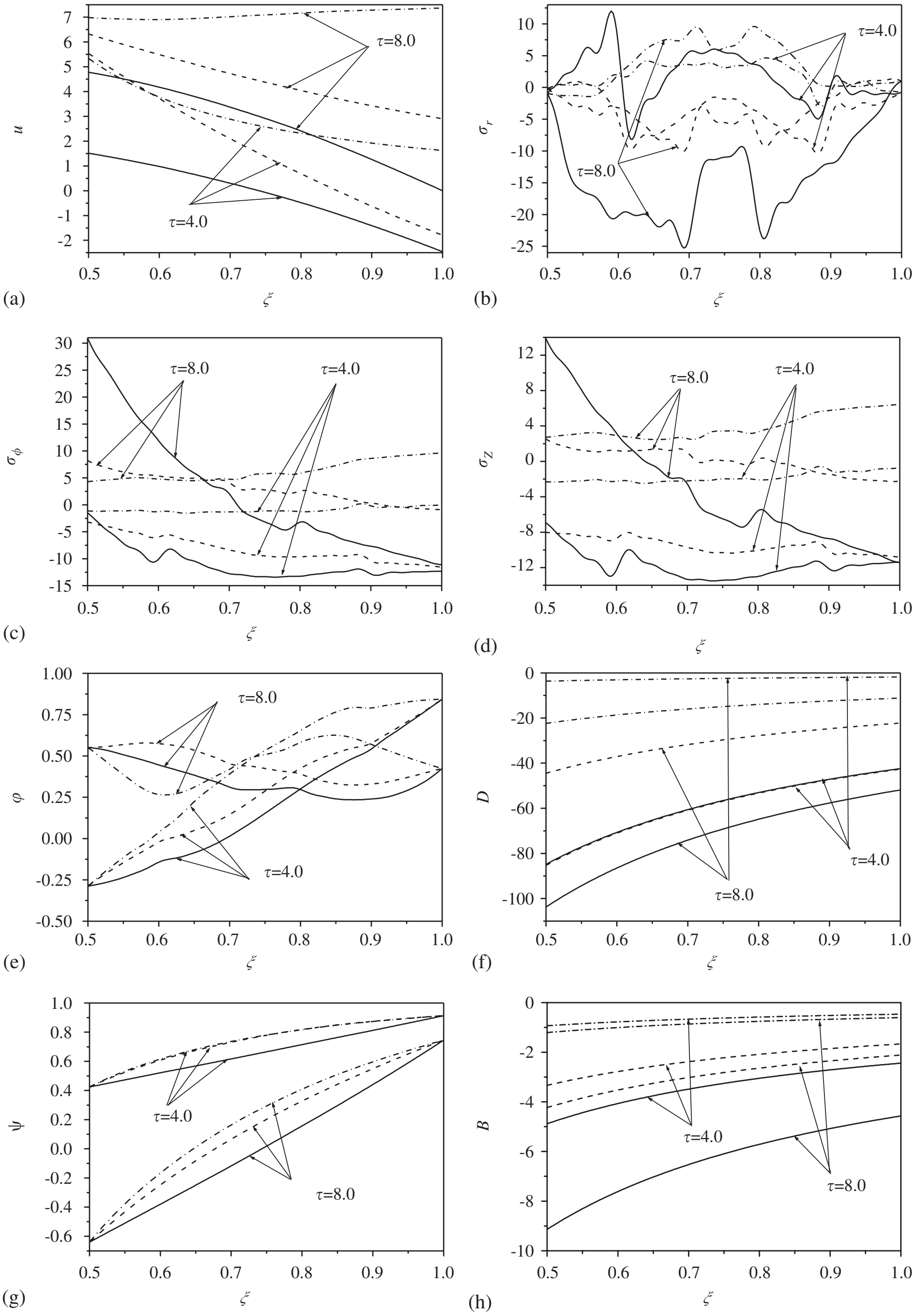

Fig. 5. Distributions of all non-dimensional components along ξ at two determined time $\tau = 4.0$ and 8.0 for dynamic combined loads. Solid line (——) for case of $N = -1$, dash line (– –) for case of $N = 0$, dash dot line (–·–·-) for case of $N = 1$.

by calculation and analysis, which are taken according to the corresponding loads and physical properties.

Acknowledgements

This work is supported by National Natural Science Foundation of China and Yu-Ying Foundation of Hunan University.

Appendix A. Definition of constants

$$G_1 = A_0\left[\alpha_3 \int_s^1 \frac{1}{\eta^{N+1}}(\eta - s)^m \,\mathrm{d}\eta + \beta_3(1-s)^m\right], \tag{A.1a}$$

$$G_2 = B_0\left[\alpha_3 \int_s^1 \frac{1}{\eta^{N+1}}(\eta - 1)^m \,\mathrm{d}\eta - \frac{\beta_3}{s^N}(s-1)^m\right], \tag{A.1b}$$

$$A_i = \alpha_3 \int_s^1 \frac{1}{\eta^{N+1}} R_i(\eta)\,\mathrm{d}\eta + \beta_3\left[R_i(1) - \frac{1}{s^N} R_i(s)\right] \quad (i = 1, 2, 3, \ldots), \tag{A.1c}$$

$$H_1 = A_0\left[\alpha_4 \int_s^1 \frac{1}{\eta^{N+1}}(\eta - s)^m \,\mathrm{d}\eta + \beta_4(1-s)^m\right], \tag{A.2a}$$

$$H_2 = B_0\left[\alpha_4 \int_s^1 \frac{1}{\eta^{N+1}}(\eta - 1)^m \,\mathrm{d}\eta - \frac{\beta_4}{s^N}(s-1)^m\right], \tag{A.2b}$$

$$B_i = \alpha_4 \int_s^1 \frac{1}{\eta^{N+1}} R_i(\eta)\,\mathrm{d}\eta + \beta_4\left[R_i(1) - \frac{1}{s^N} R_i(s)\right] \quad (i = 1, 2, 3, \ldots), \tag{A.2c}$$

$$Y_{1i} = -\frac{\beta_2}{N_i}\int_s^1 I_1(\xi) R_i \xi \,\mathrm{d}\xi, \qquad Y_{2i} = -\frac{\beta_2}{N_i}\int_s^1 I_2(\xi) R_i \xi \,\mathrm{d}\xi, \tag{A.3a}$$

$$Z_{1i} = -\frac{\beta_2}{N_i}\int_s^1 L_1(\xi) R_i \xi \,\mathrm{d}\xi, \qquad Z_{2i} = -\frac{\beta_2}{N_i}\int_s^1 L_2(\xi) R_i \xi \,\mathrm{d}\xi, \tag{A.3b}$$

$$V_{d1} = \beta_3\left(G_1 + \frac{G_2}{s^{N+1}}\right) + \sum_{i=1}^{\infty} A_i Y_{2i} + \begin{cases} \delta \ln s, & N = 0, \\ \dfrac{\delta}{2N}\left(1 - \dfrac{1}{s^{2N}}\right), & N \neq 0, \end{cases} \tag{A.4a}$$

$$V_{b1} = \beta_4\left(G_1 + \frac{G_2}{s^{N+1}}\right) + \sum_{i=1}^{\infty} A_i Z_{2i} - \begin{cases} \delta g \ln s, & N = 0, \\ \dfrac{\delta g}{2N}\left(1 - \dfrac{1}{s^{2N}}\right), & N \neq 0, \end{cases} \tag{A.4b}$$

$$W_{d1i} = \frac{A_i}{\omega_i}(Y_{1i} - Y_{2i}\omega_i^2), \tag{A.4c}$$

$$W_{b1i} = \frac{A_i}{\omega_i}(Z_{1i} - Z_{2i}\omega_i^2), \tag{A.4d}$$

$$V_{d2} = \beta_3\left(H_1 + \frac{H_2}{s^{N+1}}\right) + \sum_{i=1}^{\infty} B_i Y_{2i} - \begin{cases} \delta g \ln s, & N = 0, \\ \dfrac{\delta g}{2N}\left(1 - \dfrac{1}{s^{2N}}\right), & N \neq 0, \end{cases} \tag{A.5a}$$

$$V_{b2} = \beta_4\left(H_1 + \frac{H_2}{s^{N+1}}\right) + \sum_{i=1}^{\infty} B_i Z_{2i} + \begin{cases} \delta \ln s, & N = 0, \\ \dfrac{\delta}{2N}\left(1 - \dfrac{1}{s^{2N}}\right), & N \neq 0, \end{cases} \tag{A.5b}$$

$$W_{d2i} = \frac{B_i}{\omega_i}(Y_{1i} - Y_{2i}\omega_i^2), \tag{A.5c}$$

$$W_{b2i} = \frac{B_i}{\omega_i}(Z_{1i} - Z_{2i}\omega_i^2), \tag{A.5d}$$

$$a_{11} = V_{d1} + \sum_{i=1}^{\infty} A_i(A_{2i} - Y_{2i}), \qquad a_{12} = V_{b1} + \sum_{i=1}^{\infty} A_i(A_{3i} - Z_{2i}), \tag{A.6a}$$

$$a_{21} = V_{d2} + \sum_{i=1}^{\infty} B_i(A_{2i} - Y_{2i}), \qquad a_{22} = V_{b2} + \sum_{i=1}^{\infty} B_i(A_{3i} - Z_{2i}), \tag{A.6b}$$

$$b_1 = \varphi_1(0) - \varphi_2(0) - G_1 p_1(0) - \frac{G_2}{s^N} p_2(0) - \sum_{i=1}^{\infty} A_i A_{1i}, \tag{A.6c}$$

$$b_2 = \psi_1(0) - \psi_2(0) - H_1 p_1(0) - \frac{H_2}{s^N} p_2(0) - \sum_{i=1}^{\infty} B_i A_{1i}, \tag{A.6d}$$

$$d_1 = \dot{\varphi}_1(0) - \dot{\varphi}_2(0) - G_1 \dot{p}_1(0) - \frac{G_2}{s^N} \dot{p}_2(0) - \sum_{i=1}^{\infty} A_i A_{4i}, \tag{A.6e}$$

$$d_2 = \dot{\psi}_1(0) - \dot{\psi}_2(0) - H_1 \dot{p}_1(0) - \frac{H_2}{s^N} \dot{p}_2(0) - \sum_{i=1}^{\infty} B_i A_{4i}. \tag{A.6f}$$

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