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A finite element analysis on random vibration of nonlinear shell structures

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Abstract

The objective of this investigation is to study the random vibration of a nonlinear geometrically shell structure by using the finite element method in conjunction with the equivalent linearization approach. When the shell structure is subjected to excessive loadings, the large deformations of the shell structure must be considered. In that sense, the stiffness of the governing equation of the shell structure is related to deflection; therefore, it is nonlinear and difficult to solve. In this study, the applied loadings to the shell structure are assumed to be a nonstationary random excitation to characterize many physical loadings such as earthquake, wind, aerodynamic and acoustic loadings. The equivalent linearization and the finite element method are adopted to perform the nonlinear random vibration analysis of the shell structures, which can be quite nonuniform and complex in geometry or nonhomogeneous in material. These obtained statistical dynamics responses are very useful for estimating the structure safety and reliability. Meanwhile, some statistical responses obtained by the present approach are checked by the Monte Carlo simulation technique.

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1. Introduction

Under the analysis of continuous criteria, the classical shell theory produces the equations that are very difficult to solve and these governing equations can be carried out only if some assumptions are made, such as Kirchhoff's hypothesis. Many researches in the analysis of complex structures are limited to linearly elastic problem. The linearly elastic behaviors of the structural systems are under the assumptions that the displacements of the finite element assemblage are infinitesimally small and that the material is linearly elastic, and then the equilibrium equations can be constructed with the nature of the boundary conditions remaining unchanged during the application of the loads on the finite element assemblage. When the large deformation of the thin plate or think shell structures is considered, nonlinear behaviors must be undertaken nowadays more often than that in the past. The finite element method has been widely used in both civil engineering and mechanical structure recently, which in conjunction with the electronic digital computer have accomplished the numerical idealization and solution of the continuous system in a matrix manner, and in effect have made possible the practical extension and application of the classical procedures to very complex engineering systems.

In general, two nonlinear behaviors are considered in the structural analysis: the geometrically nonlinear and materially nonlinear. The geometrically nonlinear problems occur when the structure is under a large deformation and the materially nonlinear effects lie in the nonlinear stress–strain relation as well. Meanwhile, in order to improve the solution accuracy, it is necessary to carry out the governing equations by iterative schemes. The well known Newton–Raphson method and the modified Newton–Raphson procedure have been well developed in many textbooks. However, many researchers have contributed to the development of nonlinear problems and the brief reviews are given as follows. Stricklin and Haisler [1] presented the assessment of the solution procedures available for the analysis of inelastic and/or large deflection structural behavior. Also, Bergan and Soreide [2] gave a brief review on some of the most important techniques used for solving nonlinear equations in structural problems. Wood and Zienkiewicz [3] studied the geometrically nonlinear behaviors of elastic inplane oriented bodies by using a modified isoparametic finite element method. In general, we often use total Lagrangian formulations to approach the problem of solid mechanics; Horrigmoe and Bergan [4] described a general formulation for geometrically nonlinear analysis of shells using the flat finite elements, which was based on the updated Lagrangian description. The implicit time integration method for solving the nonlinear dynamic problems were discussed and evaluated by Bathe and Cimento [5]. Surana [6] studied the statically geometric nonlinearity for a curved shell structure by using total Lagrangian approach. The existing shell element formulations were restricted to small nodal rotations between two successive increments. The element formulation presented by this study removed such restrictions. Chakravorty and his associates [7] used an eight-noded isoparametric finite element to investigate the free vibration behaviors of a linear doubly curved conoidal shell structure. Ramesh and Krishnamoorthy [8] investigated the application of dynamic relaxation method to the geometrically nonlinear analysis of plated and shell involving large deflection, small rotations and strain. Damatty and his associates [9] adopted the total Lagrangian approach to perform the static and dynamic analysis of shell structures based on consistent shell element. Yadav and Verma [10] studied the free vibration of composite circular cylindrical shells with random material properties. Mao and Williams [11] adopted the

nonlinear and non shallow thick shell theory to perform the nonlinear analysis of cross-ply thick cylindrical shells under axial compression. Chang et al. [12] performed the nonlinear vibration analysis of a geometrically nonlinear shell structure using the finite element approach. Vu-Quoc et al. [13] presented a finite element formulation for the dynamic analysis of the geometrically exact sandwich shells by accommodating the large deformation and large overall motion. Popov [14] demonstrated in a tutorial fashion how recent ideas and methods of bifurcation theory and nonlinear dynamics have improved the understanding of structural buckling under dynamic loads and vibrations of shells under parametric excitation, he focused on geometrically nonlinear forced vibration of circular cylindrical shells. Recently Nagai et al. [15] investigated the effects of a concentrated mass on chaotic oscillations of a shallow cylindrical shell under gravity and periodic acceleration.

One of the most widely used approximation techniques for the nonlinear problem is the equivalent linearization that the nonlinearities in system are replaced by effective linear systems. Generally speaking, this method was the statistical extension of Krylov and Bogoliubov's [16] linearization technique and was first introduced by Booton [17] and Caughey [18]. Also, equivalent linearization method has been developed in the field of mechanical and structural systems by Foster [19]. In addition, a particular form of fluid damping based on statistical averaging was performed by linearization scheme by Malhotra and Penzien [20]. Continuously, Kaul and Penzien [21] extended the previous approach to analyze the nonstationary responses of an offshore structure subjected to earthquake motion. Atalik and Utaku [22] used the stochastic linearization approach to obtain the response of nonlinear multi-degree of freedom dynamic systems under stationary excitation. Iwan and Mason [23] adopted the method of equivalent linearization to study the general problem of the response of nonlinear discrete system subjected to nonstationary random excitation. The behaviors of a class of nonlinear viscoelastic shear building structures subjected to random excitation by using the stochastic linearization technique was developed by Chang et al. [24]. In addition, Chang and his associate [25] used the linearization method in conjunction with Galerkin technique to investigate the dynamic response of a non uniform orthotropic circular plate. Eliskakoff and Fang [26] investigated the large amplitude random vibrations of a beam on elastic foundation by using a new stochastic lineralization technique. This new approach is based on requirement that the mean square deviation on the strain energy of the nonlinearly deformed beam and that of the equivalent beam in a linear state, should be minimum. Also, Elishakoff and Colajanni [27] proposed a new version for the stochastic linearization technique, which has drawn the attention of most researchers engaging in nonlinear random vibration problem. In this study, two error-free stochastic linearization techniques are elucidated, namely those based on the force linearization and energy linearization.

Applying the stochastic concepts to the structural analysis is very essential and has been well developed recently. The problems of random vibration include the randomness in the dynamic loadings that is only random with respect to time. These dynamic loadings or excitations from wind pressure, jet engine exhaust, earthquake motion or ocean waves which are random in nature can be efficiently handled by stochastic processes. In this study, the stochastic dynamic analysis of a geometrically nonlinear shell structure with random excitation is dealt with by using the finite element method in conjunction with linearization technique. From the engineering point of view, these statistical results are very important for estimating the reliability of the structure.

2. Formulations of the problem

2.1. Definition of shell element

Consider the problem of a curved thin shell structure as shown in Fig. 1, a simple and efficient finite element named Serendipity 8-nodes element shown in Fig. 2 is adopted in this investigation, which was first suggested by Cook [28]. This kind of shell element contains five degrees of freedom

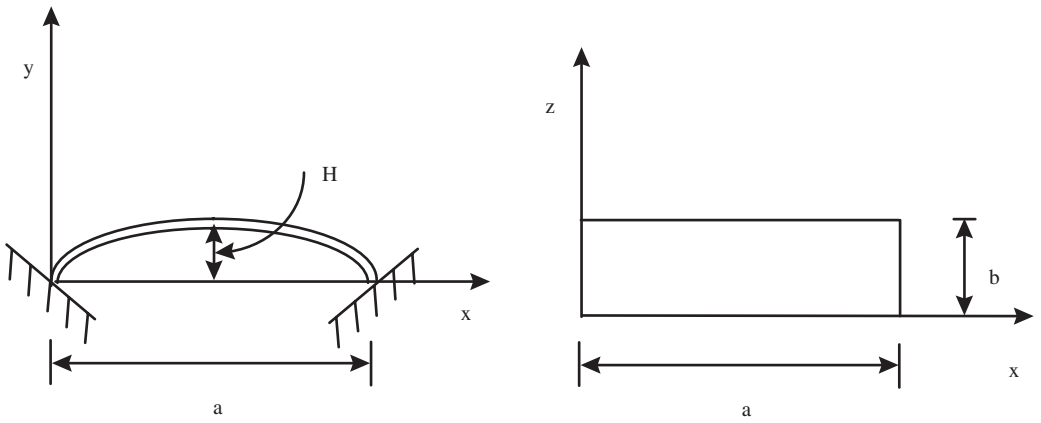

Fig. 1. The configuration of the problem.

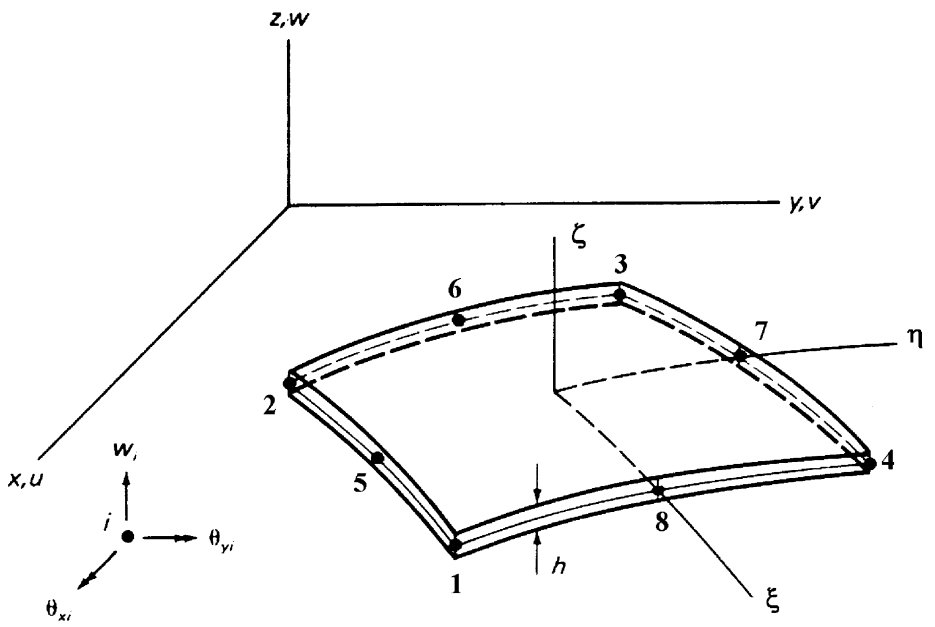

Fig. 2. Type of shell finite element used.

at each of the eight nodes: u, v, w, α, β, of which the first three are translations in the global directions and the last two are rotations about the local axes. The geometric layout of such element is presented in Fig. 3, where the coordinates of any point can be expressed as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \sum_{i=1}^{8} \mathbf{N}_i \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + \sum_{i=1}^{8} \mathbf{N}_i \varsigma \frac{h_i}{2} \begin{bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{bmatrix}, \tag{1}$$

where $\mathbf{N}_i$ are the interpolation functions, $[l_{3i}, m_{3i}, n_{3i}]^{\mathrm{T}}$ are the directional cosines of a vector $\mathbf{V}_{3i}$ that is normal to the middle surface and spans the thickness h_i of the shell at node i. It should be noted that another two vectors $\mathbf{V}_{1i}$ and $\mathbf{V}_{2i}$ are orthogonal to vector V_{3i} and to each other. The choice for the direction of one of them is arbitrary. To settle these vectors, we assume $\mathbf{V}_{2i}$ and $\mathbf{V}_{3i}$ are given, and vector $\mathbf{V}_{1i}$ is rapidly obtained by the following expression:

$$\mathbf{V}_{1i} = \mathbf{V}_{2i} \times \mathbf{V}_{3i}. \tag{2}$$

Meanwhile, the generic displacements $\mathbf{u}$ can be written in terms of the nodal displacements $\mathbf{q}_i$ as

$$\mathbf{u} = \sum_{i=1}^{8} [\mathbf{N}_i]\mathbf{q}_i, \tag{3}$$

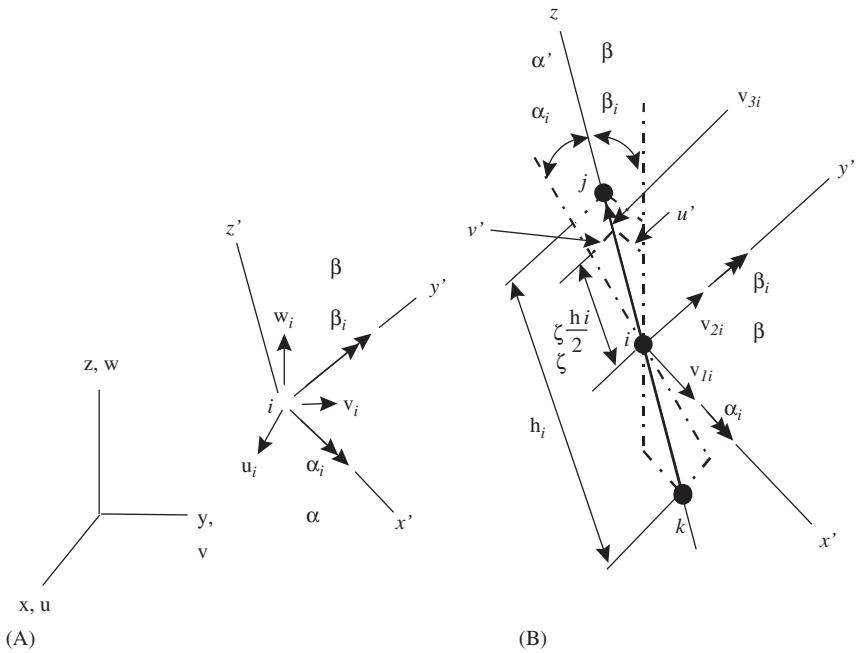

Fig. 3. The geometric layout of the node.

where

$$[\mathbf{N}_i] = \begin{bmatrix} 1 & 0 & 0 & -\frac{h_i}{2}\zeta l_{2i} & \frac{h_i}{2}\zeta l_{1i} \\ 0 & 1 & 0 & -\frac{h_i}{2}\zeta m_{2i} & \frac{h_i}{2}\zeta m_{1i} \\ 0 & 0 & 1 & -\frac{h_i}{2}\zeta n_{2i} & \frac{h_i}{2}\zeta n_{1i} \end{bmatrix}$$

$$\mathbf{q}_i = \begin{bmatrix} u_i & v_i & w_i & \alpha_i & \beta_i \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} u & v & w \end{bmatrix} \tag{4}$$

In this paper, the structure was assumed to be under an excessive loading in such a way that the large displacements and large rotations may occur, therefore, the geometric nonlinearity of the displacements must be considered in order to get more accurate results for the displacements of the structure. A total lagrangian approach is adopted throughout in which displacements are referred to the original configuration, and then the Green's strains and Piola–Kirchhoff stress formulations denote the strain and stress vectors in this study. In accordance with the thin shell theory we neglect $\varepsilon_{z'}$ and $\sigma_{z'}$, in the direction z' and the material is assumed to be linear elastic and isotropic. There exists a transformation matrix $[T]$ which is the directional cosines between the local coordinate system $(x'y'z')$ and global coordinate system (xyz). Therefore the following equations can be formulated:

$$\begin{aligned} \sigma' &= [D']\varepsilon', \\ \sigma &= [D]\varepsilon, \\ [D] &= [T][D'][T]^{\mathrm{T}}, \\ \sigma^{\mathrm{T}} &= [\,\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{yz} \quad \tau_{xz} \quad \tau_{xy}\,]^{\mathrm{T}}, \\ \varepsilon^{\mathrm{T}} &= [\,\varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy}\,]^{\mathrm{T}}, \end{aligned} \tag{5}$$

where $[D']$ is a matrix embracing the Young's modulus of elasticity E and Possion's ratio v.

Expressing the strain vector in the matrix form as follows:

$$\begin{aligned} \varepsilon &= \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_{xy} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{bmatrix} = \begin{bmatrix} u_x \\ v_y \\ w_z \\ u_y + v_x \\ v_z + w_y \\ u_y + w_x \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(u_x^2 + v_x^2 + w_x^2) \\ \frac{1}{2}(u_y^2 + v_y^2 + w_y^2) \\ \frac{1}{2}(u_z^2 + v_z^2 + w_z^2) \\ u_x u_y + v_x v_y + w_x w_y \\ u_y u_z + v_y v_z + w_y w_z \\ u_x u_z + v_x v_z + w_x w_z \end{bmatrix}, \\ &= \varepsilon_L + \varepsilon_{\mathrm{NL}}, \end{aligned} \tag{6}$$

where ε_L is the linear strain and $\varepsilon_{\mathrm{NL}}$ is the nonlinear strain.

If we define a new vector $\Gamma = [\Gamma_x, \Gamma_y, \Gamma_z]^{\mathrm{T}}$, where $\Gamma_x^{\mathrm{T}} = [u_x, v_x, w_x]$, $\Gamma_y^{\mathrm{T}} = [u_y, v_y, w_y]$ and $\Gamma_z^{\mathrm{T}} = [u_z, v_z, w_z]$, respectively, the strain vector in Eq. (6) can be simplified as

$$\varepsilon = [h]\Gamma + \tfrac{1}{2}[a]\Gamma. \tag{7}$$

2.2. Derivation of governing equation

If Γ is expressed in terms of shape functions and nodal displacements then $\Gamma = [f]q$, $\delta\Gamma = [f]\delta q$, where $[f]$ is a matrix defined purely in terms of the coordinates. Also, it can be easily shown that $[\delta a]\Gamma = [a]\delta\Gamma$. Therefore, the variation of strain will be

$$\delta\varepsilon = ([h][f] + [a][f])\delta q. \tag{8}$$

Considering the above properties, the virtual strain energy of internal stress can be obtained as

$$\begin{aligned}\delta E_{\mathrm{int}} &= \int_v \delta\varepsilon^{\mathrm{T}}\sigma \,\mathrm{d}V \\ &= \delta q^{\mathrm{T}} \int_v ([h][f] + [a][f])^{\mathrm{T}}\sigma \,\mathrm{d}V \\ &= \delta q^{\mathrm{T}} \int_v [b]^{\mathrm{T}}\sigma \,\mathrm{d}V,\end{aligned} \tag{9}$$

where $[b] = [h][f] + [a][f]$ is the nonlinear strain matrix.

Continually, we consider the virtual work of the external force as follows:

$$\delta W_{\mathrm{ext}} = \delta q^{\mathrm{T}}\mathbf{P} - \int_v (\delta u)^{\mathrm{T}}\rho\ddot{u} \,\mathrm{d}V, \tag{10}$$

where $\mathbf{P}$ is the external force, ρ is the mass density and $\ddot{u}$ is the acceleration. Substituting Eq. (3) into Eq. (10), we obtain the following:

$$\delta W_{\mathrm{ext}} = \delta q^{\mathrm{T}}\left[\mathbf{P} - \left(\rho \int_v [N]^{\mathrm{T}} \cdot [N] \,\mathrm{d}V\right)\ddot{q}\right], \tag{11}$$

where $\rho \int_v [N]^{\mathrm{T}} \cdot [N] \,\mathrm{d}V = [m]$ is the consistent mass matrix.

By using the principle of virtual work, δW_{ext} must be equal to δE_{int} when a structure reaches a state of dynamic equilibrium. Therefore, substituting Eqs. (9) and (10) into this relation and canceling the virtual displacements, then including the damping effect, we obtain the governing matrix equation of the structure after assembly as follows:

$$[\mathbf{M}]\ddot{\mathbf{U}} + [\mathbf{C}]\dot{\mathbf{U}} + \mathbf{R}(\mathbf{U}) = \mathbf{P}(t), \tag{12}$$

where $[\mathbf{M}]$ is the global consistent mass matrix of the structure, $[\mathbf{C}]$ is the global damping matrix of the structure, $\dot{\mathbf{U}}$ is the global velocity vector of the structure, $\ddot{\mathbf{U}}$ is the global acceleration vector of the structure, $\mathbf{U}$ is the global displacement vector of the structure, $\mathbf{P}$ is the global external force vector of the structure and $\mathbf{R}(\mathbf{U}) = \int_V [B]^{\mathrm{T}}\sigma \,\mathrm{d}V$ is the global vector of restoring forces of the structure that depends on the displacement field. In the present study, the geometric shapes, sizes of the structure and the material properties are assumed to be deterministic, only the external forces $\mathbf{P}(t)$ of the structure are considered to be a nonstationary random process. In addition, $\mathbf{R}$ is the global nodal restoring force vector of the structure which is nonlinear. The system will be

assumed to be initially at rest and therefore

$$\mathbf{U}(0) = \dot{\mathbf{U}}(0) = \mathbf{0}. \tag{13}$$

3. Equivalent linearization approach

The candidate equation for equivalent linear system can be assumed as

$$[\mathbf{M}]\ddot{\mathbf{U}} + [\mathbf{C}]\dot{\mathbf{U}} + \mathbf{K}[\mathbf{Q}(t)]\mathbf{U} = \mathbf{P}(t). \tag{14}$$

It is noted that Eq. (14) can be treated as a linear differential equation with nonconstant stiffness matrix, which does not depend on the displacement but on the instantaneous statistical properties of displacement, say $\mathbf{Q}(t)$. Meanwhile, if the excitations are assumed to be Gaussian with zero mean, the response of the system will be Gaussian as well and then the response can be described completely by its covariance. That is

$$\mathbf{Q}(t) = \mathbf{E}[\mathbf{U}\mathbf{U}^{\mathrm{T}}], \tag{15}$$

where $\mathbf{E}[\bullet]$ denotes the expected value. Also, the matrix $[\mathbf{M}]$ and $[\mathbf{C}]$ remain constants during the process since the material properties are deterministic and $[\mathbf{C}]$ is assumed to be proportional to matrix $[\mathbf{M}]$. The initial conditions for Eq. (14) are assumed to be identical to those in Eq. (12). According to the Priestley's model, a nonstationary random process $\mathbf{P}(t)$ with zero mean and time modulation function $A(t)$ can be expressed as

$$\mathbf{P}(t) = \mathbf{A}(t)\mathbf{r}(t), \tag{16}$$

where $\mathbf{r}(t)$ is a stationary Gaussian random process with zero mean and the power spectral density $\mathbf{S}(\omega)$. Substituting Eq. (16) into Eq. (14), the associated equivalent linear equation can therefore be written as

$$[\mathbf{M}]\ddot{\mathbf{U}} + [\mathbf{C}]\dot{\mathbf{U}} + \mathbf{K}[\mathbf{Q}(t)]\mathbf{U} = \mathbf{A}(t)\mathbf{r}(t). \tag{17}$$

Let the vector δ be the difference between the nonlinear system and the equivalent linear system, and the δ is given by

$$\delta = \mathbf{K}[\mathbf{Q}(t)]\mathbf{U} - \mathbf{R}(\mathbf{U}). \tag{18}$$

In order to obtain the optimal coefficients of matrix $\mathbf{K}$, an instantaneous measure, which is denoted by the mean square value of, δ has to be a minimum. It gives that

$$\mathbf{E}[\delta^{\mathrm{T}}\delta] = \mathbf{E}[\|\delta\|^2] \tag{19}$$

is the minimum for all $U(t)$ of the class of solutions of Eq. (14). In addition, the necessary conditions for Eq. (19) to be true is

$$\frac{\partial \mathbf{E}[\delta^{\mathrm{T}}\delta]}{\partial k_{ij}} = 0, \quad i,j = 1,2,\ldots,n, \tag{20}$$

where n is the dimension of discretized system. Consequently, substituting Eqs. (18) and (19) into Eq. (20) and performing the partial differentiation, we obtain the set of linear equations as

$$\mathbf{E}[\mathbf{U}\mathbf{U}^{\mathrm{T}}][\mathbf{K}]^{\mathrm{T}} = \mathbf{E}[\mathbf{U}\mathbf{R}^{\mathrm{T}}(\mathbf{U})]. \tag{21}$$

Nevertheless, it can be known that if $\mathbf{U}$ is a jointly Gaussian random process with zero mean and also $\mathbf{U}$ is sufficiently smooth such that the first partial derivatives of $\mathbf{R}(\mathbf{U})$ with respect to $\mathbf{U}$ exist, then the following relation can be obtained.

$$\mathbf{E}[\mathbf{U}\mathbf{R}^{\mathrm{T}}(\mathbf{U})] = \mathbf{E}[\mathbf{U}\mathbf{U}^{\mathrm{T}}]\mathbf{E}\left[\frac{\partial \mathbf{R}^{\mathrm{T}}(\mathbf{U})}{\partial \mathbf{U}}\right]. \tag{22}$$

Substituting Eq. (22) into Eq. (21), the essential result of the stiffness matrix which based on the linearization approach can be easily achieved when the input loading $\mathbf{P}(t)$ is assumed to be a zero-mean Gaussian process. Consequently, the expression of the effective stiffness matrix is

$$[\mathbf{K}]^{\mathrm{T}} = \mathbf{E}\left[\frac{\partial \mathbf{R}^{\mathrm{T}}(\mathbf{U})}{\partial \mathbf{U}}\right]. \tag{23}$$

According to the application of the state equation, Eq. (17) can be expressed in terms of the first-order differential equation as

$$\dot{\mathbf{z}} = \mathbf{G}\mathbf{z} + \mathbf{\Psi}(t)\mathbf{r}(t), \tag{24}$$

with

$$\mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}[\mathbf{Q}(t)] & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \tag{25}$$

$$\mathbf{\Psi}(t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{A}(t) \end{bmatrix} \tag{26}$$

and

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{U}(t) \\ \dot{\mathbf{U}}(t) \end{bmatrix}, \quad \dot{\mathbf{z}}(t) = \begin{bmatrix} \dot{\mathbf{U}}(t) \\ \ddot{\mathbf{U}}(t) \end{bmatrix}, \tag{27}$$

where $\mathbf{I}$ is a $n \times n$ unity matrix, $\mathbf{A}(t)$ and $\mathbf{r}(t)$ are the time modulated function and a stationary Gaussian random process with zero mean, respectively. Furthermore, Eq. (24) is the so called state equation which has been widely used in the analysis of random vibration.

4. Determination of the covariance matrix

The state equation can be solved directly as follows. First of all, let $\mathbf{Z}(t)$ to be the matrix which satisfy the homogenous equation

$$\dot{\mathbf{Z}} = \mathbf{G}[\mathbf{Q}(t)]\mathbf{Z}, \quad \mathbf{Z}(0) = \mathbf{1} \tag{28}$$

and then by using the fundamental computation of ordinary differential equation, the solution of Eq. (24) can be expressed as

$$\mathbf{z}(t) = \mathbf{Z}(t)\int_0^t \mathbf{Z}^{-1}(k)\mathbf{\Psi}(k)\mathbf{r}(k)\,\mathrm{d}k. \tag{29}$$

It is noted that when matrix $\mathbf{A}[\mathbf{Q}(t)]$ is continuous function of $\mathbf{Q}(t)$, the solution of Eq. (29) can be determined uniquely. Also, it can be easily shown that the expected value of $\mathbf{z}(t)$ is identical to zero that satisfies the previous assumption. Then the covariance matrix $\mathbf{Q}(t)$ is obtained by using Eq. (29) and some operations of transpose, multiplying and the expectations as

$$\mathbf{Q}(t) = \int_{-\infty}^{+\infty} \mathbf{Y}(\omega, t)\bar{\mathbf{Y}}^{\mathrm{T}}(\omega, t)\mathbf{S}(\omega)\,\mathrm{d}\omega, \tag{30}$$

with

$$\mathbf{Y}(\omega, t) = \mathbf{Z}(t)\int_0^t \mathbf{Z}^{-1}(s)\mathbf{\Psi}(s)\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s, \tag{31}$$

where $\bar{\mathbf{Y}}$ denotes the complex conjugate pair of $\mathbf{Y}$.

It is of interest to note that the N-dimensional quantity $\mathbf{Y}(\omega, t)$ as defined in Eq. (31) satisfies the differential equation

$$\dot{\mathbf{Y}} = \mathbf{G}\mathbf{Y} + \mathbf{\Psi}(t)\mathrm{e}^{\mathrm{i}\omega t}, \tag{32}$$

with the initial condition

$$\mathbf{Y}(0) = \mathbf{0}. \tag{33}$$

Thus, $\mathbf{Y}(\omega, t)$ has an immediate physical significance. It is simply the response of the linear system of Eq. (24), except that the excitation $\mathbf{r}(t)$ is now harmonic and deterministic rather than random.

Nevertheless, the response of $\mathbf{Y}(\omega, t)$ tends to oscillate rapidly is easily seen when the time becomes large. In order to obtain the more accurate approach, the more evaluations and computational time are required relatively. An alternative method to approximate the covariance matrix $\mathbf{Q}(t)$ can be proposed which the solutions base on the autocorrelation function of the excitation in time domain, rather than the previous spectrum density function in frequency domain.

The derivations of the differential equation which include the covariance matrix $\mathbf{Q}(t)$ presented in time domain can be shown easily, for which $\mathbf{Q}(t)$ must satisfy the following relation:

$$\dot{\mathbf{Q}} = \mathbf{G}\mathbf{Q} + (\mathbf{G}\mathbf{Q})^{\mathrm{T}} + \mathbf{\Phi}(t)\mathbf{\Psi}^{\mathrm{T}}(t) + \mathbf{\Psi}(t)\mathbf{\Phi}^{\mathrm{T}}(t) \tag{34}$$

with the initial condition

$$\mathbf{Q}(0) = \mathbf{0}, \tag{35}$$

where

$$\mathbf{\Phi}(t) = \mathbf{Z}(t)\int_0^t \mathbf{Z}^{-1}(t)\mathbf{\Psi}(\tau)\mathbf{E}[\mathbf{r}(\tau)\mathbf{r}(t)]\,\mathrm{d}\tau, \tag{36}$$

$\mathbf{E}[\mathbf{r}(\tau)\mathbf{r}(t)]$ expresses the autocorrelation function of the excitation. Both Eqs. (34) and (36) are solved simultaneously to obtain the covariance quantities. The integration of Eq. (36) should be performed numerically at each time step and this may result in a lot of errors and the computational costs cannot be omitted, however, this scheme is better than those obtained from the integration under the frequency domain in Eq. (30).

In fact, Eq. (34) is nonlinear because matrix $\mathbf{G}$ depends on $\mathbf{Q}(t)$ and a practical iteration procedure is required to obtain the responses which can be summarized as:

(a) Assign an initial value to the instantaneous covariance matrix.
(b) Using Eq. (23) to construct the matrix of the system.
(c) Solve Eqs. (34) and (36) simultaneously for the new instantaneous covariance matrix.
(d) Repeat (b) to (c) until a specified tolerance is satisfied.

5. Monte Carlo simulation

Simulation of nonstationary Gaussian random process $\mathbf{P}(t)$ with zero mean can be formulated by

$$\mathbf{P}(t) = \sqrt{2}\sum_{k=1}^{N}[\mathbf{S}_P(\omega_k)\Delta\omega]^{1/2}\mathrm{Cos}(\omega_k t - \phi_k)\mathbf{\Psi}(t), \tag{37}$$

where $\mathbf{\Psi}(t)$ is a deterministically modulating time function as defined previously. Nevertheless, for the case of a random process of Gaussian white noise with zero mean, Eq. (37) cannot apply straightforwardly and therefore we must turn to alternative formulation or procedure. A model similar to that employed by Clough and Penzien [29] approximates the white noise process. For analytical purpose one may wish to generate sample function which approach white noise. This procedure can be carried out digitally by first sampling a sequence of pairs of statistically independent random numbers $x_1, x_2, \ldots, x_n$, all of which have a uniform probability distribution over the range $0<x<1$. A new sequence of pairs of statistically independent random numbers $y_1, y_2, \ldots, y_n$,, are generated by using the relations

$$\begin{aligned} y_i &= (-2\ln x_i)^{1/2}\mathrm{Cos}(2\pi x_{i+1}),\\ y_{i+1} &= (-2\ln x_i)^{1/2}\mathrm{Sin}(2\pi x_{i+1}). \end{aligned} \tag{38}$$

It can easily be shown that Y_i's process a Gaussian distribution with zero mean and variance of unity.

When the sequence of Y_i's are generated, the modulating random force is given by

$$\mathbf{P}(t_i) = (2\pi S_0/\Delta t)^{1/2} y_i \mathbf{\Psi}(t_i), \quad 1 \leqslant i \leqslant n, \tag{39}$$

where Δt is the time interval between the successive time steps and the power spectral density $S(\omega)$ of the new process can be generated by

$$S(\omega) = S_0 \frac{6 - 8\mathrm{Cos}(\omega\Delta t) + 2\mathrm{Cos}(2\omega\Delta t)}{(\omega\Delta t)^4}. \tag{40}$$

It is obvious that $S(\omega)$ will close to S_0 when ω approaches to zero with constant Δt.

A complete ensemble of m time, the solution of Eq. (17) can be obtained for each different sample function. Based on these different dynamic responses, the statistical functions can be evaluated consequently.

6. Numerical analysis

In order to verify the formulations that were stated previously, a numerical example is presented and the configuration is shown in Fig. 1. The geometric measurements and the material properties of the shell structure are assumed to be $a = 8\,\text{m}$, $b = 1\,\text{m}$, $h = 0.01\,\text{m}$, $E = 70\,\text{Gpa}$, $v = 0.33$, $\rho = 2710\,\text{kg/m}^3$, $H = 0.2\,\text{m}$, where a, b, h, E, v, ρ, H denote the dimensions, thickness, Young's modulus of elasticity, Poisson's ratio, mass density and the height of center, respectively. The boundary conditions of the shell structure are assumed to be simply supported or clamped at both shorter sides while the other two longer sides are assumed to be free, respectively. In the numerical computations, the whole shell structure is divided into 32×16 shell elements, which is adequate to reach the convergence of the results. This can be verified since the standard deviation of the displacement at the center of shell in the present mesh is almost the same as those based on 24×16 elements mesh or 16×16 elements mesh. In fact, the relative errors of the numerical results among these three different meshes were less than 0.1%. For the purpose of carrying out the numerical computation of some response quantities and their statistics, the computer programs are coded on HP 715/100 to this aim. The computational cost of the presented study based on the specified computer and 32×16 elements mesh was approximately 40 min. The computation time was pretty high due to several reasons as follows. Firstly, we dealt with the random vibration analysis, therefore, the determination of covariance matrix of the nodal displacements was quite necessary and the size of the matrix was the square of the numbers of degrees of freedom; secondly, several iterations must be performed until a specified tolerance is satisfied, since we were solving the nonlinear problem, it took approximately 5–7 iterations during each time step in our particular numerical example. Needless to say, the computational cost for Monte Carlo simulation was tremendously huge; it was approximately 10 h against 40 min based on stochastic equivalent linearization technique proposed by the present study.

In the present study, the nonstationary excitation model is assumed to be a product of the deterministic time modulation function and the stationary random process. The deterministic time modulation function is given by (41) and presented in Fig. 4.

$$A(t) = \frac{e^{-\alpha t} - e^{-\beta t}}{\gamma}. \tag{41}$$

In Eq. (41), $\gamma = \max(e^{-\alpha t} - e^{-\beta t})$ for $t > 0$ is a normalizing constant which can be shown to be

$$\gamma = \left(\frac{\alpha}{\beta}\right)^{\alpha/\beta-\alpha} - \left(\frac{\alpha}{\beta}\right)^{\beta/\beta-\alpha}. \tag{42}$$

Both α and β are constants and equal to 0.8/s and 1.6/s, respectively. Then the maximum value of $A(t)$ occurs at $t = 0.91\,\text{s}$, as plotted in Fig. 4. Furthermore, the random process $\mathbf{r}(t)$ in Eq. (17) is assumed to be a stationary Gaussian with zero mean and its corresponding spectrum, as illustrated in Fig. 5, is assumed to be the white noise spectral density function with magnitude S_0 and S_0 is equal to $0.01\,\text{m}^2/\text{s}^3$. Incidentally, the damping matrix $[C]$ of Eq. (14) is assumed to be proportional to the mass matrix $[\mathbf{M}]$, that is, $[\mathbf{C}] = 2\bar{\xi}\omega_0[\mathbf{M}]$ where $\bar{\xi} = 0.05$ is the damping ratio and $\omega_0 = 5.323\,\text{rad/s}$ is the fundamental natural frequency of the shell structure.

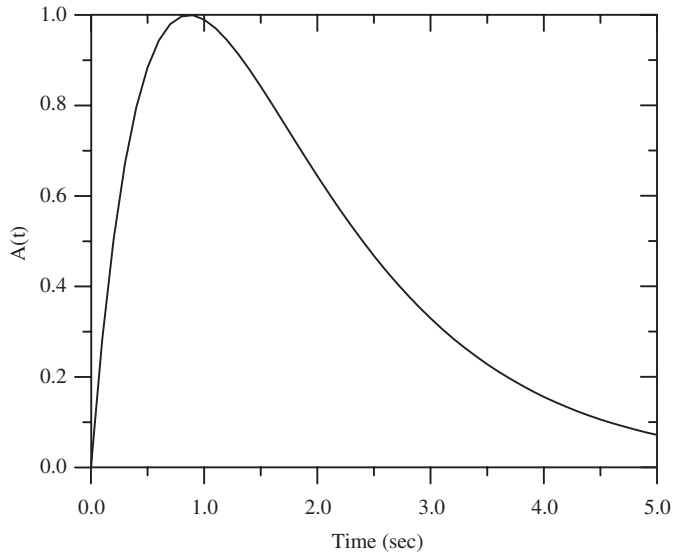

Fig. 4. The time modulation function.

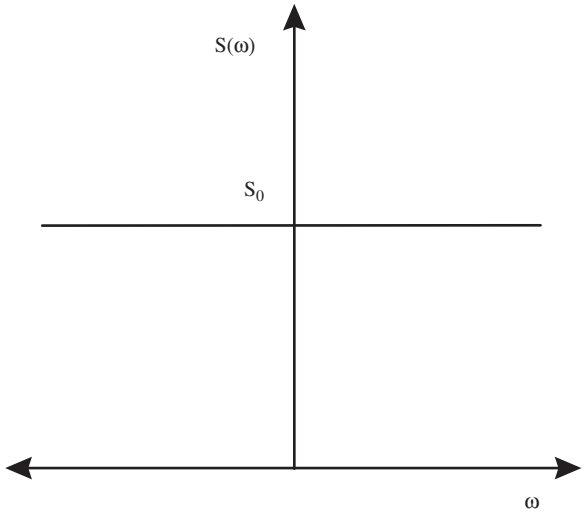

Fig. 5. The spectral density function of the excitation.

In general, the maximum statistic response may occur commonly at the center of the shell under the uniform distributed load. Therefore, all results indicated in this investigation are focused on this particular position of the shell structure. The standard deviation of the displacement $\mathbf{U}(t)$, the standard deviation of the velocity $\dot{\mathbf{U}}(t)$ and the correlation coefficient $\rho_{U\dot{U}}$ for $\mathbf{U}(t)$ and $\dot{\mathbf{U}}(t)$ are presented in Figs. 6–8. In order to check the accuracy of the stochastic equivalent linearization

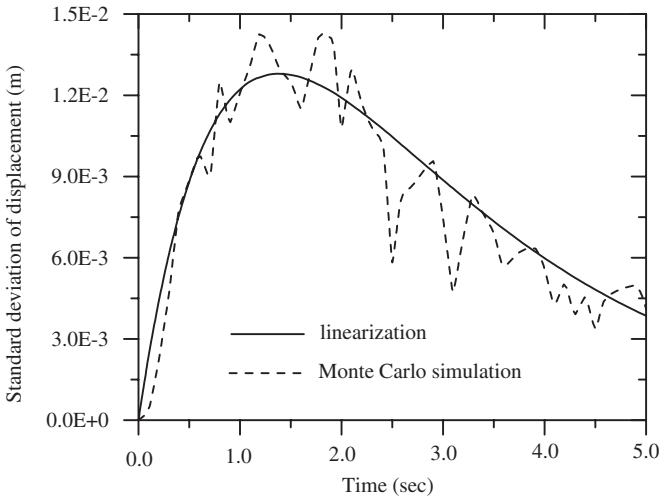

Fig. 6. The standard deviation of the displacement at center of the shell (simply supported).

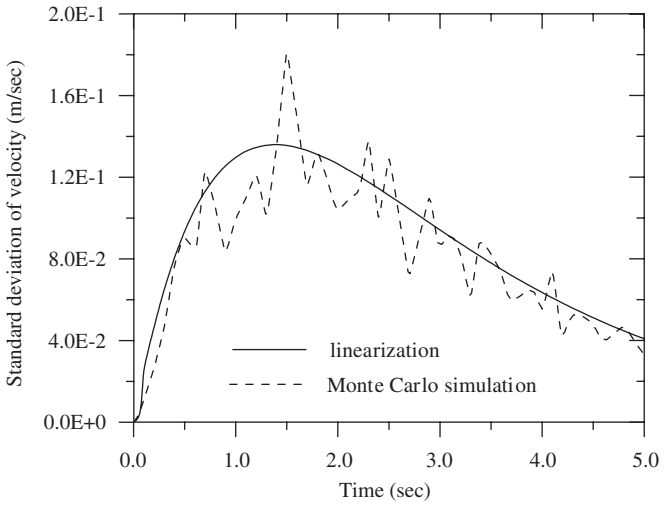

Fig. 7. The standard deviation of the velocity at center of the shell (simply supported).

technique, the Monte Carlo simulation with 200 samples is adopted to perform the simulation analysis. As it can be seen from Figs. 6 to 8, the results from the equivalent linearization and Monte Carlo simulation show a fairly good agreement. However, some discrepancy between these two techniques is unavoidable. In addition, the physical significance of the correlation coefficient $\rho_{U\dot{U}}$ for $\mathbf{U}(t)$ and $\dot{\mathbf{U}}(t)$ is that if $\rho_{U\dot{U}}$ is large and positive, the values of $\mathbf{U}(t)$ and $\dot{\mathbf{U}}(t)$ tend to be both large or both small relative to their respective means; otherwise, if $\rho_{U\dot{U}}$ is small or zero, there is little or no relationship between the values of $\mathbf{U}(t)$ and $\dot{\mathbf{U}}(t)$. In Fig. 8, the displacement and velocity of the shell structure is evidently relative to each other at the beginning of the excitation

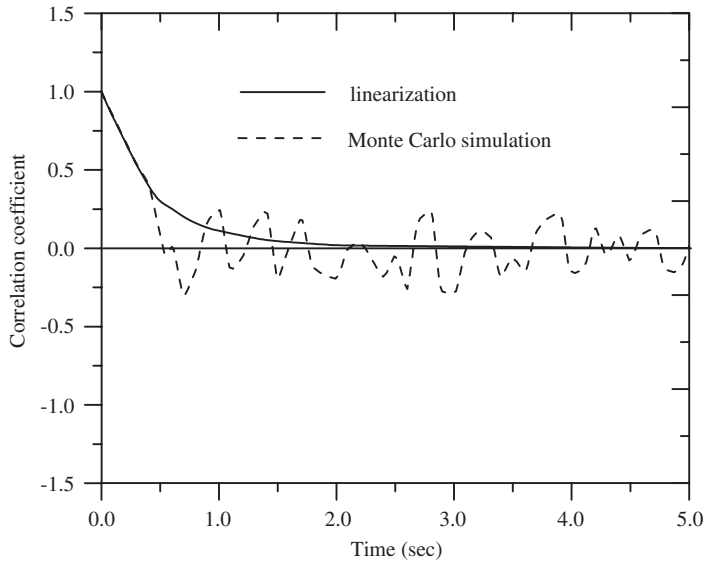

Fig. 8. The correlation coefficient for $U(t)$ and $\dot{U}(t)$ at center of the shell (simply supported).

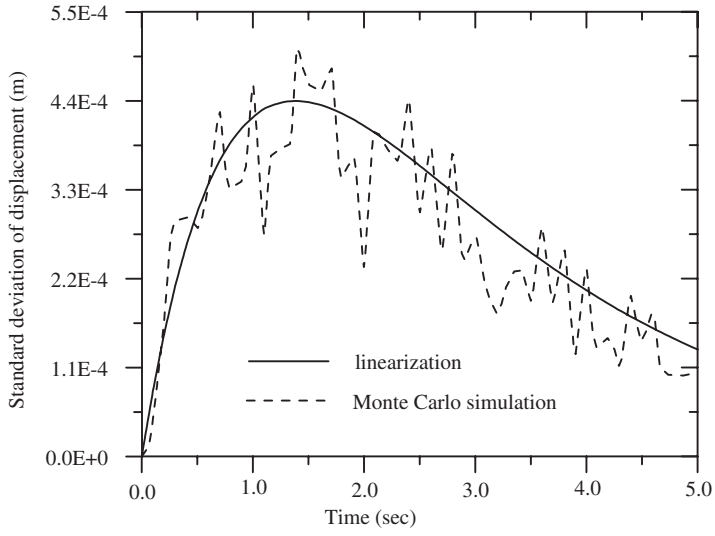

Fig. 9. The standard deviation of the displacement at center of the shell (clamped).

applied, and then it decreases rapidly with nonlinear manner which is expected. Meanwhile, all statistical responses in the clamped case, as presented in Figs. 9–11 have the similar phenomena but with smaller quantities than those in the simply supported case which is quite reasonable. From the engineering point of view, these statistical results play an important role in estimating the structural safety and reliability.

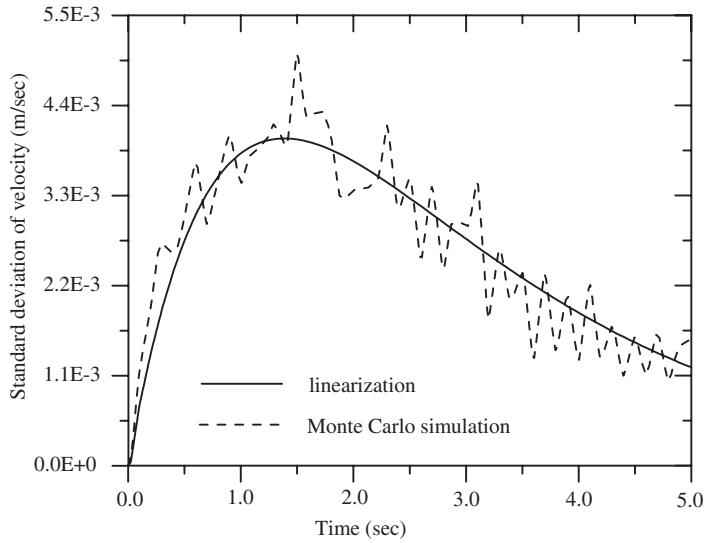

Fig. 10. The standard deviation of the velocity at center of the shell (clamped).

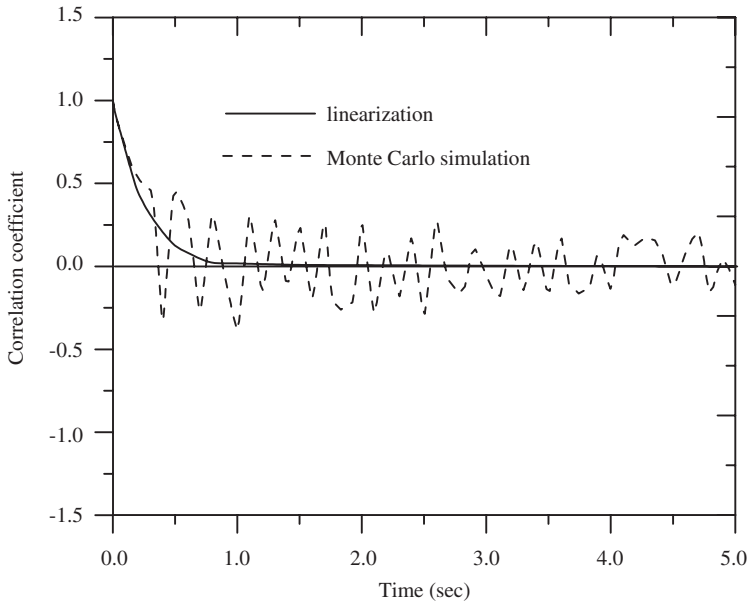

Fig. 11. The correlation coefficient for $U(t)$ and $\dot{U}(t)$ at center of the shell clamped).

7. Summary

In this investigation, a nonlinear displacement description is extended to the dynamic analysis of a geometrically nonlinear shell structure by using the finite element method, which improved

the traditional restrict of small increments during the period of deformation. Particularly, after a series of numerical analysis, the present finite element formulations are very effective and rapid with respect to the convergence in iteration. Some statistical responses of the geometrically nonlinear shell structure, which is subjected to a nonstationary random excitation, are studied by means of the stochastic equivalent linearization technique. The Monte Carlo simulation is adopted to check the accuracy of these results that shows a fairly good comparison. Finally, it should be emphasized that these statistically dynamical responses are very useful for estimating the reliability of the structure.

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References

[1] J.A. Stricklin, W.E. Haisler, Evaluation of solution procedure for material and/or geometrically nonlinear structural analysis, *AIAA* 11 (1973) 292–299.
[2] P.G. Bergan, T. Soreide, A comparative study of different numerical solution techniques as applied to a nonlinear structural problem, *Computer Methods in Applied Mechanics and Engineering* 2 (1973) 185–201.
[3] R.D. Wood, O.C. Zienkiewicz, Geometrically nonlinear finite element analysis of beams, frames, arches and axisymmetric shells, *Computer & Structures* 7 (1976) 725–735.
[4] G. Horrigmore, P.G. Bergan, Nonlinear analysis of free-form shells by flat finite elements, *Computer Methods in Applied Mechanics and Engineering* 16 (1978) 11–35.
[5] K.J. Bathe, A.P. Cimento, Some practical procedures for the solution of nonlinear finite element equations, *Computer Methods in Applied Mechanics and Engineering* 22 (1979) 56–85.
[6] K.S. Surana, Geometrically nonlinear formulation for the curved shell elements, *International Journal for Numerical Methods in Engineering* 19 (1983) 581–615.
[7] D. Chakrovororty, J.N. Bandyopadhyay, P.K. Sinha, Finite element free vibration analysis of conoidal shells, *Computer & Structures* 56 (1994) 975–978.
[8] G. Ramesh, C.S. Krishnamoorthy, Geometrically non-linear analysis of plates and shallow shells by dynamic relaxation, *Computer Methods in Applied Mechanics and Engineering* 123 (1995) 15–32.
[9] A.A. El Damatty, R.M. Korol, F.A. Mirza, Large displacement extension of consistent shell element for static and dynamic analysis, *Computers & Structures* 62 (6) (1997) 943–960.
[10] D. Yadav, N. Verma, Free vibration of composite circular cylindrical shells with random material properties. Part I: General theory, *Composite Structures* 41 (1998) 331–338.
[11] R. Mao, F.W. Williams, Nonlinear analysis of cross-ply thick cylindrical shells under axial compression, *International Journal of Solids Structures* 35 (17) (1998) 2151–2171.
[12] T.-P. Chang, H.-C. Chang, Nonlinear vibration analysis of geometrically nonlinear shell structures, *Mechanics Research Communications* 27 (2) (2000) 173–180.
[13] L. Vu-Quoc, H. Deng, X.G. Tan, Geometrically exact sandwich shells: The dynamic case, *Computer Methods in Applied Mechanics and Engineering* 190 (2001) 2825–2873.
[14] A.A. Popov, Parametric resonance in cylindrical shells: a case study in the nonlinear vibration of structural shells, *Engineering Structures* 25 (2003) 789–799.
[15] K. Nagai, S. Maruyama, M. Oya, T. Yamaguchi Chaotic oscillations of a shallow cylindrical shell with a concentrated mass under periodic excitation,, *Computers & Structures* 82 (2004) 2607–2619.

[16] N. Krylov, N. Bogolinbov, *Introduction to Nonlinear Mechanics*, Kiev, translation by Princeton University Press, Princeton, NJ, 1943.
[17] R.C. Booton, The analysis of nonlinear control systems with random inputs, *IRE Translation of Circuit Theory* 1 (1954) 32–34.
[18] T.K. Caughey, Equivalent linearization techniques, *Acoustical Society of America, Journal* 35 (1963) 1706–1711.
[19] E.T. Foster, Semilinear random vibration in discrete system, *Journal of Applied Mechanics, ASME* 35 (3) (1968) 560–564.
[20] A.K. Malhotra, J. Penzien, Nondeterministic analysis of offshore structures, *Journal of Engineering Mechanics Division, American Society of Civil Engineers* 96 (1970) 985–1003.
[21] M.K. Kaul, J. Penzien, Stochastic seismic analysis of yielding offshore towers, *Journal of Engineering Mechanics Division, American Society of Civil Engineers* 100 (1974) 1025–1038.
[22] T.S. Atalik, S. Utku, Stochastic linearization of multi-degree-of-freedom nonlinear systems, *International Journal of Earthquake Engineering Structure* 4 (1976) 411–420.
[23] W.D. Iwan, A.B. Mason, Equivalent linearization for systems subjected to nonstationary random excitation, *International Journal of Nonlinear Mechanics* 15 (1980) 71–82.
[24] T.P. Chang, T. Mochio, E. Samaras, Seismic response analysis of nonlinear structures, *Probabilistic Engineering Mechanics* 1 (3) (1986) 157–166.
[25] T.P. Chang, J.L. Ke, Nonlinear random vibration analysis of a nonuniform orthotropic circular plate, *Computer & Structures* 60 (1) (1994) 113–123.
[26] I. Elishakoff, J. Fang, Random vibration analysis of nonlinearly deformed beam by a new stochastic linearization technique, *International Journal of Solids Structures* 32 (11) (1995) 1571–1584.
[27] I. Elishakoff, P. Colajanni, Stochastic linearization critically re-examined, *Chaos, Solitons, Fractals* 8 (12) (1997) 1957–1972.
[28] R.D. Cook, D.S. Malkus, E. Plesha, *Concepts and Application of Finite Element Analysis*, Third ed., Wiley, New York, 1989.
[29] R.W. Clough, J. Penzien, *Dynamics of Structures*, Second ed, McGraw-Hill, New York, 1993.