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Journal of Sound and Vibration 291 (2006) 275–284
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Accuracy of the finite difference method in stochastic setting

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Received 28 April 2005; received in revised form 28 April 2005; accepted 7 June 2005
Available online 6 September 2005

Abstract

In this paper we study the accuracy of the finite difference method when the finite difference method is applied to approximately analyze the structure.

1. Introduction

Analysis of stochastic structures is concerned with determination of the reliability, namely of the probability that a specific mission is fulfilled.

In this context, the unreliability or probability of failure that equals unity minus reliability must be extremely small. This fact immediately poses a question if the approximate methods allow for accurate evaluation of the extremely small unreliability of the structure.

It makes sense to look for the possibility of analytical evaluation of the accuracy associated with the calculation of structural reliability. Discretization error appears to be best to be investigated in the context of the problem that has an exact analytical solution. Fortunately, such a solution is derivable, in the deterministic context, and appear to be in need of the stochastic generalization. Already Lagrange and Rayleigh evaluated the vibration frequencies of the string with N beads of equal mass on it [1] (see also Refs. [2–4]).

Discrete vibration approximations of the uniform beam were studied by Livesley [5], Leckie and Lindberg [6], Yoo and Haug [7] and Weaver [8]. In this study we evaluate analytically

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.06.038

the natural frequency as a function of N-number of segments in the finite different approximation.

The obtained formula is instrumental in analytical evaluation of the structural unreliability and comparing it with the "exact" one, the latter naturally derived within the Bernoulli–Euler theory of beams.

2. Analysis using first-order finite difference method

The differential equation for transversal vibrations of a uniform and homogeneous beam is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where E is the beam modulus of elasticity, I is the inertia moment, A is the cross-sectional area, ρ is the density, x is the axial coordinate, t is the time and w is the transverse displacement.

In the case of free vibrations, we set $w(x,t) = W(x)\sin\omega t$; thus, Eq. (1) can be written in the following form:

$$\frac{\mathrm{d}^4 W}{\mathrm{d}x^2} - \frac{\rho A}{EI}\omega^2 W = 0. \tag{2}$$

The ordinary differential equation (2) can be solved using the first-order central difference method. By this means the differential equation is replaced by an equivalent finite difference equation for any nodal point i under the condition of uniform nodal points spacing. It has the following expression:

$$W_{i-2} - 4W_{i-1} + \left(6 - \frac{\rho A}{EI}h^4\omega^2\right)W_i - 4W_{i+1} + W_{i+2} = 0 \tag{3}$$

in which i is an arbitrary nodal point within the beam, h is the uniform nodal spacing given by the ratio between the total length of the bar L and the number N of segments.

The solution of the difference equation (3) with constant coefficients can be obtained by setting

$$W_i = A\lambda^i. \tag{4}$$

Substituting Eq. (4) into Eq. (3) and manipulating the resulting expression one gets the following equation in λ:

$$\left(\frac{1}{\lambda} + \lambda\right)^2 - 4\left(\frac{1}{\lambda} + \lambda\right) + \left(4 - \frac{\rho A}{EI}h^4\omega^2\right) = 0. \tag{5}$$

Eq. (5) has the solutions:

$$\lambda_{1,2} = 1 - \frac{h^2\omega}{2}\sqrt{\frac{\rho A}{EI}} \pm \sqrt{\left(1 - \frac{h^2\omega}{2}\sqrt{\frac{\rho A}{EI}}\right)^2 - 1},$$

$$\lambda_{3,4} = 1 + \frac{h^2\omega}{2}\sqrt{\frac{\rho A}{EI}} \pm \sqrt{\left(1 + \frac{h^2\omega}{2}\sqrt{\frac{\rho A}{EI}}\right)^2 - 1}. \tag{6}$$

By letting

$$\vartheta=\cos^{-1}\left(1-\frac{h^2}{2}\sqrt{\frac{\rho A}{EI}}\omega\right), \tag{7}$$

the solutions in Eq. (6) can be rewritten in the following form:

$$\lambda_{1,2}=\cos\vartheta\pm\mathrm{i}\sqrt{\sin\vartheta},$$
$$\lambda_{3,4}=2-\cos\vartheta\pm\sqrt{(2-\cos\vartheta)^2-1}. \tag{8}$$

The general solution for W_i takes the form

$$W_i=C_1\cos\mathrm{i}\vartheta+C_2\sin\mathrm{i}\vartheta+C_3\lambda_3^i+C_4\lambda_4^i, \tag{9}$$

in which λ_3 and λ_4 have the expressions in Eq. (8) and C_1, C_2, C_3 and C_4 are arbitrary constants of integration.

To determine the four constants of integration we have to fix the boundary conditions, two at each end of the beam. For a simply supported beam at both ends we have:

$$W_0=W_N=0;\quad W_{-1}=-W_1;\quad W_{N+1}=-W_{N-1}. \tag{10}$$

For a clamped beam at both ends the boundary conditions read:

$$W_0=W_N=0;\quad W_{-1}=W_1;\quad W_{N+1}=W_{N-1}. \tag{11}$$

Let us consider the case of a simply supported beam; the substitution of boundary conditions (10) into Eq. (9) yields:

$$C_1+C_3+C_4=0,$$
$$C_1\cos N\vartheta+C_2\sin N\vartheta+C_3\lambda_3^N+C_4\lambda_4^N=0,$$
$$2C_1\cos\vartheta+C_3\left(\frac{1}{\lambda_3}+\lambda_3\right)+C_4\left(\frac{1}{\lambda_4}+\lambda_4\right)=0,$$
$$2C_1\cos\vartheta\cos N\vartheta+2C_2\cos\vartheta\sin N\vartheta+C_3\lambda_3^N\left(\frac{1}{\lambda_3}+\lambda_3\right)+C_4\lambda_4^N\left(\frac{1}{\lambda_4}+\lambda_4\right)=0. \tag{12}$$

Since the equations are homogeneous, the condition to obtain a solution different from the trivial one is that the determinant of the coefficient of C_1, C_2, C_3 and C_4 must vanish. The determinant of the coefficients after some manipulations is obtained as follows:

$$(\cos\vartheta-1)^2\left[\left(2-\cos\vartheta+\sqrt{\cos\vartheta^2-4\cos\vartheta+3}\right)^N-\left(2-\cos\vartheta-\sqrt{\cos\vartheta^2-4\cos\vartheta+3}\right)^N\right]\sin N\vartheta=0. \tag{13}$$

In order that the product to be zero each factor or one of them must be zero. Thus, either

$$\left(2-\cos\vartheta+\sqrt{3-4\cos\vartheta+\cos\vartheta^2}\right)^N-\left(2-\cos\vartheta-\sqrt{3-4\cos\vartheta+\cos\vartheta^2}\right)^N=0, \tag{14}$$

or

$$\cos\theta - 1 = 0, \tag{15}$$

or

$$\sin N\vartheta = 0. \tag{16}$$

Satisfaction of Eq. (14) leads to

$$\cos\vartheta^2 - 4\cos\vartheta + 3 = 0, \tag{17}$$

whose solutions are cos $\vartheta = 3$ and cos $\vartheta = 1$; Eq. (15) leads to the solution cos $\vartheta = 1$, naturally, cos $\vartheta = 3$ is an inadmissible solution; the condition cos $\vartheta = 1$, bearing in mind Eq. (7) leads to

$$\frac{h^2}{2}\sqrt{\frac{\rho A}{EI}}\omega = 0, \tag{18}$$

that would constitute a nonsensical conclusion $\omega = 0$.

The above implies that we consider Eq. (16) as a physically admissible condition with the solution

$$N\vartheta = k\pi, \quad k = 1, 2, 3, \ldots . \tag{19}$$

Keeping in mind expression (7) we evaluate cos ϑ as follows:

$$\cos\vartheta = \cos\frac{k\pi}{N} = 1 - \frac{h^2}{2}\sqrt{\frac{\rho A}{EI}}\omega. \tag{20}$$

Using trigonometric relations we obtain

$$\sqrt{\frac{\rho A}{EI}}\frac{h^2}{2}\omega = 2\sin^2\frac{k\pi}{2N}, \tag{21}$$

where k should be set equal to unity for the first natural frequency. Since h is the length of each of the N segment and it is equal to the ratio between the length L of the beam and the total number N of segments, the beam's fundamental frequency has the following expression:

$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}}\left(\frac{\sin\pi/2N}{\pi/2N}\right)^2. \tag{22}$$

When N goes to infinity we obtain the exact expression for the frequency of a simply supported beam in free vibrations.

3. Probabilistic analysis

We want to study the case of a beam with elastic modulus as a continuous random variable with given probability density function (PDF) fixing other parameters as deterministic quantities.

Since the main objective is to avoid resonance phenomenon, the natural frequency of the beam must be less than an excitation frequency ω_0,

$$\omega_1 < \omega_0. \tag{23}$$

From the expression of fundamental frequency (Eq. (22)) we see that since E is a random variable so is the fundamental frequency Ω_1, where we reserve capital letters to the random variables. The randomness of E necessities the careful analysis of the possible values of the modulus of elasticity that satisfy inequality in Eq. (23).

We can introduce the reliability R defined as the probability of the event expressed in Eq. (23):

$$R = \text{Prob}(\Omega_1 < \omega_0). \tag{24}$$

Keeping in mind the expression of the approximate natural frequency in Eq. (22), from its definition (Eq. (24)), the reliability can be written as

$$R_{\text{approx}} = \text{Prob}\left[\frac{\pi^4}{L^4}\frac{EI}{\rho A}\left(\frac{\sin \pi/2N}{\pi/2N}\right)^4 < \omega_0^2\right], \tag{25}$$

or

$$R_{\text{approx}} = F_E\left[\frac{L^4}{\pi^4}\frac{\rho A \omega_0^2}{I}\left(\frac{\pi/2N}{\sin \pi/2N}\right)^4\right], \tag{26}$$

where $F_E(e)$ is the probability distribution function of E. Once the expression of the reliability R is known we are able to solve the design problem of the beam, under the consideration that the structure performs satisfactorily if the reliability is greater or equal than a codified reliability value r_0:

$$R \geqslant r_0, \quad 0 < r_0 \leqslant 1. \tag{27}$$

The same problem can be treated in terms of the unreliability of the structure, defined as the probability of failure as follows:

$$P_f = 1 - R \tag{28}$$

and should satisfy the following inequality

$$P_f \leqslant p_0, \tag{29}$$

where p_0 is the tolerable level of probability of failure.

The objective of reliability analysis of a structure is to keep the probability of failure extremely small. If the probability density function for the random variable E is known, we can deduce an expression for a design parameter, like the length of the beam L, from the approximate analysis. It is easy to note that this parameter is a function of the number of elements N and the specified value for r_0.

In the case under consideration we know the exact expression for the natural frequency for the simply supported beam given by the well-known expression:

$$\omega_{\text{exact}} = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}}. \tag{30}$$

This means we are able to evaluate the exact reliability R_{exact} for the beam.

The substitution of the approximate parameter $L = L(N, r_0)$ into the reliability based on utilizing expression (30) leads to a general expression for the "*actual*" reliability, according to parameters N and r_0.

The comparison between the actual reliability and the required reliability r_0 allows us to evaluate the accuracy of FDM, in the stochastic setting.

4. Numerical example

The random variable modulus of elasticity is probabilistically characterized by an exponential distribution:

$$f_E(e) = \begin{cases} 0, & e < 0, \\ a\ \exp[-ae], & e \geqslant 0,\ a > 0, \end{cases}$$

the average $M[E] = 1/a$ and $\mathrm{Var}[E] = 1/a^2$, where $M[\bullet]$ means mathematical expectation.

Therefore, the expression of *approximate* reliability is given by:

$$R_{\text{approx}} = 1 - \exp\left[-\frac{1}{M[E]}\frac{\omega_0^2 \rho A}{I}\frac{L^4}{\pi^4}\left(\frac{\pi/2N}{\sin \pi/2N}\right)^4\right]. \tag{31}$$

In design setting ($R_{\text{appox}} = r_0$), the length of the beam has the following expression:

$$L_{\text{approx}} = L(N, r_0) = \pi \cdot \sqrt[4]{M[E]\frac{I}{\omega_0^2 \rho A}\ln\frac{1}{1 - r_0}}\left(\frac{\sin \pi/2N}{\pi/2N}\right). \tag{32}$$

Taking into account the expression of the exact solution for the natural frequency of a simply supported beam (Eq. (30)), the *exact* reliability is given by

$$R_{\text{exact}} = \mathrm{Prob}\left(\frac{\pi^4}{L^4}\frac{EI}{\rho A} < \omega_0^2\right), \tag{33}$$

or

$$R_{\text{exact}} = 1 - \exp\left[-\frac{1}{M[E]}\frac{L^4}{\pi^4}\frac{\omega_0^2 \rho A}{I}\right]. \tag{34}$$

Keeping in mind the expression of the approximate length of the beam (Eq. (32)), the actual reliability is given by

$$\begin{aligned} R_{\text{actual}} = R_{\text{actual}}(N, r_0) &= R_{\text{exact}}|_{L = L_{\text{approx}}} \\ &= 1 - \exp\left[-\frac{1}{M[E]}\frac{L_{\text{approx}}^4}{\pi^4}\frac{\omega_0^2 \rho A}{I}\right]. \end{aligned} \tag{35}$$

At last the substitution of Eq. (32) in Eq. (35) then yields

$$R_{\text{actual}} = 1 - \exp\left[\left(\frac{\sin \pi/2N}{\pi/2N}\right)^4 \ln(1 - r_0)\right]$$
$$= 1 - (1 - r_0)^{[\{\sin(\pi/2N)]/(\pi/2N)^4}. \tag{36}$$

The evaluation of R_{actual} for increasing number of N leads to values smaller than the codified r_0.

In the following graphics (Fig. 1a–d) percentage errors between R_{actual} and r_0 versus N are depicted, fixing the value r_0 equal, respectively, to 0.90, 0.99, 0.999 and 0.9999.

For $r_0 = 0.90$ the error goes from 1.76% for $N = 5$ ($R_{\text{actual}} = 0.884$) to 0.43% for $N = 10$ ($R_{\text{actual}} = 0.896$) to 0.188% for $N = 15$ ($R_{\text{actual}} = 0.898$).

The corresponding value of R_{actual} for $r_0 = 0.99$ and $N = 5$ is 0.98658 ($\varepsilon = 0.345\%$); always for $N = 5$ and $r_0 = 0.999$ and 0.9999 we have, respectively, $R_{\text{actual}} = 0.9984$ ($\varepsilon = 0.056\%$) and 0.99982 ($\varepsilon = 0.008\%$).

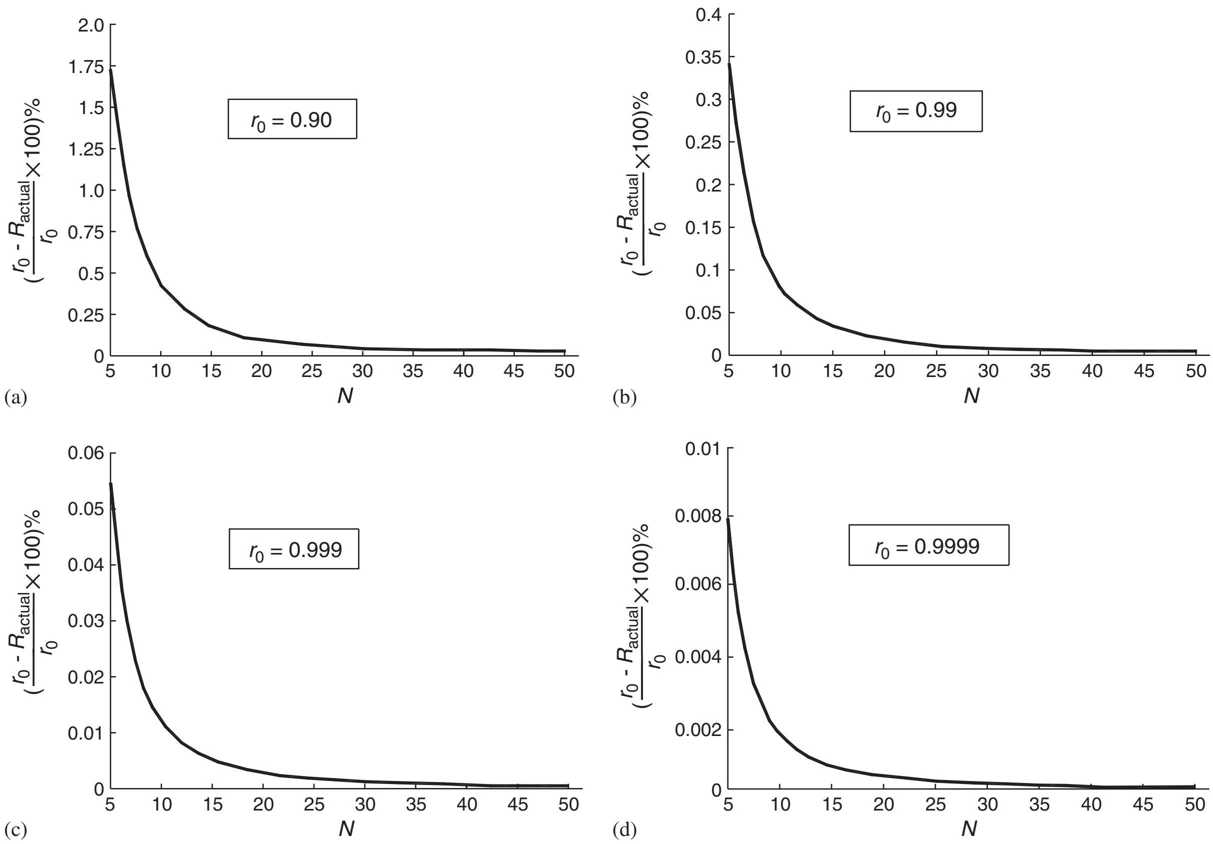

Fig. 1. Percentage error in reliability: evaluation versus the discretization parameter N, for various codified reliabilities r_0: (a) for $r_0 = 0.90$; (b) for $r_0 = 0.99$; (c) for $r_0 = 0.999$; (d) for $r_0 = 0.9999$.

From the analysis of the figures unexpected results are obtained, namely, we observe the decrease of the error (for a fixed value of N) when increasing the codified value of reliability r_0.

The *actu*al probability of failure from its definition (Eq. (28)) is related with the codified probability of failure p_0 by the following relation:

$$P_{f,\,\mathrm{actual}} = p_0^{\delta}, \tag{37}$$

where

$$\delta = \left(\frac{\sin \pi/2N}{\pi/2N}\right)^4. \tag{38}$$

Note that the exponent δ in expression (38) for increasing number of segments N, takes values close but always smaller than unity; since p_0 smaller than unity, the actual probability of failure is always greater than the requested value.

Graphics in Fig. 2 depict the variation of the actual probability of failure versus N for various levels of the codified value p_0 for the probability of failure.

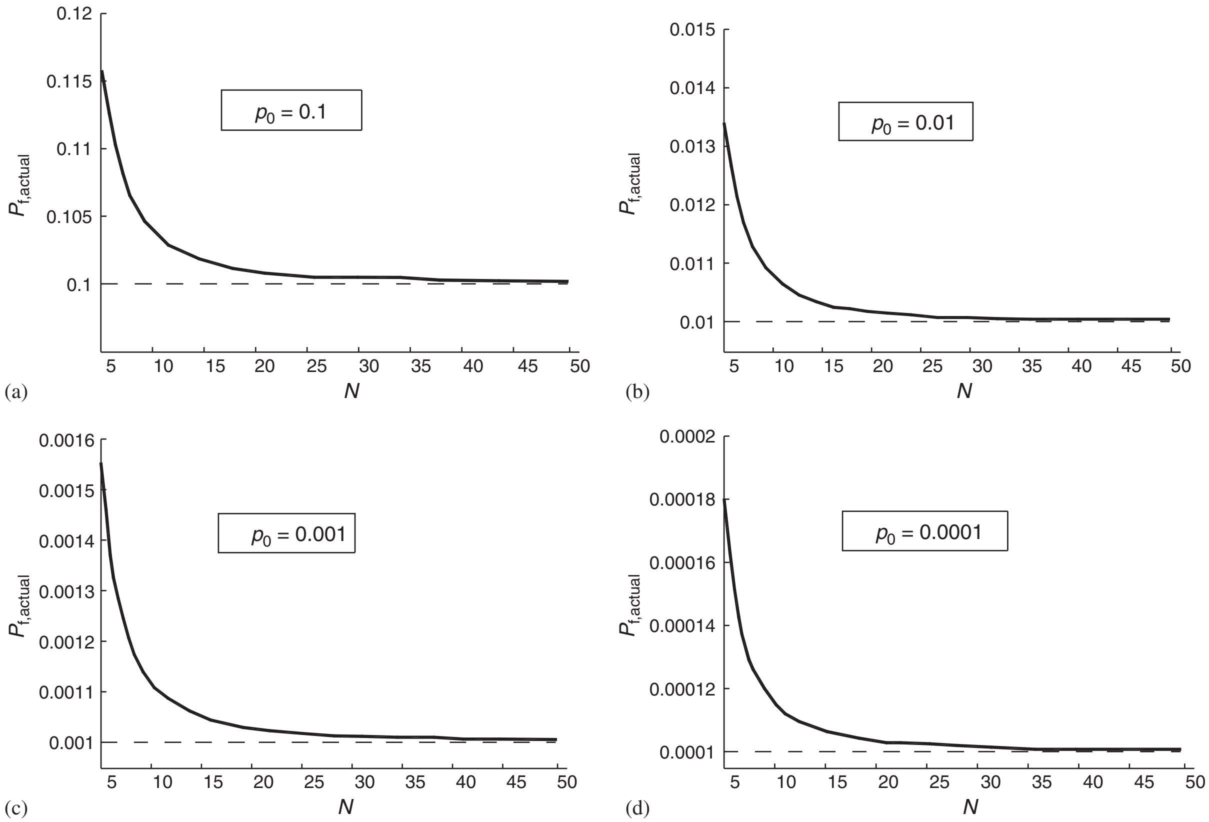

Fig. 2. Variation of the *actual* probability of failure versus N, for various levels of p_0: (a) for $p_0 = 0.1$; (b) for $p_0 = 0.01$; (c) for $p_0 = 0.001$; (d) for $p_0 = 0.0001$.

Once one fixes the codified value for $p_0 = 0.1$, one gets $P_{f,\mathrm{actual}} = 1.15847p_0$ for $N = 5$ ($\delta = 0.93612$), $P_{f,\mathrm{actual}} = 1.03831p_0$ for $N = 10$ ($\delta = 0.983672$) and $P_{f,\mathrm{actual}} = 1.01692p_0$ for $N = 15$ ($\delta = 0.992713$).

For $p_0 = 0.001$, we obtain $P_{f,\mathrm{actual}} = 1.55471p_0$ for $N = 5$, $P_{f,\mathrm{actual}} = 1.1194p_0$ for $N = 10$ and $P_{f,\mathrm{actual}} = 1.05162p_0$ for $N = 15$.

When increasing the number of segments N, for a codified value of p_0, the value of $P_{f,\mathrm{actual}}$ decreases, maintaining its value greater than p_0, reaching the equality $P_{f,\mathrm{actual}} = p_0$ only in the case when N tends to infinity.

The value of actual probability of failure should be less than the required one, but results demonstrate that the values obtained are not in safe side for.

Using other distributions (Rayleigh distribution or an uniform distribution) for the random variable representing the modulus of elasticity leads to analogous qualitative results.

5. Discussion and conclusion

According to Ditlevsen [9], "decisions based on structural reliability analysis depend, naturally, on the mathematical model which is set up for the analysis by the engineer. Moreover, if careful real life decisions are to be made, it is necessary that considerations about the uncertainty of the model itself are quantified within the model". This paper investigates the accuracy of the finite difference method by analytical means utilizing the problem which allows the "closed-form" solution of the finite difference equations. The analogous study on the finite element probabilistic vibration analysis is underway and will be reported elsewhere.

Acknowledgements

Isaac Elishakoff appreciates the partial financial support of the J. M. Rubin foundation of the Florida Atlantic University. Roberta Santoro appreciates the Ministry of Education of Italy for financial support of her stay at the Florida Atlantic University as a Visiting Scholar.

References

[1] A.M. Filimonov, P.F. Kurchanov, A.D. Myshkis, Some unexpected results in the classical problem of vibrations of the string with *n* beads when n is large, *Comptes Rendus de l' Academie des Sciences Paris*, Serie I 313 (1991) 961–965.

[2] A.M. Filimonov, Continuous approximations of difference operators, *Journal of Difference Equations and Applications* 2 (1996) 411–422.

[3] I.V. Andianov, J. Awrejcewicz, On the Average continuous representation of an elastic discrete medium, *Journal of Sound and Vibration* 264 (2003) 1187–1194.

[4] M. Gürgöze, A. Özer, Forced response of uniform *n*-mass oscillators and interesting series, *Journal of Sound and Vibration* 173 (2) (1994) 283–288.

[5] R.K. Livesley, The equivalence of continuous and discrete mass distributions in certain vibration problems, *Quarterly Journal of Mechanics and Applied Mathematics* 8 (3) (1995) 353–360.

[6] F.A. Leckie, G.M. Lindberg, The effect of lumped parameters on beam frequencies, *Aeronautical Quarterly* 14 (1963) 1910–1918.
[7] W.S. Yoo, E.J. Haug, Dynamics of articulate structures, Part 1: theory, *Journal of Structural Mechanics* 14 (1) (1986) 105–126.
[8] W. Weaver Jr, Dynamics of discrete-parameter structures, in: W.A. Shaw (Ed.), *Development on Theoretical and Applied Mechanics*, vol. 2, Pergamon Press, New York, 1965, pp. 629–651.
[9] O. Ditlevsen, Model uncertainty in structural reliability, *Structural Safety* 1 (1982) 73–86.