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Vibration of viscoelastic beams subjected to an eccentric compressive force and a concentrated moving harmonic force

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Abstract

The problem of lateral vibration of a beam subjected to an eccentric compressive force and a harmonically varying transverse concentrated moving force is analyzed within the framework of the Bernoulli–Euler beam theory. The Lagrange equations are used to examine the free vibration characteristics of an axially loaded beam and the dynamic response of a beam subjected to an eccentric compressive force and a moving harmonic concentrated force. The constraint conditions of supports are taken into account by using the Lagrange multipliers. In the study, trial function denoting the deflection of the beam is expressed in a polynomial form. By using the Lagrange equations, the problem is reduced to the solution of a system of algebraic equations. Results of numerical simulations are presented for various combinations of the value of the eccentricity, the eccentric compressive force, excitation frequency and the constant velocity of the transverse moving harmonic force. Convergence studies are made. The validity of the obtained results is demonstrated by comparing them with exact solutions based on the Bernoulli–Euler beam theory obtained for the special cases of the investigated problem.

1. Introduction

Lateral vibration of beams under axial loading has been of practical interest in recent years. The influences of axial loading on the vibration characteristics of beams have been well investigated.

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The earliest work on the effect of axial force on lateral vibration of beams was done by Timoshenko and Young [1]. Fryba [2] presented in his book analytical solution, for vibration of a simply supported beam subjected to an axial force and a moving force. Bokaian [3,4] investigated the effect of the compressive and tensile axial forces on natural frequencies and mode shapes of a uniform single-span beam for various end conditions. Virgin and Plaut [5] determined analytically the steady-state linear response of uniform elastic beams subjected to a distributed, harmonically varying transverse force. The transverse force varies harmonically in time and spatially is distributed along the beam, and the static axial force may be tensile or compressive. Kukla [6] analyzed the free vibration of axially loaded beams with concentrated masses and intermediate elastic supports, using Green functions. In the case of a beam with N elastic supports and concentrated masses, the frequency equation was expressed by means of an Nth order determinant. A recent paper by Luo [7] analyzed the lateral vibration of an infinite uniform Bernoulli–Euler beam under axial loads by setting up a model of infinite beam with a harmonically varying transverse concentrated force at the centre of the beam. Nallim and Grossi [8] presented a simple variational approach based on the use of the Rayleigh–Ritz method with the characteristic orthogonal polynomial shape functions for the determination of free vibration frequencies of beams with several complicating effects. Lee [9] utilized Hamilton's principle to solve the dynamic response of a beam with intermediate point constraints subjected to a moving load by using the vibration modes of a simply supported beam as the assumed modes. Abu-Hilal and Mohsen [10] studied the dynamic response of elastic homogenous isotropic beams with different boundary conditions subjected to a constant force moving with accelerating, decelerating and constant velocity types of motion. Zheng et al. [11] considered the dynamic response of the continuous beams subjected to moving loads by using the modified beam vibration functions. Dugush and Eisenberg [12] examined vibrations of non-uniform continuous beams under moving loads by using both the modal analysis method and the direct integration method. Zhu and Law [13] analyzed the dynamic response of a continuous beam under moving loads using Hamilton's principle and eigenpairs obtained by the Ritz method.

In order to use the cross section of beams effectively, pre-tensioning and post-tensioning are used for long-span beams in civil engineering applications. When pre-tensioning or post-tensioning is used, the tensile stresses of the beams are reduced or vanished completely, which are desired situations in the design of long-span bridges. Although the present problem is of great practical interest to engineers designing structural and mechanical systems such as prestressed beams of viaducts of roadways, railways and bridges, it has not been dealt with as far as the authors know.

In the present study, the problem is analyzed by using the Lagrange equations with the trial function in polynomial form denoting the deflection of the beam for determining the dynamic response of a beam subjected to an eccentric compressive force and a concentrated moving harmonic force with constant axial speed. The eccentric compressive force can be resolved into a force and a couple at the center of the cross section of the beam. The constraint conditions of the supports are taken into account by using Lagrange multipliers. By using the Lagrange equations, the problem is reduced to a system of algebraic equations, and they are solved by using the direct time integration method of Newmark [14]. The convergence study is based on the numerical values obtained for various numbers of polynomial terms. Results given in this paper may be useful for further investigations in this field.

2. Theory and formulations

Consider a simply supported beam of length L, modulus of elasticity E, moment of inertia I, and mass of per unit length ρ. The beam is subjected to an eccentric compressive force and a transverse concentrated moving harmonic force, as shown in Fig. 1. The constraint conditions are satisfied by using Lagrange multipliers. When there is no concentrated moving harmonic load on the beam at time $t = 0$, the beam is at rest in a bent configuration under the eccentric compressive load. In this study, total displacements, including the displacements of the beam at rest, are considered. A harmonic force $Q(t)$ is applied in the w-direction, moving from left to right with a prescribed constant speed in the axial direction. The assumptions made in the following formulation are that transverse deflections are small, so that the dynamical behavior of the beam is governed by the Bernoulli–Euler beam theory. Moreover, all the transverse deflections occur in the same plane, defined by the x and w axes. Origin of the x and w axes is chosen at the midpoint of the total length of the beam, as shown in Fig. 1.

According to the Bernoulli–Euler beam theory, the elastic strain energy of the beam at any instant due to bending is expressed as an integral in Cartesian coordinates:

$$U = \frac{1}{2}\int_{-L/2}^{L/2} E\,I(x)\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 \mathrm{d}x, \tag{1}$$

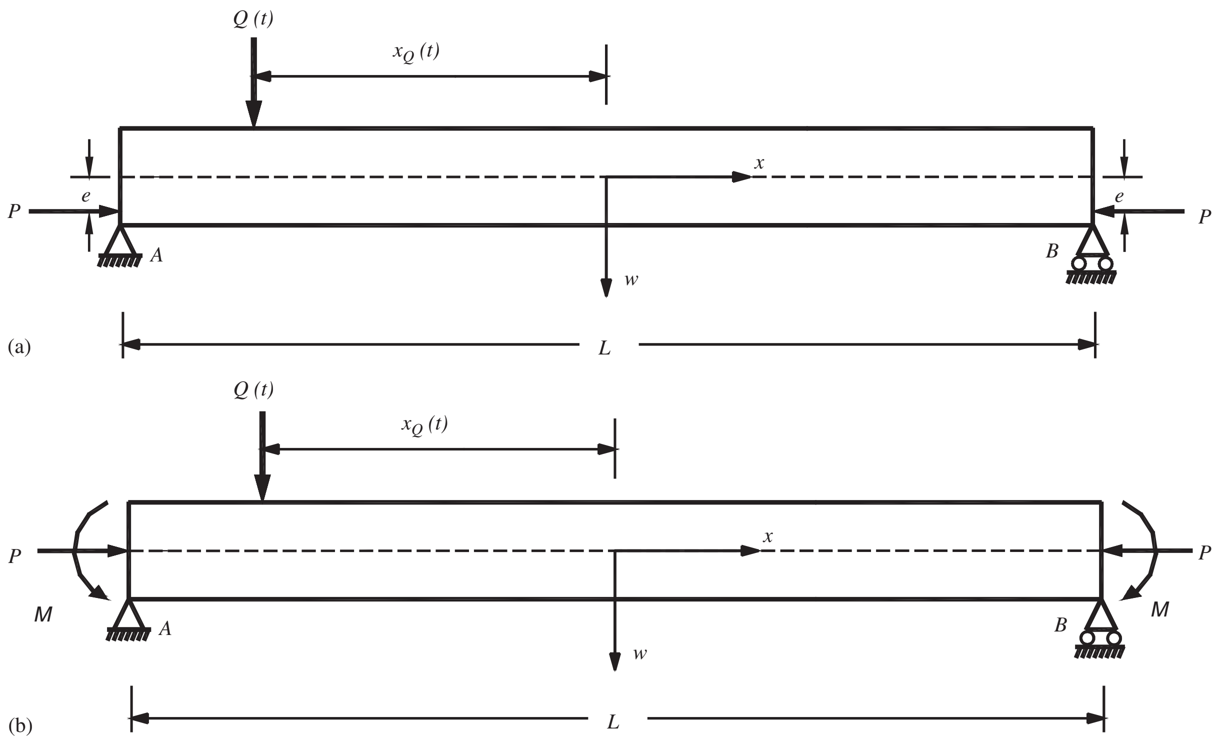

Fig. 1. (a) A simply supported beam subjected to an eccentric compressive force and a moving harmonic force, (b) transferring the eccentric compressive force to the center of cross section of the beam as a compressive axial force and a couple.

where E, $I(x)$ and $w(x,t)$ are the Young's modulus, the moment of inertia of the cross section of the beam and the total displacement function of the beam.

Neglecting rotary inertia effects, the kinetic energy of the beam at any instant is

$$T = \frac{1}{2}\int_{-L/2}^{L/2} \rho \left(\frac{\partial w(x,t)}{\partial t}\right)^2 \mathrm{d}x, \tag{2}$$

where ρ is the mass of the beam per unit length. The Kelvin–Voigt model for the material is used. In this case, the dissipation function of the beam at any instant is

$$R = \frac{1}{2}\int_{-L/2}^{L/2} r_i \left(\frac{\partial^2 \dot{w}(x,t)}{\partial x^2}\right)^2 \mathrm{d}x, \tag{3}$$

$$r_i = \gamma_2 E\, I(x), \tag{4}$$

where r_i and γ_2 are the coefficient of internal damping of the viscoelastic beam and proportionality constant of internal damping, respectively. By transferring the eccentric compressive force to the gravity center of the cross section of the beam, an axial force and a couple are obtained, which are shown in Fig. 1b. The potential of the external forces and couples at any instant can be written as follows

$$V = -Q(t)\, w\left(x_Q(t), t\right) - \frac{P}{2}\int_{-L/2}^{L/2} \left(\frac{\partial w(x,t)}{\partial x}\right)^2 \mathrm{d}x + M\frac{\partial w(x_A,t)}{\partial x} - M\frac{\partial w(x_B,t)}{\partial x}, \tag{5a}$$

$$Q(t) = F\,\sin(\Omega t), \tag{5b}$$

$$M = Pe, \tag{5c}$$

where F is the amplitude of the moving harmonic force, Ω is the excitation frequency of the moving harmonic force, P is the axial compressive force, e is the eccentricity of the eccentric compressive force and $x_Q(t)$ is the location of the moving harmonic force at any instant and expressed as

$$x_Q(t) = vt - L/2, \quad -\frac{L}{2} \leqslant x_Q(t) \leqslant \frac{L}{2}, \quad 0 \leqslant t \leqslant \frac{L}{v}, \tag{6}$$

where v is the velocity of the moving harmonic force along the axial direction. The expressions of Eq. (5a) are calculated at any instant. In other words, they are not related with the work done between time interval $t = 0$ and any time t. In this sense, in the first term of Eq. (5a), $Q(t)$ is the magnitude of the force at any instant and $\mathrm{w}(x_Q(t), t)$ is the displacement just under the load again at the same instant. The functional of the problem is given below:

$$I = T - (U + V). \tag{7}$$

It is known that some expressions satisfying geometrical boundary conditions are chosen for $w(x,t)$ and, by using the Lagrange equations, the natural boundary conditions are also satisfied. Therefore, by using the Lagrange equations and by assuming the displacement function $w(x,t)$ to be representable by a series of admissible functions and adjusting the coefficients in the series to satisfy the Lagrange equations, an approximate solution is found for the displacement function. By applying the Lagrange equations, the trial function $w(x,t)$ is approximated by space-dependent polynomial terms x^0, x^1, x^2, ..., x^{M-1}, and time-dependent generalized displacement coordinates $a_m(t)$. Thus,

$$w(x,t)=\sum_{m=1}^{M} a_m(t)x^{m-1}, \tag{8}$$

where $w(x,t)$ is the dynamic response of the beam subjected to an eccentric compressive force and a concentrated moving harmonic transverse force. The constraint conditions of the supports are satisfied by using the Lagrange multipliers. The constraint conditions are

$$\beta_i w(x_{Si},t)=0, \quad i=1,2, \tag{9}$$

where x_{Si} denotes the location of the ith support. In Eq. (9), β_i quantities are the Lagrange multipliers and in the considered problem they are support reactions. The Lagrange multipliers formulation of the considered problem requires us to construct the Lagrangian functional as follows:

$$L=I+\beta_i w(x_{Si},t), \quad i=1,2, \tag{10}$$

which attains its stationary value at the solution $(w(x_{Si},t),\beta_i)$. The generalized damping force Q_{D_r} can be obtained from the dissipation function by differentiating R with respect to $\dot{a}_k$

$$Q_{D_r}=-\frac{\partial R}{\partial \dot{a}_k}, \quad k=1,2,3,\ldots,M+2, \tag{11}$$

where the dot above is the derivative with respect to time. Then, using the Lagrange equations

$$\frac{\partial L}{\partial a_k}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{a}_k}+Q_{D_r}=0, \quad k=1,2,3,\ldots,M+2, \tag{12}$$

and introducing

$$a_{M+i}=\beta_i, \quad i=1,2 \tag{13}$$

yields the following equation:

$$\boldsymbol{A}\boldsymbol{a}+\boldsymbol{B}\dot{\boldsymbol{a}}+\boldsymbol{C}\ddot{\boldsymbol{a}}=\boldsymbol{D}, \tag{14}$$

where

$$\begin{aligned}
A_{km}&=\int_{-L/2}^{L/2} EI(x)\left(x^{k-1}\right)''\left(x^{m-1}\right)''\,\mathrm{d}x-P\int_{-L/2}^{L/2}\left(x^{k-1}\right)'\left(x^{m-1}\right)'\,\mathrm{d}x, \quad k,m=1,2,3,\ldots,M,\\
A_{km}&=x_{Sm}^{k-1}, \quad k=1,\ldots,M, \quad m=M+1,\ M+2,\\
A_{km}&=x_{Sk}^{m-1}, \quad k=M+1,\ M+2, \quad m=1,\ldots,M,\\
A_{km}&=0, \quad k,m=M+1,\ M+2,
\end{aligned}$$

$$
\begin{aligned}
B_{km} &= \int_{-L/2}^{L/2} r_i \left(x^{k-1}\right)'' \left(x^{m-1}\right)'' \mathrm{d}x, \quad k,m = 1,2,3,\ldots,M,\\
B_{km} &= 0, \quad k,m = M+1,\ M+2,\\
C_{km} &= \int_{-L/2}^{L/2} \rho x^{k-1} x^{m-1}\, \mathrm{d}x, \quad k,m = 1,2,3,\ldots,M,\\
C_{km} &= 0, \quad k,m = M+1,\ M+2,\\
D_k &= Q x_Q^{k-1} - \mathsf{M}\left(x_A^{k-1}\right)' + \mathsf{M}\left(x_B^{k-1}\right)', \quad k = 1,2,3,\ldots,M,\\
D_k &= 0, \quad k = M+1,\ M+2,
\end{aligned} \tag{15}
$$

where $(x^k)'$, $(x^k)''$ are the first and the second derivatives of x^k. **A**, **B**, **C** are the matrices that do not depend on time, but **D** depends on time; namely, x_Q^k depends on time.

For free vibration analysis, the time-dependent generalized displacement coordinates can be expressed as follows:

$$a_m(t) = \bar{a}_m \mathrm{e}^{\mathrm{i}\omega t}. \tag{16}$$

By substituting Eq. (16) into Eq. (14), when the damping matrix of the beam **B** and the external forces matrix **D** are taken as zero in Eq. (14), this situation results in a set of linear homogeneous equations that can be expressed in the following matrix form:

$$\mathbf{A}\bar{\mathbf{a}} - \omega^2 \mathbf{C}\bar{\mathbf{a}} = \mathbf{0}, \tag{17}$$

where ω is the natural frequency of the beam. By using the direct time integration method of Newmark [14], Eq. (14) is solved and, a_m, $\dot{a}_m$, $\ddot{a}_m$ and β_i coefficients are obtained for any instant t between $0 \leqslant t \leqslant L/v$. Then, the displacements, velocities and accelerations at the considered point and instant can be determined by using Eq. (8).

3. Numerical results

The dynamic response of a simply supported beam subjected to an eccentric compressive force and a concentrated moving harmonic transverse force is calculated numerically. The total length of the beam is $L = 20\,\mathrm{m}$, mass of the beam per unit length is $\rho = 1000\,\mathrm{kg/m}$, inertia moment of the cross section is $I = 0.08824\,\mathrm{m}^4$, the Young's modulus is $E = 34{,}000\,\mathrm{MPa}$ ($EI = 3 \times 10^9\,\mathrm{Nm}^2$) and magnitude of the concentrated moving harmonic force is $F = 100\,\mathrm{kN}$. The numerical integration is performed by using Gaussian quadrature. In the following figures, $w(x_Q(t), t)$ is the deflection under the moving harmonic force. Damping ratio is considered in the form given in Ref. [10] as follows:

$$\xi = \frac{\gamma_1 + \gamma_2 \omega_k^2}{2\omega_k}. \tag{18}$$

In all the calculations, the value of ξ is taken constant as 0.05. In Eq. (18), γ_1 is the proportionality constant of the external damping, γ_2 is the proportionality constant of the internal damping, and ω_k is the natural circular frequency of the kth mode, respectively. It is known that external damping is very small with respect to internal damping. Therefore, external damping is ignored in this study. In the case of $\xi = 0.05$, by using Eqs. (4) and (18), the damping coefficient r_i is obtained.

As far as the authors know, there are no existing results for beams subjected to eccentric compressive forces and moving harmonic transverse forces. Therefore, a short investigation of free vibration of the considered simply supported beam subjected to an axial force is done for comparing the obtained results with the existing exact results of free vibration characteristics of the simply supported beam subjected to an axial force. The natural frequencies of the beam are determined by calculating the eigenvalues ω of the frequency by Eq. (17). In Table 1, the calculated natural frequencies are compared with those of Timoshenko and Young [1] presented by the formula $\omega_i = \left(ai^2\pi^2/l^2\right)\sqrt{1 - Pl^2/i^2EI\pi^2}$ (where $a = \sqrt{EI/\rho}$) and Fyrba [2]. The convergence is tested by taking the number of terms $(M+2) = 12, 14, 16, 18$. It is seen that the present converged values show excellent agreement with those of Timoshenko and Young [1], and Fyrba [2]. Also, it is seen from Table 1 that compression tends to reduce the free vibration frequencies while tension tends to increase them, as was observed in Refs. [1–5].

It is observed from Table 1 that the natural frequencies decrease as the number of the polynomial terms increase: It means that the convergence is from above. By increasing the number of the polynomial terms, the exact value can be approached from above. It should be remembered that energy methods always overestimate the fundamental frequency; so with more refined

Table 1
Convergence study of natural frequencies ω_i of the simply supported beam and comparison of the obtained results with the existing exact results for the special case of the problem

Axial force (kN)	Determinant size	ω_1	ω_2	ω_3	ω_4	ω_5
$P = 0$	10	42.7362	170.9466	384.7235	684.7015	1155.6048
	12	42.7362	170.9466	384.6307	683.7969	1073.2305
	14	42.7362	170.9466	384.6295	683.7860	1068.5211
	16	42.7362	170.9466	384.6295	683.7860	1068.4161
$P = 1 \times 10^3$	10	42.4468	170.6573	384.4354	684.4134	1155.3253
	12	42.4468	170.6573	384.3414	683.5076	1072.9425
	14	42.4468	170.6573	384.3414	683.4979	1068.2318
	16	42.4468	170.6573	384.3414	683.4979	1068.1268
$P = -1 \times 10^3$	10	43.0242	171.2347	385.0123	684.9901	1155.8836
	12	43.0242	171.2347	384.9183	684.0855	1073.5186
	14	43.0242	171.2347	384.9183	684.0745	1068.8092
	16	43.0242	171.2347	384.9183	684.0745	1068.7042
Timoshenko and Young [1]	$P = 0$	42.7366	170.9465	384.6297	683.7862	1068.4160
	$P = 1 \times 10^3$	42.4469	170.6576	384.3409	683.4975	1068.1273
	$P = -1 \times 10^3$	43.0243	171.2349	384.9183	684.0748	1068.7046
Fyrba [2]	$P = 0$	42.7366	170.9465	384.6297	683.7862	1068.4160
	$P = 1 \times 10^3$	42.4469	170.6576	384.3409	683.4975	1068.1273
	$P = -1 \times 10^3$	43.0243	171.2349	384.9183	684.0748	1068.7046

analyses, the exact value can be approached from above. Convergence study indicates that the calculated values are converged to within three significant figures on the right-hand side of the decimal point.

In Ref. [8], dimensionless frequency coefficients λ_i for the first five modes are calculated for the simply supported beam which is subjected to an axial tensile force. In addition, dimensionless axial force coefficient S is also defined for the free vibration analysis. In Ref. [8], the dimensionless frequency and axial force coefficients λ_i and S are defined as $\lambda_i = \sqrt{(\rho A/EI)}\omega_i L^2$ and $S = sL^2/EI$, where ρ, A and s are mass density, cross-sectional area and the axial tensile force, respectively. The other parameters are already defined in the present study. In Ref. [8], the dimensionless frequency coefficients λ_i are obtained in tabular form for $S = 5$, 10, 20.

In order to compare the obtained results with the above-mentioned results of Ref. [8], the parameters P, EI and L are selected conveniently for obtaining the values of S as 5, 10, 20, which were given in Ref. [8]. For this purpose, the axial tensile force P is taken as 3.75×10^4, 7.5×10^4, 15×10^4 kN, the bending rigidity EI is taken as $EI = 3 \times 10^9\,\mathrm{N\,m^2}$, the span L of the beam is taken as $L = 20$ m. Then, the natural frequencies are obtained by using Eq. (17). These natural frequencies are substituted into the dimensionless frequency coefficient λ_i defined in Ref. [8]. The obtained dimensionless frequency coefficients are compared with those of Ref. [8] in Table 2.

Another example for comparing the obtained results with the exact solution, which is the special case of the present study, is a beam subjected to only moving harmonic force with constant velocity investigated by Timoshenko and Young [1]. The deflections of the beam are presented by the following formula:

$$y = \frac{PL^3}{EI\pi^4}\sum_{i=1}^{\infty}\sin\frac{i\pi x}{l}\left\{\frac{\sin\left(\frac{i\pi v}{l}+\omega\right)t}{i^4-(\beta+i\alpha)^2}+\frac{\sin\left(\frac{i\pi v}{l}-\omega\right)t}{i^4-(\beta-i\alpha)^2}-\frac{\alpha}{i}\left(\frac{\sin\frac{i^2\pi^2 at}{l^2}}{-i^2\alpha^2+(i^2-\beta)^2}+\frac{\sin\frac{i^2\pi^2 at}{l^2}}{-i^2\alpha^2+(i^2+\beta)^2}\right)\right\},$$

where $\alpha = vl/\pi a$ ($a = \sqrt{EI/\rho}$) is the ratio of the period $\tau = 2l^2/\pi a$ of the fundamental type of vibration of the beam to twice the time $\tau_1 = l/v$, it takes the force P to pass over the beam, $\beta = \tau/\tau_2$ is the ratio of the period of the fundamental type of vibration of the beam to the period $\tau_2 = 2\pi/\omega$ of the harmonic force. However, in the above formula, the harmonic force was taken

Table 2
Comparison of the obtained dimensionless frequency coefficients λ_i of the simply supported beam with the existing results of Ref. [8]

λ_i	$S = 5$	$S = 10$	$S = 20$
λ_1 [8]	12.114335	14.003754	17.169775
Present result	12.114334	14.003763	17.169723
λ_2 [8]	41.903908	44.196489	48.457340
Present result	41.903886	44.196481	48.457326
λ_3 [8]	91.292214	93.693136	98.319199
Present result	91.292190	93.693113	98.319196
λ_4 [8]	160.39407	162.83758	167.61621
Present result	160.39405	162.83677	167.61548
λ_5 [8]	249.23288	252.27915	256.54422
Present result	249.22752	251.71427	256.54520

as $Q(t) = P \cos \omega t$, where ω is the angular velocity of the driving wheel, and, the damping of the beam was neglected. Therefore, in order to compare the present results with the above-mentioned exact solution, the harmonic force is taken as $Q(t) = F \cos(\Omega t)$ instead of the expression in Eq. (5b), and the damping of the beam is neglected.

From here on, in the calculation of the results of the present study, 14 terms of the polynomial series are used, namely the size of the determinant is 16×16.

As seen from Fig. 2, the obtained results are in excellent agreement with the results of Timoshenko and Young [1] in the case of moving force ($\Omega = 0$), and in the case of moving harmonic force ($\Omega = 40\,\text{rad/s}$). In Fig. 2, it is almost imposible to realize the difference between the curves of Timoshenko and Young [1], and the present study. They are very close to each other.

In Fig. 3, the effect of axial compressive force on the lateral vibrations of a beam under the moving harmonic force is analyzed for various values of excitation frequencies Ω at constant speed $v = 20\,\text{m/s}$. It is seen from this figure that the effect of axial compressive force is very small and can be ignored. Moreover, as seen in Fig. 3c, for $\Omega = 40\,\text{rad/s}$, which is very close to the first eigenvalue ($\omega_1 = 42.7366\,\text{rad/s}$) of the considered beam, the response of the beam is very large relative to the other cases and this situation resembles the resonant case. After the resonant frequency of the external harmonic force, the displacements decrease with increase in the frequency of the external harmonic force in the considered frequency range.

Figs. 4 and 5 show the effect of the eccentric compressive force for various compressive forces and excitation frequency Ω at constant speed $v = 20\,\text{m}$ for $e = 0.25$ and $0.5\,\text{m}$, respectively. In contrast to the axial force, the eccentric compressive force has a strong effect on the response of the beam as seen in Figs. 4 and 5, which show almost the same characteristics. This is because of the end moments, which are due to the eccentricity of the compressive force. The eccentricity of the compressive force affects the behavior of the beam significantly. The displacements given in Figs. 4, 5 and 7–12 can be obtained by superposing the displacements caused by the end moments shown in Fig. 1b and the displacements caused by the axial load P and moving harmonic load $Q(t)$ shown in Fig. 1b. When $P = 0$, this situation is the same with the moving harmonic force situation without compressive force. This case is shown in Figs. 4 and 5 with solid lines. With increase in the compressive force, namely increase in the end moments due to the eccentricity of the compressive force, positive deflections decrease and the absolute values of negative deflections increase. The absolute values of deflections increase remarkably with increase in the compressive force. For the values of $\Omega = 60, 80\,\text{rad/s}$ and after $P = 1000\,\text{kN}$, all displacements of the beam occur in the negative region and they reach very high values with increase in the value of axial force P. This is a desirable situation in the prestressed concrete beams of viaducts and bridges. Meanwhile, it is important to note that the effect of the negative displacement, which means tension stresses at the upper side of neutral axis of the beam, should be taken into account in the design of such systems.

The deflections increase with increase in the force frequency Ω until the first characteristic frequency value of the beam. In the values of force frequency Ω greater than the characteristic frequency value of the beam ω_1, the deflections become small with increase in Ω. The above-mentioned comments are valid for Fig. 5 for which e is taken as $0.5\,\text{m}$. It can be noted for this case that the absolute values of the negative displacements become approximately two times larger than the deflection values of Fig. 4.

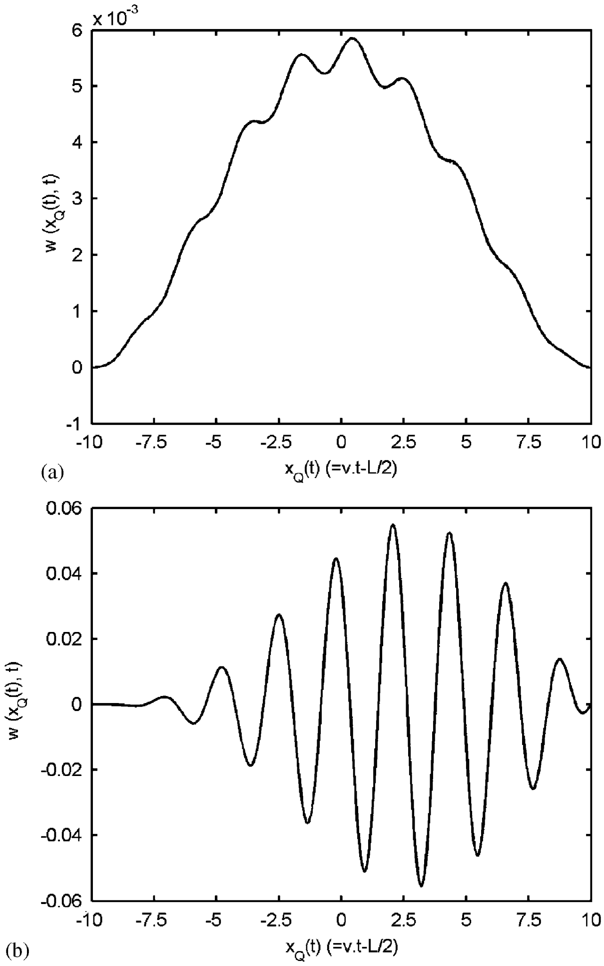

Fig. 2. Deflections under the moving harmonic force for constant velocity $v = 15\,\mathrm{m/s}$ for the excitation frequencies: (a) $\Omega = 0$, (b) $\Omega = 40\,\mathrm{rad/s}$, (——) Ref. [1], (– – – –) present study.

Figs. 6–8 show the effect of the velocity of the moving harmonic force on the deflections of the beam for various values of the excitation frequency Ω and velocities $v = 10, 20, 40, 60, 80\,\mathrm{m/s}$ at constant eccentric compressive force $P = 1500\,\mathrm{kN}$ for $e = 0$, 0.25, 0.5 m, respectively. It can be clearly seen from Figs. 6–8 that, velocity has an important effect on the response of the beam. It is seen from Fig. 6a that, when $\Omega = 0$, the maximum deflection is obtained for $v = 80\,\mathrm{m/s}$. It can be concluded from Figs. 6–8 that, when the value of the force frequency Ω is close to the first eigenvalue of the beam, the lower values of the velocity cause greater positive values of deflections and greater absolute values of negative deflections than those of the higher values of velocity. Otherwise, higher values of velocities give higher absolute values of deflections than those of the lower values of velocity. Also, the cycles of the deflections in the considered beam length decrease

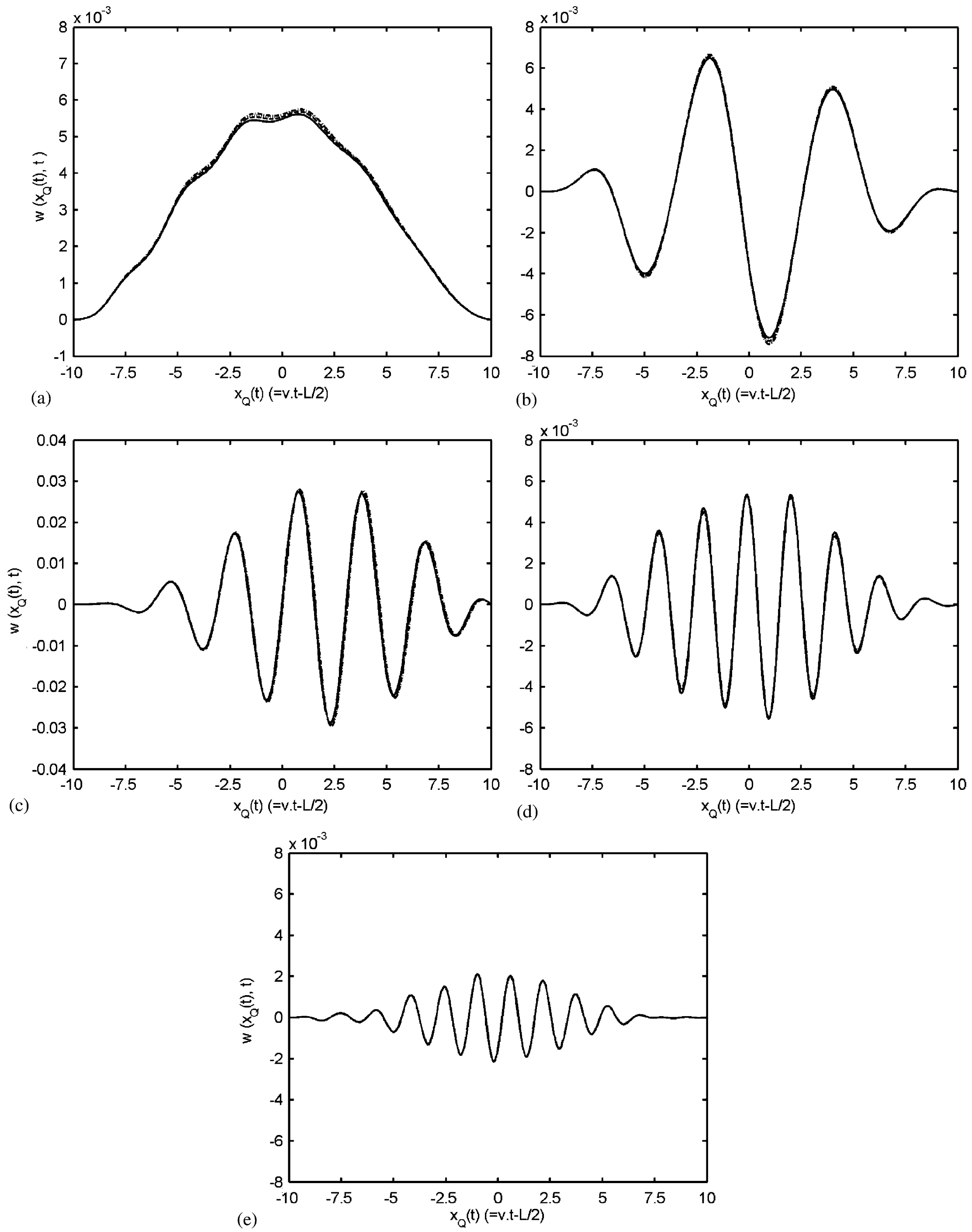

Fig. 3. Deflections under the moving harmonic force for various compressive forces P for $e = 0$, $v = 20\,\mathrm{m/s}$ for (a) $\Omega = 0$, (b) $\Omega = 20\,\mathrm{rad/s}$, (c) $\Omega = 40\,\mathrm{rad/s}$, (d) $\Omega = 60\,\mathrm{rad/s}$, (e) $\Omega = 80\,\mathrm{rad/s}$, (——) $P = 0$, (– – –) $P = 1000\,\mathrm{kN}$, (· · · · · · ·) $P = 1500\,\mathrm{kN}$, (– · – · –) $P = 2000\,\mathrm{kN}$.

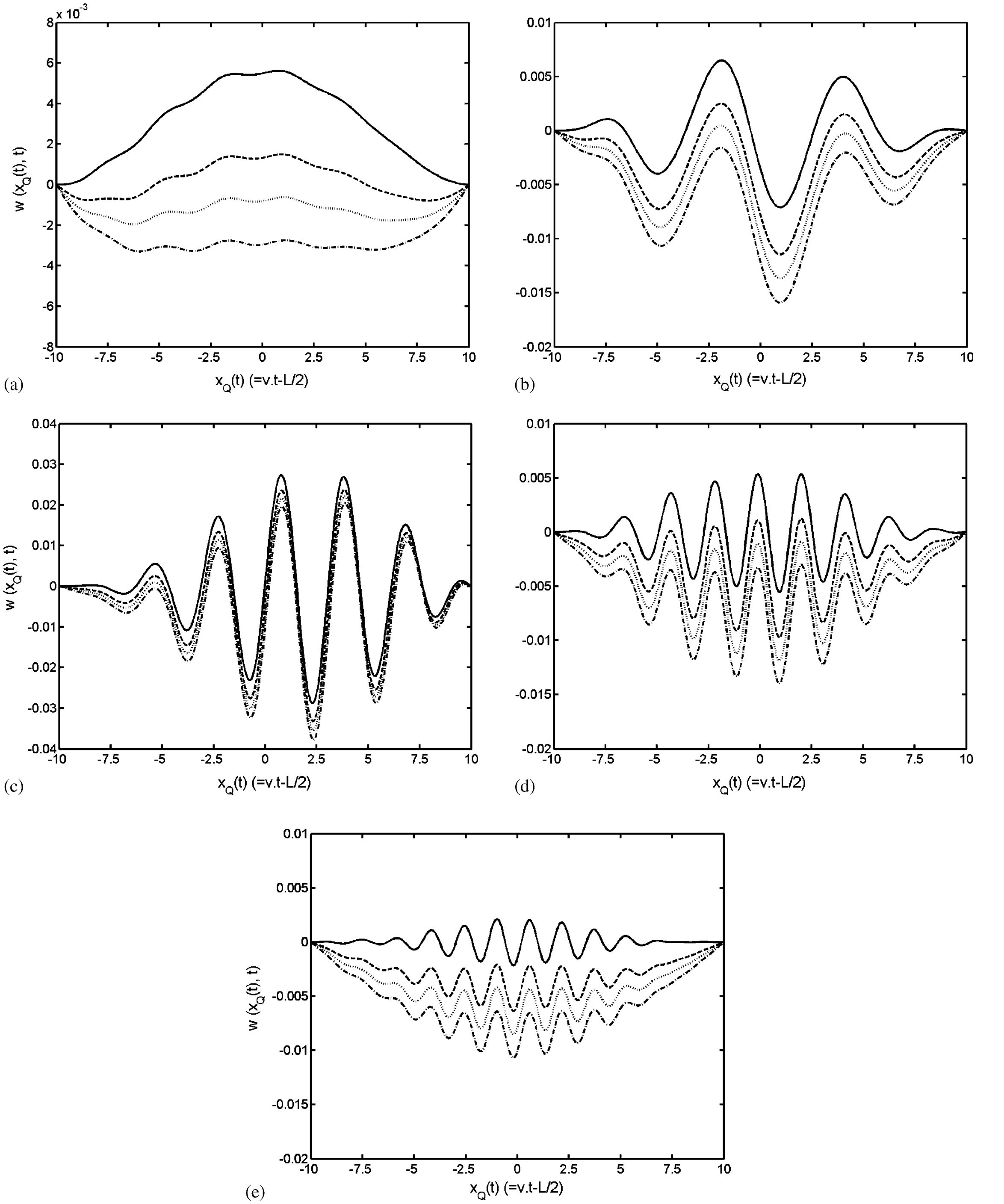

Fig. 4. Deflections under the moving harmonic force for various compressive forces P for $e = 0.25\,\text{m}$, $v = 20\,\text{m/s}$ for (a) $\Omega = 0$, (b) $\Omega = 20\,\text{rad/s}$, (c) $\Omega = 40\,\text{rad/s}$, (d) $\Omega = 60\,\text{rad/s}$, (e) $\Omega = 80\,\text{rad/s}$, (——) $P = 0$, (– – –) $P = 1000\,\text{kN}$, (· · · · · · · ·) $P = 1500\,\text{kN}$, (– · – · –) $P = 2000\,\text{kN}$.

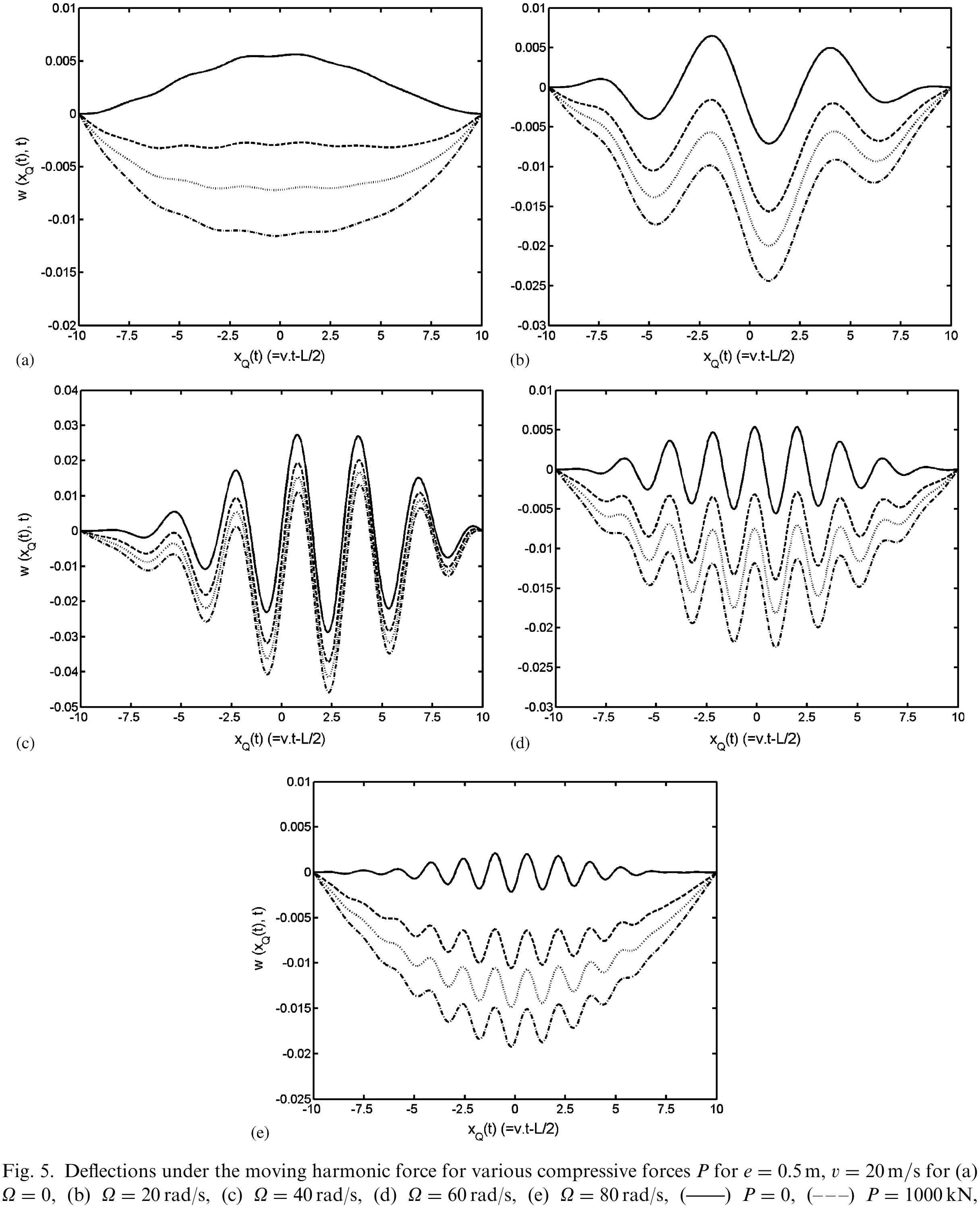

Fig. 5. Deflections under the moving harmonic force for various compressive forces P for $e = 0.5\,\text{m}$, $v = 20\,\text{m/s}$ for (a) $\Omega = 0$, (b) $\Omega = 20\,\text{rad/s}$, (c) $\Omega = 40\,\text{rad/s}$, (d) $\Omega = 60\,\text{rad/s}$, (e) $\Omega = 80\,\text{rad/s}$, (——) $P = 0$, (– – –) $P = 1000\,\text{kN}$, (· · · · · · ·) $P = 1500\,\text{kN}$, (– · – · –) $P = 2000\,\text{kN}$.

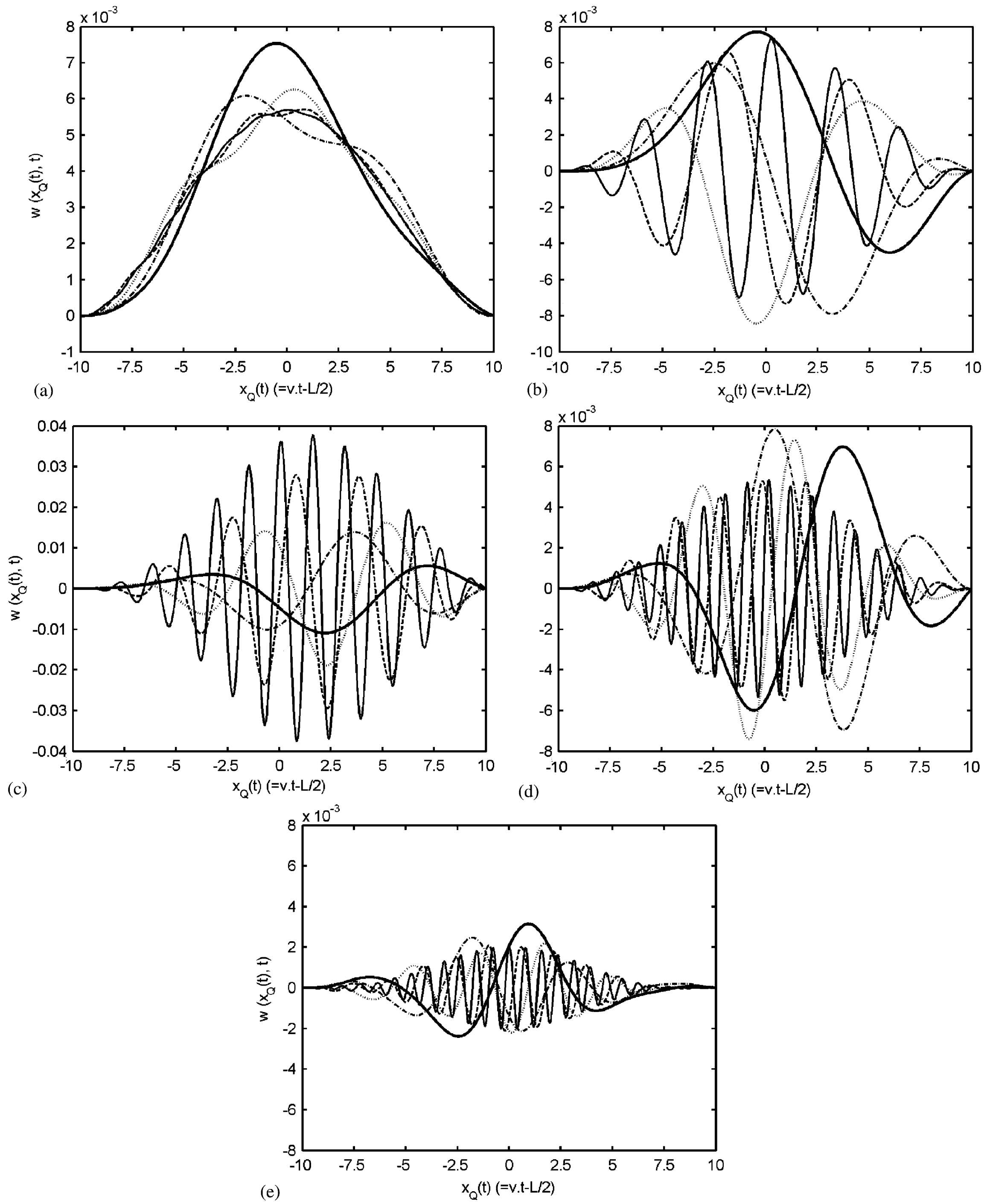

Fig. 6. Deflections under the moving harmonic force for various velocities v for $e = 0$, $P = 1500\,\text{kN}$ for (a) $\Omega = 0$, (b) $\Omega = 20\,\text{rad/s}$, (c) $\Omega = 40\,\text{rad/s}$, (d) $\Omega = 60\,\text{rad/s}$, (e) $\Omega = 80\,\text{rad/s}$, (——) $v = 10\,\text{m/s}$, (–––) $v = 20\,\text{m/s}$, (· · · · · · · ·) $v = 40\,\text{m/s}$, (– · – · –) $v = 60\,\text{m/s}$, (▬▬) $v = 80\,\text{m/s}$.

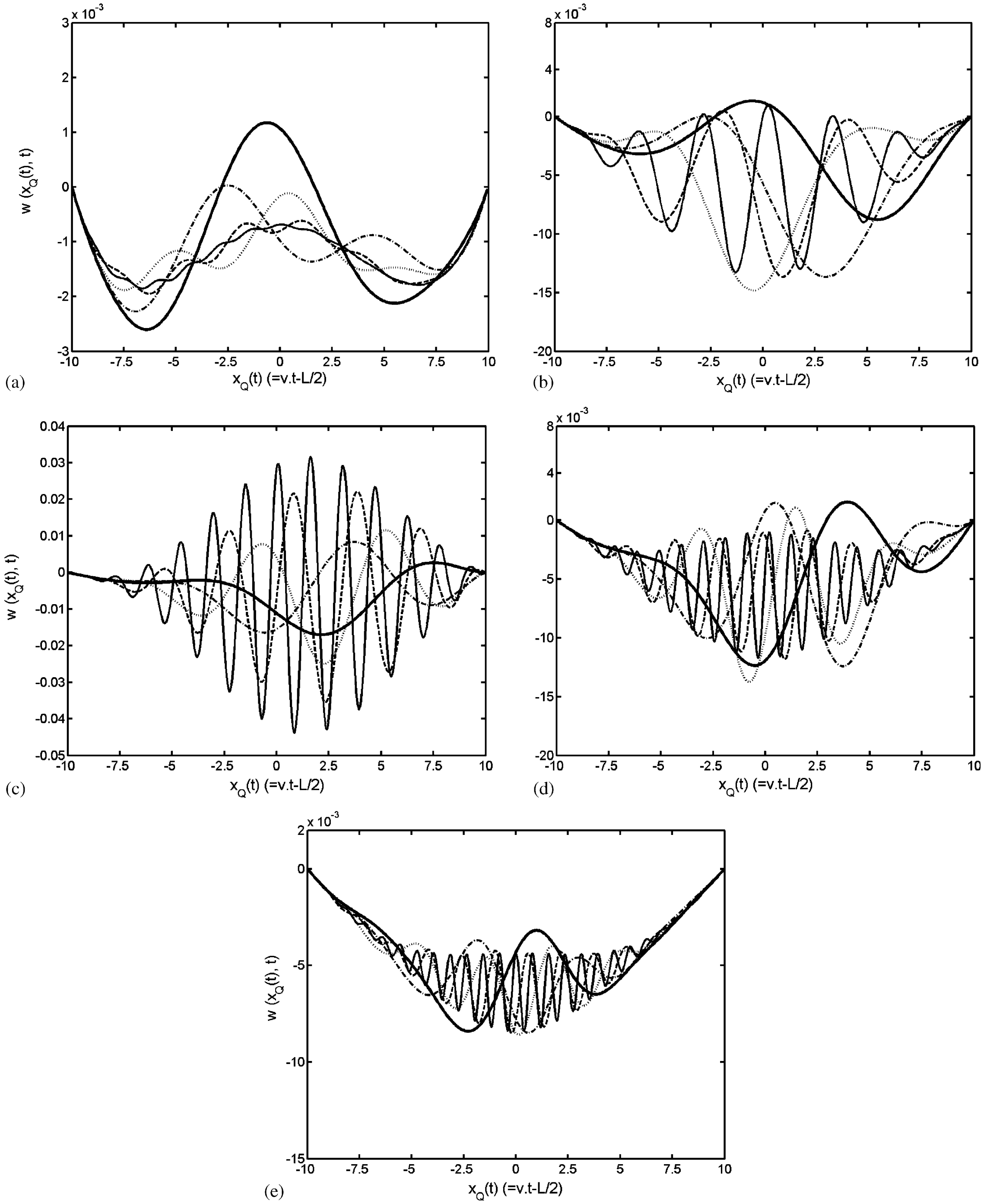

Fig. 7. Deflections under the moving harmonic force for various velocities v for $e = 0.25\,\mathrm{m}$, $P = 1500\,\mathrm{kN}$ for (a) $\Omega = 0$, (b) $\Omega = 20\,\mathrm{rad/s}$, (c) $\Omega = 40\,\mathrm{rad/s}$, (d) $\Omega = 60\,\mathrm{rad/s}$, (e) $\Omega = 80\,\mathrm{rad/s}$, (——) $v = 10\,\mathrm{m/s}$, (– – –) $v = 20\,\mathrm{m/s}$, (· · · · · · · ·) $v = 40\,\mathrm{m/s}$, (– · – · –) $v = 60\,\mathrm{m/s}$, (▬) $v = 80\,\mathrm{m/s}$.

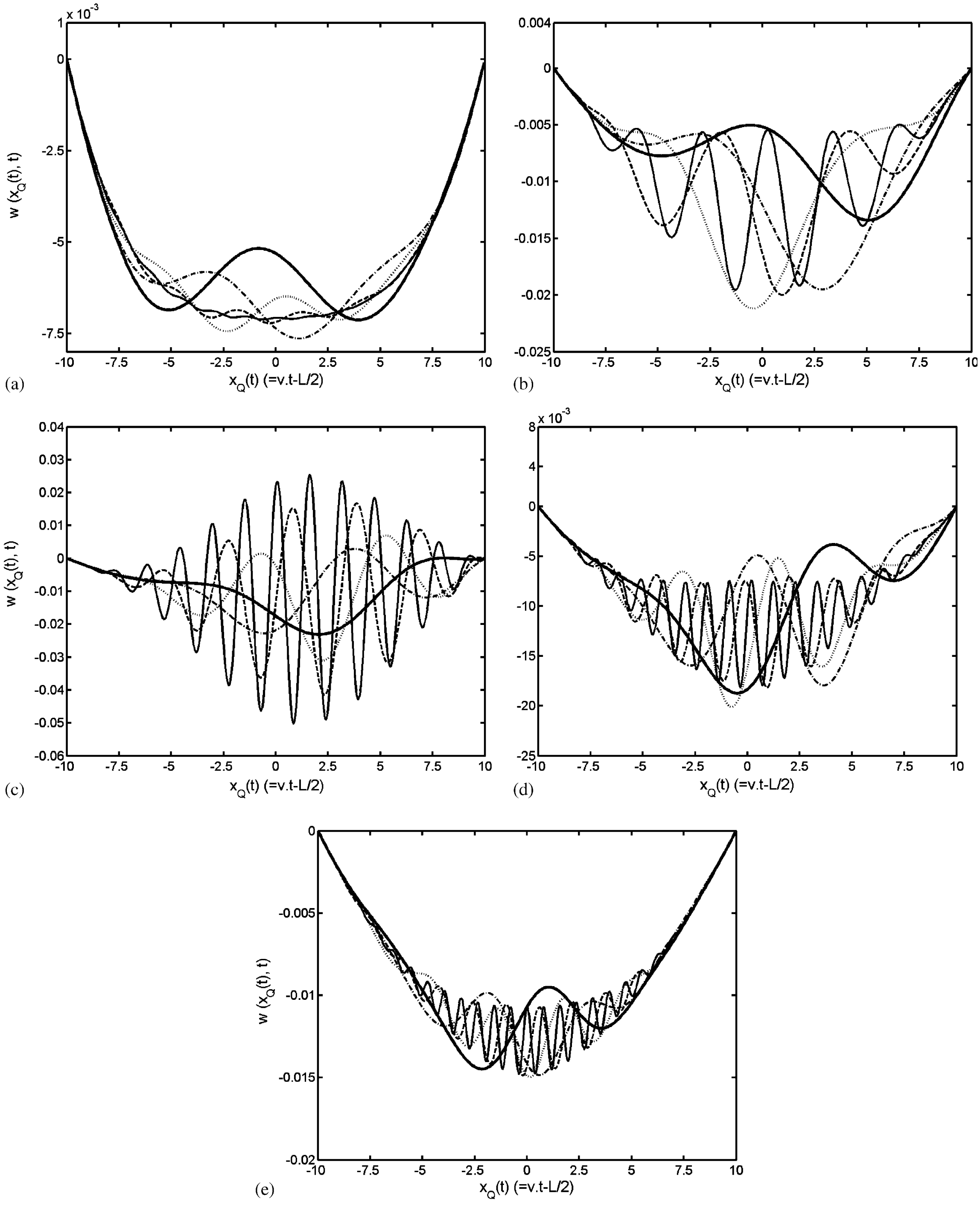

Fig. 8. Deflections under the moving harmonic force for various velocities v for $e = 0.5\,\text{m}$, $P = 1500\,\text{kN}$ for (a) $\Omega = 0$, (b) $\Omega = 20\,\text{rad/s}$, (c) $\Omega = 40\,\text{rad/s}$, (d) $\Omega = 60\,\text{rad/s}$, (e) $\Omega = 80\,\text{rad/s}$, (——) $v = 10\,\text{m/s}$, (– – –) $v = 20\,\text{m/s}$, (· · · · · · · ·) $v = 40\,\text{m/s}$, (– · – · –) $v = 60\,\text{m/s}$, (▬▬) $v = 80\,\text{m/s}$.

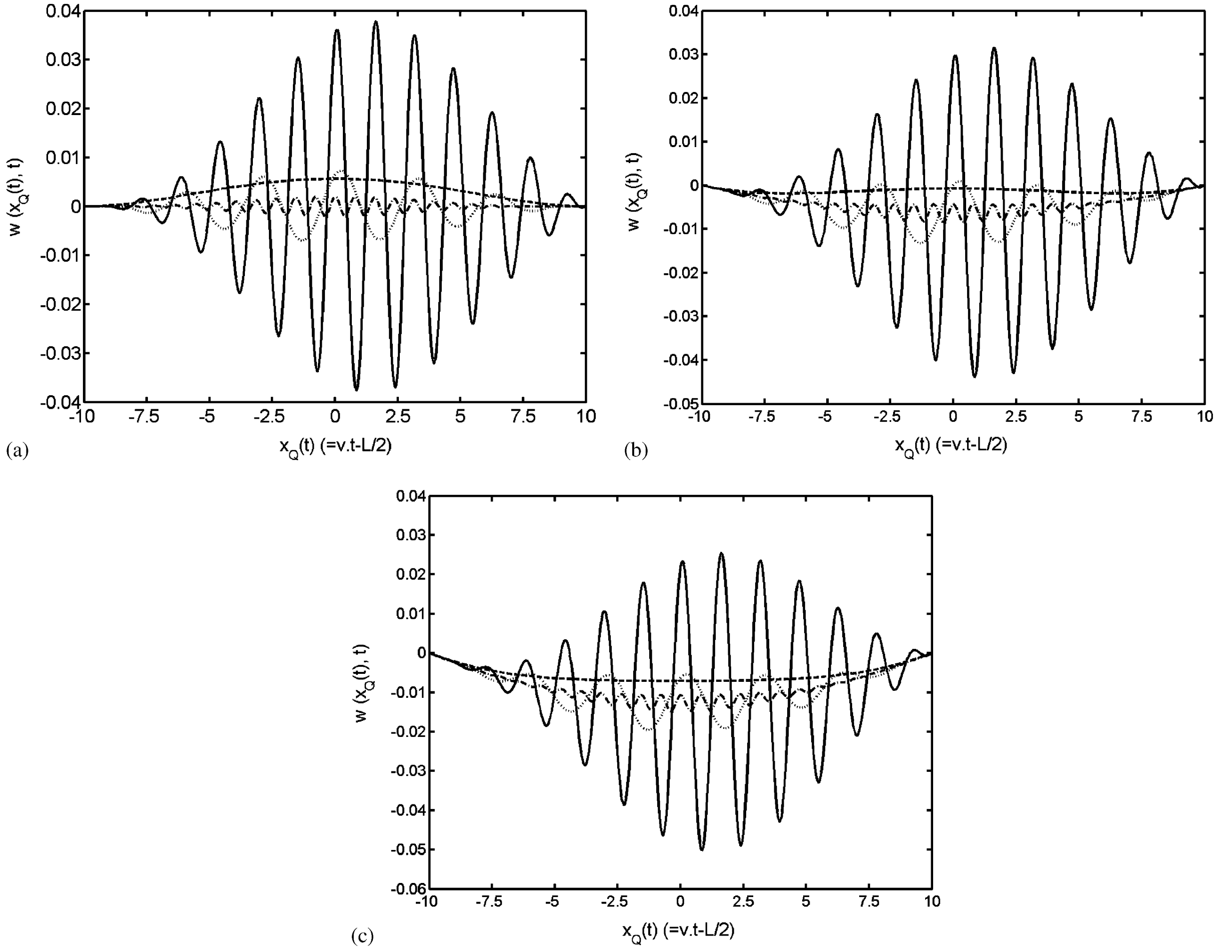

Fig. 9. Deflections under the moving harmonic force for various frequencies Ω for $v = 10\,\mathrm{m/s}$, $P = 1500\,\mathrm{kN}$ for (a) $e = 0$, (b) $e = 0.25\,\mathrm{m}$, (c) $e = 0.5\,\mathrm{m}$, (– – –) $\Omega = 0$, (· · · · · · ·) $\Omega = 20\,\mathrm{rad/s}$, (——) $\Omega = 40\,\mathrm{rad/s}$, (– · – · –) $\Omega = 80\,\mathrm{rad/s}$.

with increase in the velocity of the moving force as an expected situation. The reason for this situation can be explained as follows: As the values of the velocity increase, the acting time of the force on the beam becomes shorter. Therefore, the number of cycles of the moving harmonic force in the high values of the velocity is smaller than the number of cycles of the moving harmonic force in the low values of velocity for the same force frequency Ω. By comparing the deflections of Figs. 7 and 8, it is seen that the absolute values of the negative displacements of Fig. 8 become approximately two times larger than the deflection values of Fig. 7, which was also mentioned previously for Figs. 4 and 5.

Figs. 9–12 show the effect of the frequency of the moving harmonic force on the deflections of the beam for various values of the eccentricity of the compressive eccentric force and excitation

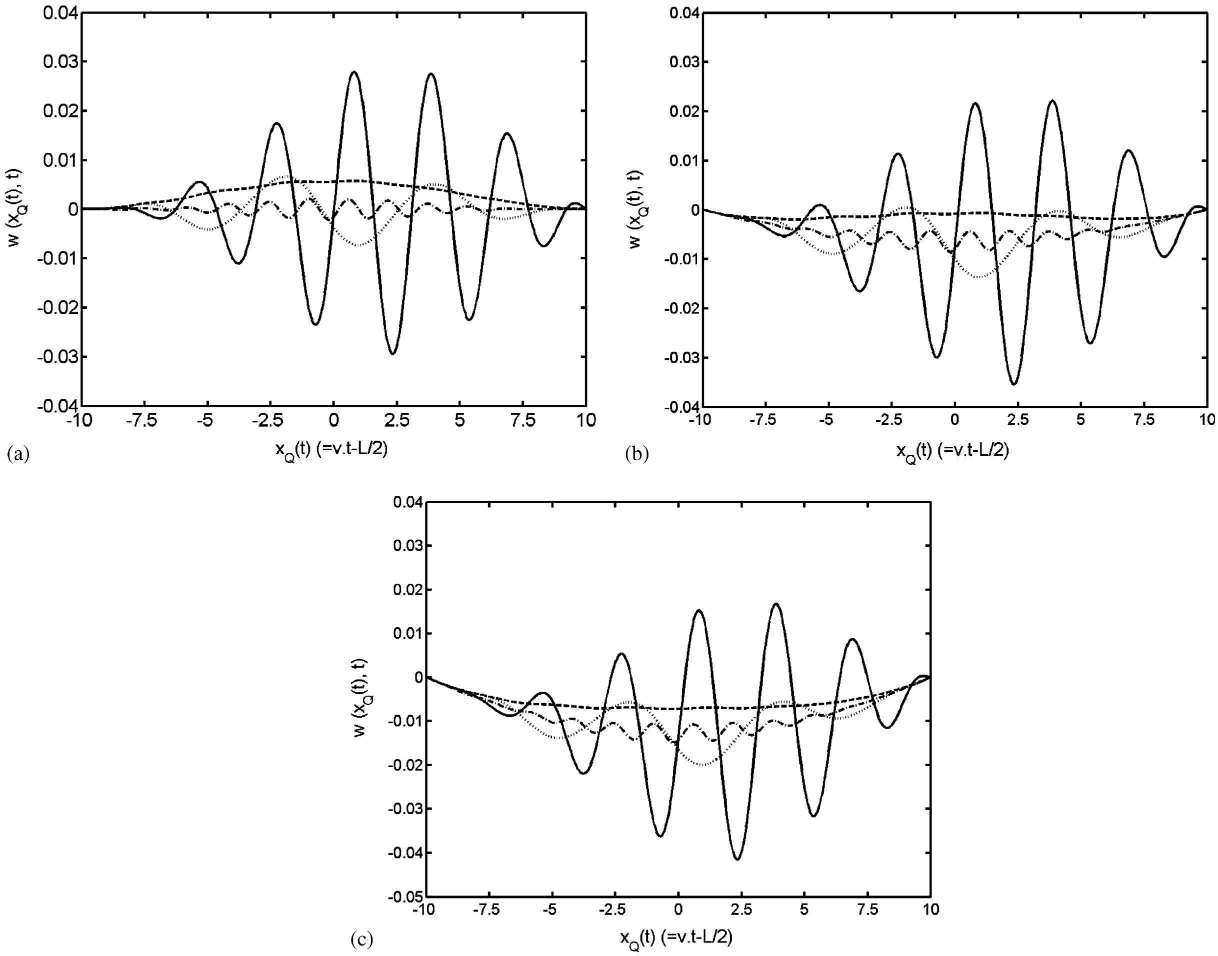

Fig. 10. Deflections under the moving harmonic force for various frequencies Ω for $v = 20\,\mathrm{m/s}$, $P = 1500\,\mathrm{kN}$ for (a) $e = 0$, (b) $e = 0.25\,\mathrm{m}$, (c) $e = 0.5\,\mathrm{m}$, (– – –) $\Omega = 0$, (· · · · · · · ·) $\Omega = 20\,\mathrm{rad/s}$, (——) $\Omega = 40\,\mathrm{rad/s}$, (– · – · –) $\Omega = 80\,\mathrm{rad/s}$.

frequencies $\Omega = 0, 20, 40, 80\,\mathrm{rad/s}$ at constant eccentric compressive force $P = 1500\,\mathrm{kN}$ for $v = 10, 20, 40, 80\,\mathrm{m/s}$, respectively. Figs. 9 and 10 show that excitation frequency of the moving harmonic force plays an important role in the deflections of the beam in the low velocities, as it was explained before. It was mentioned before that increase in the force frequency Ω until the first characteristic value of the beam causes increase in the deflections of the beam. If the excitation frequency is greater than the first eigenvalue of the beam, increase in the frequency causes decrease in the maximum absolute values of deflections of the beam. Figs. 9 and 10 also show that increase in the eccentricity causes decrease in the values of the positive deflections and increase in the absolute values of negative deflections, which was mentioned before while commenting on Figs. 4 and 5.

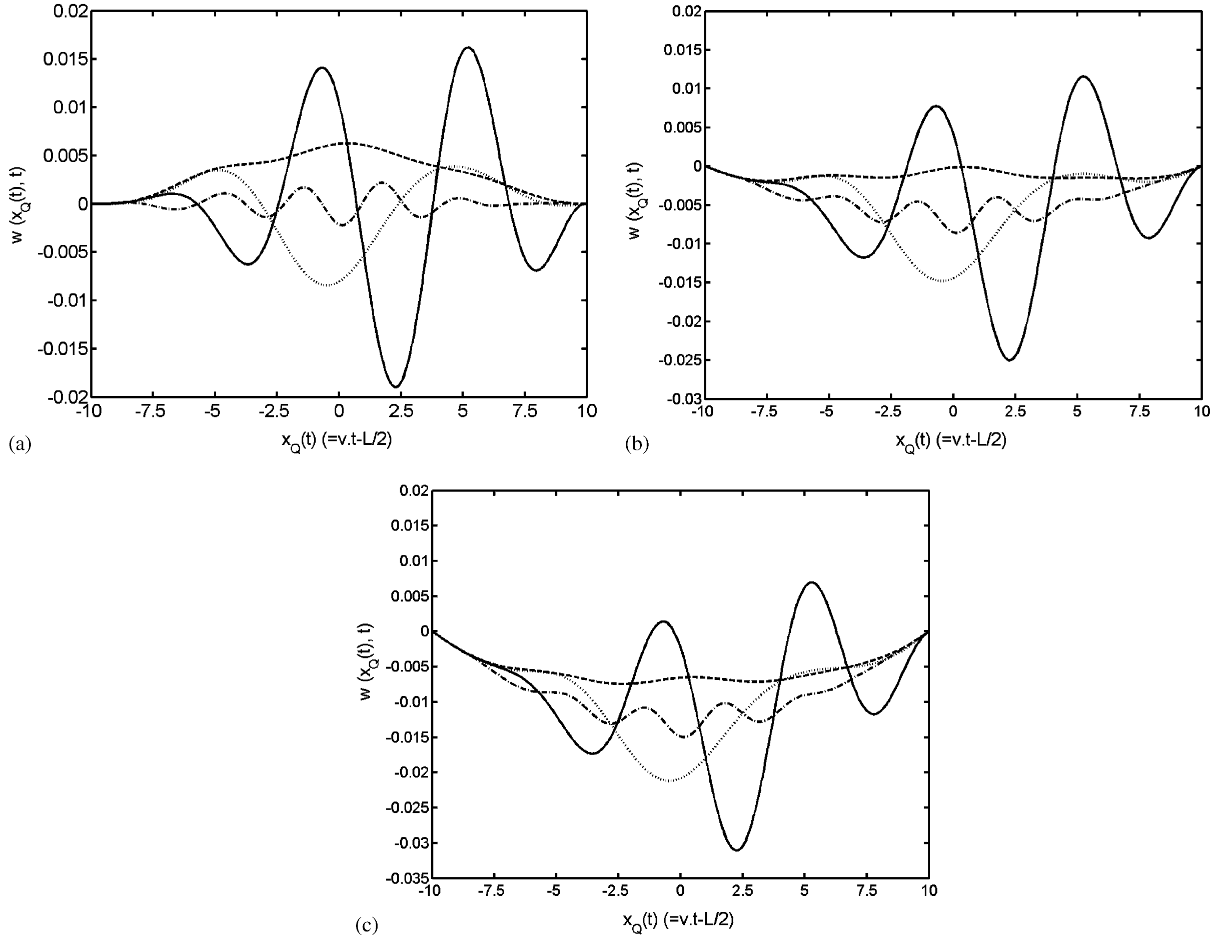

Fig. 11. Deflections under the moving harmonic force for various frequencies Ω for $v = 40\,\text{m/s}$, $P = 1500\,\text{kN}$ for (a) $e = 0$, (b) $e = 0.25\,\text{m}$, (c) $e = 0.5\,\text{m}$, (---) $\Omega = 0$, (·······) $\Omega = 20\,\text{rad/s}$, (——) $\Omega = 40\,\text{rad/s}$, (-·-·-) $\Omega = 80\,\text{rad/s}$.

4. Conclusions

Dynamic deflections of a simply supported beam subjected to an eccentric compressive force and a moving harmonic force with various constant velocities have been investigated. The obtained natural frequencies of the simply supported beam subjected to an axial force are compared with the exact results. To use the Lagrange equations with the trial function in the polynomial form and to satisfy the constraint conditions by use of the Lagrange multipliers is a very good way of studying the dynamic behavior of beams. Numerical calculations have been conducted to clarify the effects of the three important parameters: effect of the eccentric compressive force, axial velocity of the moving harmonic force and the excitation frequency of the

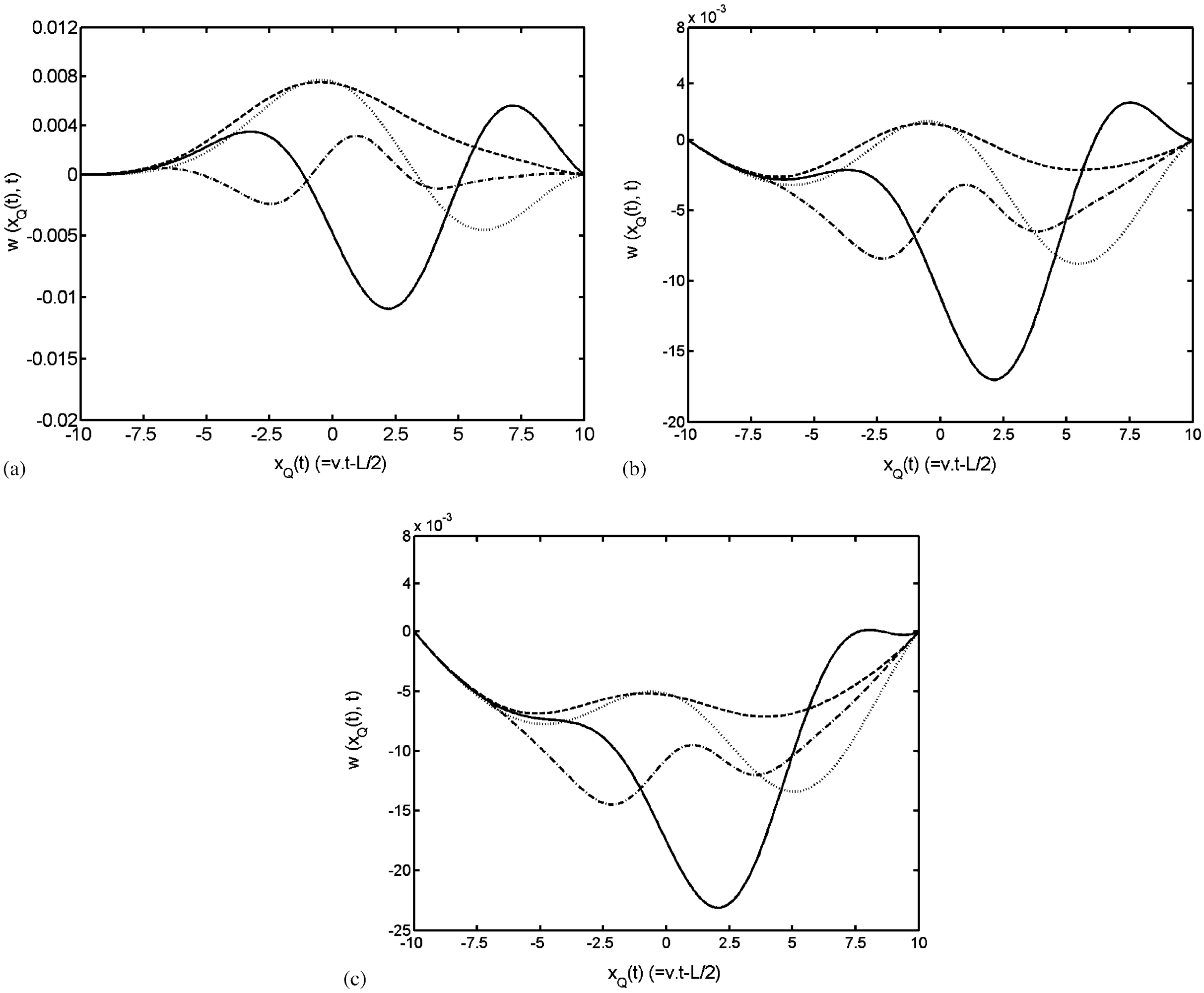

Fig. 12. Deflections under the moving harmonic force for various frequencies Ω for $v = 80\,\mathrm{m/s}$, $P = 1500\,\mathrm{kN}$ for (a) $e = 0$, (b) $e = 0.25\,\mathrm{m}$, (c) $e = 0.5\,\mathrm{m}$, (– – –) $\Omega = 0$, (· · · · · · ·) $\Omega = 20\,\mathrm{rad/s}$, (——) $\Omega = 40\,\mathrm{rad/s}$, (– · – · –) $\Omega = 80\,\mathrm{rad/s}$.

moving force for the constant damping ratio $\zeta = 0.05$. It is observed from the investigations that eccentric compressive force, axial velocity and frequency of the moving harmonic force have a very important influence on deflections of the beam.

The main purpose of the study is to investigate the influence of eccentric compressive force on deflections of prestressed beams. It is shown that the eccentricity of compressive force plays an important role in the responses of prestressed beams under moving harmonic forces.

All of the obtained results are very accurate and may be useful for design purposes and a better understanding of the behavior of the structural systems such as prestressed beams of viaducts of roadways and railways, and also bridges under moving harmonic loads.

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