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Particularities of Newton's method in space frame force determination, utilising eigenpair functions

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Abstract

Redundant space frame dynamics are modelled through the structural eigenproblem, which is formulated to account for the geometric stiffness due to self-equilibrating membrane forces. The resulting mathematical eigenvalues and eigenvectors, respectively indicative of natural frequencies and modes of oscillation, are then assumed to be continuous functions of a scalar factor upon a normalised set of forces in equilibrium. Newton's method—or sensitivity analysis—provides a means for minimising the difference between mathematical and physically observed eigenvalues, whereupon the axial forces are inferred from the model. Issues of convergence and root uniqueness are anticipated. Eigenvalue coalescence, in which eigenvalues transitorily coalesce to permute their arbitrarily numerical ordering, is seen to define non-smooth eigenvalue functions and hence encumber Newton's method; mode tracing strategies to overcome this are discussed. The existence of various degrees of eigenvalue degeneracy, owing to frame cyclic symmetry or otherwise, is anticipated in terms of the influence upon Newton's method. If the degree of frame static redundancy is greater than one and there exist a number of linearly independent force distributions, then the required force determination is a multidimensional Newton method. Problems encountered in these manifold dimensions, including those arising from frame spatial periodicity and the adversity of Newton's method, are overcome. Emphasis is placed upon minimising the dimensionality of Newton's method through enforcement of equilibrium constraints. Illustrative numerical simulations are given and some applications of the method are proposed.

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Nomenclature

d	degree of eigenvalue degeneracy
f	number of space frame members, member force
$\mathbf{f}$	member force vector
$\mathbf{G}$	geometric stiffness matrix
$\hat{\mathbf{G}}$	geometric stiffness matrix shell
$\mathbf{I}$	identity matrix
$\mathbf{J}$	Jacobian of eigenvalue derivatives
k	iteration number
$\mathbf{K}$	constant stiffness matrix
$\mathbf{L}$	equilibrium matrix
$\hat{\mathbf{L}}$	reduced, full rank equilibrium matrix
m	number of linear equations
$\mathbf{w}$	modal mass vector
$\mathbf{M}$	mass matrix
n	dimensionality of Newton's method
N	model order
p	force parameter, factor upon $\mathbf{f}$
$\mathbf{p}$	n-vector of force parameters
Δp, $\Delta\mathbf{p}$	Newton excursion
r	modal designation
$\mathscr{R}$	Euclidean space
$\mathbf{U}$	matrix of left singular vectors
$\mathbf{V}$	matrix of right singular vectors
$\boldsymbol{\Gamma}$	transformation operator orthogonalising eigenvectors
$\bar{\boldsymbol{\Gamma}}$	transformation operator aligning eigenvector bases
$\boldsymbol{\Theta}$	Boolean permutation operator upon eigenpairs
κ	number of linearly independent force distributions
λ	root-standardised eigenvalue
$\boldsymbol{\lambda}$	vector of eigenvalues λ
$\bar{\bar{\lambda}}$	eigenvalue
$\bar{\bar{\boldsymbol{\lambda}}}$	vector of eigenvalues $\bar{\bar{\lambda}}$
$\boldsymbol{\Lambda}$	diagonal matrix of degenerate eigenvalues of order d
σ	singular value
$\boldsymbol{\Sigma}$	diagonal matrix of singular value reciprocals
ϕ	eigenvector
$\boldsymbol{\Phi}$	matrix of eigenvectors
ω	natural frequency

Superscripts, subscripts

$+$	Moore–Penrose matrix pseudoinverse
$*$	quasi-root designation
i	vector entry designation, matrix row designation
j	matrix column designation
prm	ambiguous root exactly satisfying eigenvalue constraints
R	root designation
T	matrix transpose

Vector and matrix operations

$\lvert\ \rvert$	modulus
$\lVert\ \rVert_2$	Euclidean vector norm
tr	matrix trace
$\rightarrow$	set permutation

1. Introduction

The dynamic response of axially loaded bars has long been understood. Lord Rayleigh [1], as early as 1877, derived the equations governing the longitudinal, torsional and lateral free vibrations of bars. Longitudinal and torsional vibrations are in nature simpler than lateral vibrations, being governed by the one-dimensional wave equation; lateral vibrations may be considered through the balance of potential and kinetic elemental energies. Rayleigh's work went on to include the effects of longitudinal tension in the equations of motion, most significantly the raising of natural frequencies. More comprehensive accounts of the effects of axial load on beam dynamics can be found in more recent papers: Shaker [2], Bokaian [3,4] and Liu et al. [5]; Stephen [6] presents corresponding upper and lower bound approximations. These, extending the work of

Rayleigh [1], include the effects of compressive as well as tensile force. The relationship between the fundamental natural frequency and the Euler buckling load is made apparent in that progressive compression leads to the fundamental dynamic and Euler buckling modes coalescing, whereupon the former vanishes. Galef [7] studies the vibrations of compressed beams and develops a simple formula relating the compressed beam frequency to the magnitude of compression, each, respectively, normalised to the unloaded frequency and Euler buckling load. Virgin and Plaut [8] investigate the effect of axial load on the forced vibrations of beams.

While the case of an isolated bar or beam is easily appreciable, the effects on the global dynamics of collective, axially loaded members in a framework are not intuitive. Howson and Williams [9] investigate the frequencies of an H-frame comprising Timoshenko members, under axial load, and find near-parabolic frequency–load relationships. Alpay and Utku [10] develop an analysis tool through the finite-element method to study the dynamic response of various systems under the influence of pre-stress. Xiaocheng [11] investigates an approach for the analysis of the random response of pre-stressed structures, where pre-stress can arise from static and thermal loads; there is a further interaction with dynamic load. Mead [12] gives an extensive study of the effects of self-equilibrating forces on the frequencies and mode shapes of two specific plane frames: a pair of parallel beams and a six-member cross-braced square. The axial force is taken far beyond the values required to buckle the first frame member in its fundamental Euler mode. What is immediately apparent is that the fundamental frequency of a frame, much like that of an isolated member, vanishes upon buckling of the most susceptible member. The two vanishing points of the fundamental frequency, corresponding to the respective buckling loads of oppositely loaded frame members, cause there to be a point of stationarity of the frequency somewhere between. This serves to show that the frame fundamental frequencies are global systems coupling the analogous, single member, linear or quasi-linear eigenvalue–load relationship systems. An earlier work by Przybylski et al. [13] investigates the aforementioned parallel beam configuration, experimentally and with the perturbation method formulated on Hamilton's principle, and, while only concentrating on a narrow range of axial force, is in places supportive of the findings of Mead [12]. Amongst other things, the additional effect of external axial load is investigated. Lieven and Greening [14] study experimentally the effect of pre-stress on the modal behaviour of a six-member plane frame, in nature that investigated by Mead [12].

The effects of pre-stress on a narrow frequency range of a cyclic, indeterminate frame are investigated by Holnicki-Szulc and Haftka [15] through the dynamic eigenproblem. Their aim is to control vibration through a derivate-based optimisation scheme, not by displacing frequencies from a specific range—as it is stated that it is unlikely that these can be distanced enough in the spectrum so that their effects become negligible—but to manipulate the mode shapes by pre-stressing in order to mitigate amplitudes in specified regions. It is felt that the actuators that pre-exist in adaptive structures could be utilised for this endeavour. Ultimately the desire is to coincide nodes with critical regions, but the authors appreciate that this is not possible to achieve with a considerable number of modes. Baycan et al. [16] set forth the theory of pre-stressing to control the frequencies of frameworks. If such works are thought of as forward problems in the sense of force prescription for dynamic-based requirements, then the work presented in this paper is the inverse problem of force determination with dynamic constraints, that is to say the dynamic characteristics of the physical framework in question.

The determination of axial force in single struts via inference from observed natural frequencies has been outlined by Stephens [17] and, exacting this, Lurie [18], who also sought to determine the degree of end fixity. Sundararajan [19] has also contributed by proposing a simple secant method of approximating the fundamental frequency of an axially loaded beam, or other simple structure, by using a reference structure of the same stiffness distribution. Such means of force identification are appealing in that they are passive and effective, but they do not analogise in an obvious way to assemblies of members in frameworks.

With much work done in the field of evaluating eigenpair derivatives, gradient-based search algorithms are available for the task of system identification through the dynamic eigenproblem and Newton's method. That is to say, given an eigenproblem dependent upon a parameter, the root parameter value—that of the physical structure—might be converged upon iteratively from some arbitrary parameter value nominated for the eigenproblem. Newton's method is a powerful gradient method in which the function value and derivative are used to make an affine model of the function to evaluate an excursion to the proceeding iterate, hopefully progressing to the problem root. Presently, the functions of force are chosen to be the eigenvalues or natural frequencies. The study of eigenpair derivatives is vast in the literature, but those works of present interest owe to Wittrick [20], Fox and Kapoor [21], Nelson [22], Ojalvo [23], Mills-Curran [24] and Dailey [25]. Essentially, in the present paper, computation of eigenvalue derivatives is based on the equation of Fox and Kapoor [21] and that of Ojalvo [23] is used to compute the eigenvector derivatives where they are needed.

Identification of framework loading through Newton's method has previously been documented by Greening and Lieven [26], who employ sensitivity analysis to determine in turn each of the six-member forces of their previous planar frame without consideration of an equilibrium constraint. Present attention is turned to space frames—frames occupying space, in opposition to plane frames, which are confined to a plane. In the axially loaded space frame context, Newton's method is presented with a number of difficulties. The phenomenon of eigenvalue coalescence, in which eigenvalues transitorily coalesce and permute the arbitrary ordering of eigenpairs, is seen to define non-smooth eigenvalue functions and hence encumber Newton's method. A number of mode-tracing strategies have been suggested to overcome this and retain the physically pertinent, smooth eigenpair functions of force. These include the computationally expensive perturbation expansions of the eigenproblem—as discussed by Eldred et al. [27]—and those utilising the consistency of the eigenvectors across points of eigenvalue coalescence, which include the works of Ting et al. [28], Gibson [29], Eldred et al. [30,31] and Kim and Kim [32]. The non-smooth, numerically ordered eigenvalue functions will have many abrupt changes in gradient sign, which increase the number of ambiguous roots, and therefore if convergence is achieved at all with untraced modes the likelihood of discovering the true root is greatly lessened. And so mode tracing is seen to be a necessary measure if correct convergence is to ensue. Overcoming eigenvalue coalescence necessitates that not only the eigenvalues at the root, that is to say those of the physical frame, be measured, but also the mode shapes, for it is these that will form the reference by which all eigenvalue functions are traced.

Eigenvalue coalescence can be thought of as a special, limiting case of the general phenomenon of eigenvalue loci veering, which is discussed by Leissa [33], Petyt and Fleischer [34], Kuttler and

Sigilitto [35], Perkins and Mote [36], Pierre [37], Natsiavas [38] and Liu [39] amongst others. Here, the eigenvalues do not intersect but their loci rapidly veer away from one another. The narrow region in the vicinity of the veering, where two separate dynamic systems couple, displays abrupt variations of the eigenvectors of the respective loci, making mode tracing problematic. A routine to overcome this is suggested in a paper by Bahra and Greening [40] and is employed here where appropriate; Ting et al. [28] also have suggestions. The routine in reference [40] augments the assurance with which modes can be traced by forward and backward casting the eigenvector freedoms, based on knowledge of their values and derivatives. This also serves to extend the permissible parameter perturbation suggested by an iterative excursion that preserves eigenvector consistency. Other considerations highlighted by Bahra and Greening [40] in the tracing of modes in permanently degenerate eigensystems, as arise in the axial loading of cyclically symmetric space frames, are noted.

If the degree of frame static redundancy is greater than one and there exist a number of linearly independent force distributions, then the required numerical iteration is a multidimensional Newton method. It is seen that the greater the dimensionality of the iteration, the more problematic is the employment of Newton's method. Emphasis is therefore placed on minimising this dimensionality, and this is done through arguments based on satisfying equilibrium constraints.

A fundamental aspect of the proposed method of force identification is that it utilises the physics of space frames. Typically, strain gauges are used to monitor force. These require that the forces are known at the instant of gauge application and so it is sensible to apply them when the structure is in a zero load state. Problems would arise if the strain gauges were to delaminate with the structure experiencing some degree of loading, as the datum to which to reset them would be unknown. Provided that an accurate eigenproblem for a given space frame can be formulated, force identification through minimising the difference between mathematical and physically observed eigenvalues circumvents this problem.

2. Axial force dependency of frame dynamics

Upon the axial loading of an isolated member, all of its natural frequencies will experience a mutual modification to their magnitudes. If the load is tensile, this modification will be an increase; if compressive, a decrease, with progressive loading eventually leading to the vanishing of the fundamental dynamic mode at static buckling, where the dynamic and static deflection modes coalesce. Further, the frequency–load relationship is parabolic for a simply supported beam, whose fundamental vibration and buckling modes are identical, and near-parabolic for other support conditions (Fig. 1). Consequently, the eigenvalues, which by Eq. (3) are seen to be directly proportional to the squares of the frequencies, are linear or quasi-linear functions of axial force.

While the eigenvalues can exhibit significant variations with respect to load, for an isolated member, the eigenvectors or mode shapes, notwithstanding amplitude, are stationary or quasi-stationary. The modes shapes, as the frequencies, *will* be affected by the degree of end fixity of the member—in terms of the degree of curvature. The classic results for the frequencies of an axially loaded beam with various end fixity conditions are shown in Fig. 1. The fundamental frequencies

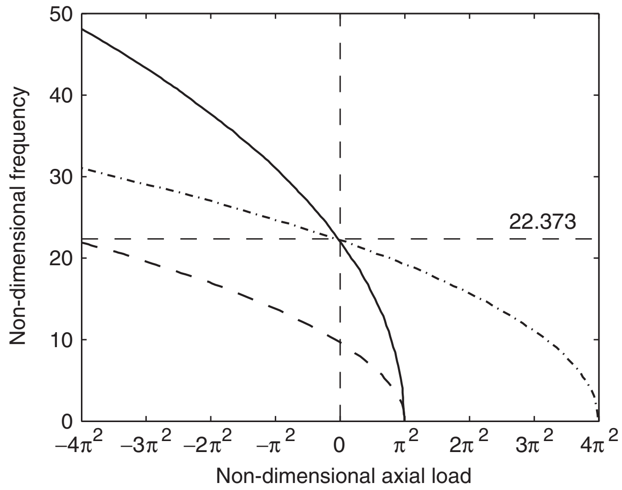

Fig. 1. Frequency–force relationships for single, axially loaded, prismatic beams with various degrees of end fixity (—— free–free; – · – · – fixed–fixed; – – – – simple–simple); frequency and load, respectively, non-dimensionalised as $\varpi = \omega(ml^4E^{-1}I^{-1})^{1/2}$ and $\rho = pl^2E^{-1}I^{-1}$, where m, l, E, I, ω and p are, respectively, mass per unit length, length, Young's modulus, second area moment, frequency and axial force (compression positive).

and axial forces presented are, respectively, non-dimensionalised as

$$\varpi = \omega\left(ml^4E^{-1}I^{-1}\right)^{1/2}$$

and

$$\rho = pl^2E^{-1}I^{-1},$$

where m, l, E, I, ω and p are, respectively, mass per unit length, length, Young's modulus, second area moment, frequency and axial force. For detailed analysis of the dynamics of axially loaded beams, the reader is referred to Shaker [2] and Bokaian [3,4].

In space frames, a number of members exist to form a global, coupled system, the response of which is dependent upon the relative stiffnesses of the members and their connectivity at the joints. Consequently, the eigenvalues of the global system will not be linear or quasi-linear functions of some parameter indicative of the distribution of forces and neither will the eigenvectors exhibit stationarity with respect to load. However, since the global system couples a number of distinct, local systems, there exists an analogy between the dynamic–load relationship of a framework and those relationships of its constituent members isolated, albeit a non-intuitive one.

Many single-beam dynamic systems are connected in a framework so that their linear or quasi-linear eigenvalue–force loci exhibit coupling. Indeed, the analogies that exist between the global and local systems and similarities would become apparent if the respective loci were superimposed: the intersection of the independent, single-beam loci would mark the focus for the veering of the global, coupled system, the loci of which would be asymptotic to the superimposed single-beam loci. Mead [12] notes that the beam subsystems to which a global,

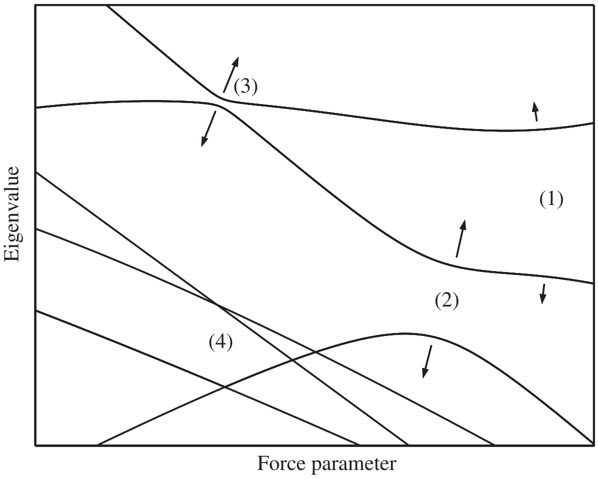

Fig. 2. Eigenvalue loci veering and eigenvalue coalescence: (1) weak veering; (2) intermediate veering; (3) strong veering and (4) various coalescences.

fixed–jointed frame system is analogous have end fixity of some degree less that fixed–fixed, since the beams of a space frame are not held in an absolute sense but only with respect to one another.

Veering can exist in varying degrees and Liu [39] proposes certain measures of this: the eigenvector derivative and eigenvalue second derivative. Veering is seen to be more pronounced the closer the eigenvalues. Also, Perkins and Mote [36] suggest measures for distinguishing a veering from a coalescence. While veering can be an actual phenomenon of a mathematical model, it can also be caused by model discretisation and unsuitable, (continuous) mathematical modelling, as indicated by Leissa [33]. The effects of discretisation on veering, as well as geometric symmetry and system non-adjointness are demonstrated by Perkins and Mote [36]. Since the intricate global system of a space frame involves couplings of multiple subsystems, the veerings of loci are not simple. Fig. 2 illustrates a typical series of veering events for a group of space frame eigenvalue–force loci; it is possible for certain global systems to be completely uncoupled, and these have their eigenvalue loci intersecting in a smooth coalescence.

3. Theoretical development

Of interest are the distributions of axial force that arise in redundant, free–free space frames, induced by member strains. If the degree and nature of the static redundancy is such that a number of linearly independent axial force distributions exist, then any state of frame equilibrium can be described by a set of scalar force parameters, factors upon each of the superimposed distributions known a priori. This automatically enforces an equilibrium constraint and minimises the dimensionality of the Newton iteration. The approach of Greening and Lieven [26] required that there be as many updating parameters as frame members and made no use of an equilibrium constraint; since the number of linearly independent force distributions can never exceed the number of frame members, such an approach is more computationally prohibitive than that here suggested.

Newton's second law of motion for the displacement $\mathbf{x}(t)$ can be expressed for the free vibration of an undamped, axially loaded frame in terms of its distributions of mass $\mathbf{M}$, structural stiffness $\mathbf{K}$ and geometric stiffness $\mathbf{G}$ as

$$\mathbf{M}\frac{\partial^2 \mathbf{x}(t)}{\partial t^2} + \mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{x}(t) = 0. \tag{1}$$

The geometric stiffness is a function of the distribution of axial forces as all of its entries are linear functions of some scalar parameter characterising an entire force distribution. The geometric stiffness matrix presently used accounts for the interaction of axial force and torsion and further the geometric interaction of axial force and both bending in the principal directions and flexural shear. The structural and geometric stiffness matrices are consistent in the sense that for every first-order term in the former, there is a corresponding second-order term in the latter.

Assuming the frame is capable of oscillating in simple harmonic motion at any particular frequency, substitution of the displacement solution into the differential equation (1) leads to the general eigenproblem, which is dependent upon a parameter since the geometric stiffness is a function of axial force

$$((\mathbf{K} + \mathbf{G}) - \lambda_r \mathbf{M})\phi_r = \mathbf{0}, \quad r = 1,\ 2,\ \ldots,\ N, \tag{2}$$

where the rth natural frequency

$$\omega_r = \frac{\sqrt{\lambda_r}}{2\pi}. \tag{3}$$

The eigenvectors are indicative of the modes of vibration and are assumed to be mass-orthogonal and further normalised such that

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi} = \mathbf{I}, \tag{4}$$

where $\mathbf{\Phi}$ is the complete set of eigenvectors.

Presently, only free–free space frames are considered so that the total stiffness matrix $(\mathbf{K}+\mathbf{G})$ is singular. Consequently, the characteristic polynomial of Eq. (2) has a zero at its origin of multiplicity equal to the defect in the total stiffness matrix. That is to say, there are six zero eigenvalues that relate to the six rigid body modes of a free–free structure; these are presently ignored so that modal designation commences with the initial strain mode.

3.1. Eigenpair derivatives

The expression for the eigenvalue derivative is taken from Fox and Kapoor [21] and can be seen to derive as follows. Differentiation of the eigenproblem (2) leads to

$$((\mathbf{K} + \mathbf{G}) - \lambda_r \mathbf{M})\frac{\partial \phi_r}{\partial p} = \mathbf{M}\phi_r \frac{\partial \lambda_r}{\partial p} - \left(\frac{\partial}{\partial p}(\mathbf{K} + \mathbf{G}) - \lambda_r \frac{\partial}{\partial p}\mathbf{M}\right)\phi_r, \tag{5}$$

whence an expression for the eigenvalue derivative is obtained upon pre-multiplication by ϕ_r^{T}, noting the orthogonality conditions, as

$$\frac{\partial \lambda_r}{\partial p} = \phi_r^{\mathrm{T}}\left(\frac{\partial}{\partial p}(\mathbf{K} + \mathbf{G}) - \lambda_r \frac{\partial}{\partial p}\mathbf{M}\right)\phi_r. \tag{6}$$

Further noting that both the mass and structural stiffness of a frame are stationary with respect to the scalar force parameter p, and that all entries, zero or otherwise, of the geometric stiffness matrix $\mathbf{G}$ are linear functions of p,

$$\frac{\partial \lambda_i}{\partial p} = \phi_i^{\mathrm{T}} \hat{\mathbf{G}} \phi_i, \tag{7}$$

where $\hat{\mathbf{G}}$ can be defined as

$$\hat{\mathbf{G}} = \frac{\mathbf{G}}{p}, \quad p \neq 0. \tag{8}$$

The matrix $\hat{\mathbf{G}}$ can be thought of as the geometric stiffness *shell*, independent of, and therefore stationary with respect to, p. It is apparent then from Eq. (7) that the evaluation of the eigenvalue derivative involves only the evaluation of the eigenvectors associated with the eigenvalue, and is therefore computationally inexpensive.

If a space frame exhibits cyclic symmetry, there will exist a permanent, two-fold degeneracy of certain eigenvalues with respect to loading and the associated eigenvectors will not necessarily be orthogonal to one another. Degeneracy is typically two-fold since all of the eigenvectors are accountable as linear sums of the eigenvector pair associated with the degenerate eigenvalues. Indeed, any linear combination of theses eigenvectors is itself a nullspace of Eq. (2). The eigenvectors associated with degenerate eigenvalues will not necessarily be orthogonal to one another and consequently, condition (4) will not hold. There will exist a pair of orthogonal eigenvectors at degeneracy, and these may be found through a transformation matrix, which is the set of eigenvectors of the auxiliary eigenproblem introduced by Ojalvo [23],

$$\left(\begin{bmatrix} \phi_r & \phi_{r+1} \end{bmatrix}^{\mathrm{T}} \hat{\mathbf{G}} \begin{bmatrix} \phi_r & \phi_{r+1} \end{bmatrix} \right) \boldsymbol{\Gamma} = \boldsymbol{\Gamma} \left(\frac{\partial \boldsymbol{\Lambda}}{\partial p} \right). \tag{9}$$

Orthogonality of the eigenvectors is then achieved through post-multiplication by $\boldsymbol{\Gamma}$; mass-orthonormality needs to be reinforced thereafter. In the case of transitory coalescence, this also serves to orientate the eigenvectors at degeneracy to the adjacent eigenvectors immediately following separation of the eigenvalues. With permanent degeneracy, such orientation is non-sensical and so an orthogonal set of eigenvectors can also be found through Gram–Schmidt orthogonalisation.

Nelson [22] gives an efficient direct method to compute the eigenvector derivatives; the direct evaluation of the eigenvector derivative in the case of permanent eigenvalue degeneracy is given by Ojalvo [23] in an extension to this work. Summaries are presented by Bahra and Greening [40], along with further considerations for mode tracing in degenerate eigensystems. Essentially, this latter point involves vector alignment of the arbitrarily orientated eigenvector bases. Alignment is also important for differentiability of the eigenvectors to be defined, since it enforces continuity.

In the transitory case of degeneracy, where the eigenvalues become numerically very similar near coalescence, the computation for the eigenvalue derivatives only fails if the eigenvectors of Eq. (7) do not form an orthogonal set—it is assumed that an orthogonal set of vectors is always enforceable through the transformation matrix $\boldsymbol{\Gamma}$. Computation of the eigenvector derivatives at eigenvalue coalescence, however, requires further treatment. In the present context, the event of an iterate settling upon such a point is a highly improbable event, and so is not anticipated. For the sake of interest, work extending the method of evaluating eigenvector derivatives owing to Nelson [22] has been done by Mills-Curran [24] and Dailey [25] to account for this coalescence; Irwanto et al. [41] develop a method for cyclic structures which utilises the symmetry characteristics to provide a more efficient alternative.

3.2. Newton's method

Assuming that the geometric stiffness matrix is continuously dependent upon some force-indicative scalar parameter p, a factor upon a distribution of axial forces, and that the eigenvalues are likewise continuous functions of the extent of loading of a force distribution, Newton's fundamental theorem of calculus states that, for $\lambda : \mathscr{R}^n \rightarrow \mathscr{R}^m, \quad m \geq n$,

$$\boldsymbol{\lambda}(\mathbf{p}^k + \Delta\mathbf{p}) = \boldsymbol{\lambda}(\mathbf{p}^k) + \int_{\mathbf{p}^k}^{\mathbf{p}^k+\Delta\mathbf{p}} \mathbf{J}(\mathbf{p})\mathrm{d}\mathbf{p}. \tag{10}$$

The multidimensional case is taken as general, in which $\mathbf{J} \in \mathscr{R}^{m\times n}$ is the Jacobian of eigenvalue derivatives and the order is governed by the n linearly independent force distributions existent in a space frame and indicated by the n-tuple $\mathbf{p}$; $\boldsymbol{\lambda}$ is an $\mathbf{m}$-tuple of eigenvalue functions such that

$$\boldsymbol{\lambda}(\mathbf{p}^k) = \bar{\bar{\boldsymbol{\lambda}}}(\mathbf{p}^k) - \bar{\bar{\boldsymbol{\lambda}}}(\mathbf{p}^R), \tag{11}$$

where $\bar{\bar{\boldsymbol{\lambda}}}(\mathbf{p}^k)$ and $\bar{\bar{\boldsymbol{\lambda}}}(\mathbf{p}^R)$ are, respectively, the eigenvalues at the current iterate and at the root. In a first-order approximation of the indefinite integral in Eq. (10), an affine model $\boldsymbol{\xi}$ of the eigenvalue functions can be expressed as

$$\boldsymbol{\xi}(\mathbf{p}^k + \Delta\mathbf{p}) = \boldsymbol{\lambda}(\mathbf{p}^k) + \mathbf{J}(\mathbf{p}^k) \cdot \Delta\mathbf{p}. \tag{12}$$

Solving for the root of the affine model leads to the evaluation of the Newton excursion

$$\Delta\mathbf{p} = -\mathbf{J}^{+}(\mathbf{p}^k) \cdot \boldsymbol{\lambda}(\mathbf{p}^k), \tag{13}$$

$$\Delta\mathbf{p} = -\mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^{\mathrm{T}} \cdot \boldsymbol{\lambda}(\mathbf{p}^k), \tag{14}$$

where $\boldsymbol{\Sigma}$ is the $n \times m$ diagonal matrix of the reciprocals of singular values and $\mathbf{U}$ and $\mathbf{V}$ are, respectively, the left and right singular vector matrices of the Jacobian. The $(k+1)$th iterate is then

$$\mathbf{p}^{k+1} = \mathbf{p}^k + \Delta\mathbf{p}. \tag{15}$$

The inverse of the Jacobian is necessarily the Moore–Penrose pseudoinverse if the system of linear equations (13) is overdetermined. Note that in that instance, the fixed points of the Newton operator do not necessarily correspond to the zeros of $\boldsymbol{\lambda}$, but to a least-squares minimised solution. Overdetermination is a key measure in defining root uniqueness by overcoming ambiguity (repeated system roots).

Whether the root of the affine model approximates the root of the eigenvalue function depends upon the suitability of the approximation—whether the current iterate is in the neighbourhood of the sought root. A higher-order approximation may of course be made of the indefinite integral in Eq. (10). Brandon [42,43] has explored the significance of making a second-order approximation in the context of modal design and has duly compared such an iteration to Newton's method; the indications are that in certain instances, the second-order derivatives are beneficial to the rate of iteration. The computational effort of function modelling may, however be unjustified if an affine modelling is adequate. Indeed, within the neighbourhood of a root, Newton's method can exhibit quotient quadratic convergence for a determined system of nonlinear equations and there are means for upholding convergence away from the root.

3.3. Force identification

Let $\mathbf{p}^*$ be the well-converged solution approximating the root $\mathbf{p}^R$ to some specified tolerance—root uniqueness is discussed ahead. Given the number of frame members f, let $\mathbf{L} \in R^{f \times f}$ be the square equilibrium matrix established by a set of column vectors giving each member force in a specific distribution; these distributions are arbitrarily chosen to have a maximum absolute frame force of positive unity; present convention takes axial tension as a positive force. Owing to frame topology and the number of static redundancies, $\mathbf{L}$ is likely to be rank deficient. This rank n governs the order of the system of nonlinear equations and hence the number of linearly independent force distributions that need to be sought. The form of $\mathbf{L}$ is readily known for simple frames; for more complex frameworks, a static finite element analysis may be necessary and one is suggested in Section 3.5. Once the rank of $\mathbf{L}$ is determined, the full rank matrix $\hat{\mathbf{L}}$ can be established by deleting suitable columns of $\mathbf{L}$. In cyclically symmetric frames, there will exist in $\mathbf{L}$ sets of identical columns which require deleting. Upon convergence, the solution to the member forces can be expressed as

$$\mathbf{f}^* = \hat{\mathbf{L}}\mathbf{p}^*. \tag{16}$$

The full rank of this force distribution matrix means that the solution to the force parameters

$$\mathbf{p}^* = \hat{\mathbf{L}}^+\mathbf{f}^* \tag{17}$$

is unique for a given state of frame equilibrium $\mathbf{f}^*$. Thus, a unique solution to the set of nonlinear eigenvalues functions can be expected in the sense that there is not more than one $\mathbf{p}^*$ leading to the solution $\mathbf{f}^*$ and it is this that must be sought.

3.4. Geometries exhibiting cyclic symmetry

If the matrix $\mathbf{L}$ is rank deficient and is not suitably reduced, then the Jacobian matrix of eigenvalue derivatives will be singular and its inverse will not exist. The Moore–Penrose pseudoinverse of the Jacobian in Eq. (14) could be taken with truncation of the zero singular values of the Jacobian, that is to say forcing their reciprocals to zero in the matrix $\mathbf{\Sigma}$, in order for the Newton excursion to be determined. However, there would then exist an infinite number of solutions to the force parameters, a cluster of which would exist within the pre-buckled region of the iteration space. There would be no guarantee of discovering the true root, although beneficially any of such a family of roots would lead to the correct solution to the member forces. However, the dimensionality of the iteration space would be unnecessarily large, encumbering iteration on more counts than just iterative adversity—see the end of the current subsection. It is therefore advisable to always seek the full rank equilibrium matrix $\hat{\mathbf{L}}$.

It was earlier stated that the event of an iterate encountering a point of transitory eigenvalue coalescence is not anticipated. In the case of symmetric space frames with multiple axes of spatial periodicity, the hypersurfaces representing the eigenvalue functions of the n force parameters p have an n-dimensional focus of symmetry defined as the hyperplane where all of the parameters p are equal. This owes to the spatial ambiguity; certain force distributions will be identical in every sense but orientation and will therefore be interchangeable independent variables with respect to the eigenvalue functions. At the symmetry focus, the eigenvalue functions are transitorily coalesced. The coalescence is not simple and may involve combinations of distinct and two-fold degenerate eigenvalue curves intersecting. The equation computing the Fréchet eigenvalue derivative (7) then no longer holds. Gâteaux (directional) derivatives would be computable by finite difference schemes, but would pose computational expense. If one of the eigenvalues at the focus separates immediately away from it, then an orthogonal set of eigenvectors can be found through Gram–Schmidt orthogonalisation taking the associated eigenvector as datum. However, it is an easy matter and simpler to avoid this focus of symmetry by ensuring that the vector $\mathbf{p}^0$ at which iteration commences has non-repeated values p_i, $i = 1, \ldots, n$; the event of encountering it subsequently is immensely improbable.

Further for frames possessing more than one axis of spatial periodicity—for example those based on platonic solids—there exist roots to the set of nonlinear eigenvalue functions which, although not leading to the correct solution to the member forces, give a solution that describes the actual sought force distribution at some different spatial orientation. This owes to the spatial ambiguity of the geometries of such frames in that they are identical from multiple spatial aspects. There are then a number of roots $\mathbf{p}^R_{prm}$ at which the eigenvalue constraints are exactly satisfied. These ambiguous roots are informative in that they identify the member forces but it is the designation of each member force that remains uncertain. One would not be able to confidently ascribe the found member forces. This issue would jeopardise the uniqueness of the solution $\mathbf{p}^R$ were it not for mode tracing discerning the eigenvalue functions upon which the unique solution lies. If mode tracing is not employed, then the roots $\mathbf{p}^R_{prm}$ and $\mathbf{p}^R$, which conform to the same hyperplane—since they give rise to the same frame eigenvalues and are symmetric through the symmetric focus—lie on the same eigenvalue function as defined by numerical ordering, introducing ambiguity. The above is diagrammatically described in Fig. 3, where for a frame whose eigenvalues are dependent upon two force parameters, modes 1 and 4 (numerical designations) are chosen at the root for the iteration. Since the starting surfaces are nominated according to the consistency with the root eigenvectors, stagnation at the $\mathbf{p}^R_{prm}$ roots is avoided and convergence to $\mathbf{p}^R$ ensues. Thus the modes at the point at which iteration commences are 2 and 3 (numerical designations), respectively, corresponding to those at the root. The physical interpretation of why mode tracing overcomes this ambiguity is that, although the force distributions are spatially ambiguous, the eigenvector–force distribution relationships are unique, and so the eigenvector-based mode tracing is afforded a discernment of the unique root. The solution is well converged after three iterations.

Note that if the dimensionality of the iteration space is unnecessarily large because of incomplete reduction of the equilibrium matrix, as just discussed, then even mode tracing cannot prevent convergence to an erroneous root. This is because of the potential of discovering a $\mathbf{p}^R_{prm}$ root, which, because of their abundance and locations on the eigenvalue hypersurfaces would not allow the type of discernment depicted in Fig. 3. This is the main thrust for reducing the dimensionality of Newton's method.

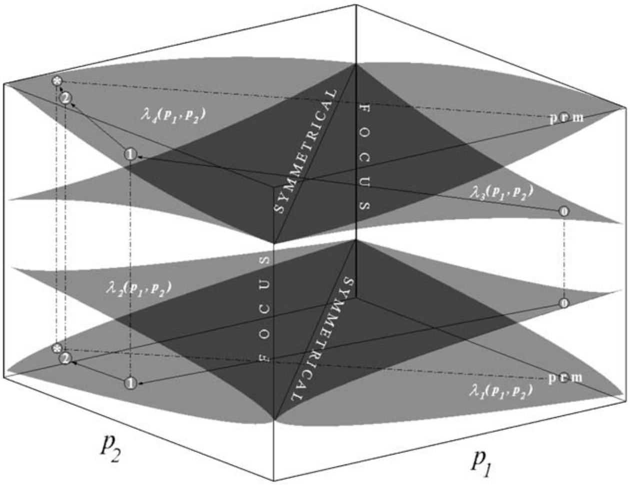

Fig. 3. Mode tracing overcoming root ambiguity in the force identification of a cyclically symmetric frame possessing two axes of spatial periodicity; functions are labelled according to default numerical ordering.

3.5. Geometries deviated from cyclic symmetry

If the geometry of a space frame is such that it marks a deviation from some regular, cyclic form to generate an irregular frame, then particularities of symmetric frames do not hold (for example, the symmetrical focus wanders into less readily defined space)—however, they are not wholly alleviated. Although the force distribution matrix **L** will have developed distinct columns, it will still be rank deficient and therefore, again, a full rank matrix $\hat{\mathbf{L}}$ needs to be developed. That is to say, particular force distributions will be expressible as linear combinations of other distributions, and therefore present surplus force parameters, wherefore they should be disregarded. The number of linearly independent force distributions in an irregular frame will be equal to that for its regular counterpart due to topology conservation.

The geometries of such frames may mean that the force distributions are not immediately apparent. In this case, a static analysis can be employed in which successive members are removed, one end restrained in 6 dof to alleviate singularity of the stiffness matrix, and a force parallel to the removed member applied at the other end. The global solution vector of displacements would then simply be the inverse of the constant stiffness matrix multiplied by the applied force vector. This would allow evaluation of member forces resulting from a particular member strain. If the equilibrium matrix for the frame, **L**, is developed through such a static analysis, computational inaccuracy may be introduced and a further, inspective process needs to be employed to establish the full rank matrix $\hat{\mathbf{L}}$.

If σ_1 and σ_k are, respectively, the first and last significant singular values of the equilibrium matrix, then $\hat{\mathbf{L}}$ should contain select columns of, for example, the conditioned matrix

$$\mathbf{L} = \begin{bmatrix} \mathbf{U}_{i,1} & \cdots & \mathbf{U}_{i,\kappa} \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_\kappa \end{bmatrix} \begin{bmatrix} \mathbf{V}_{i,1} & \cdots & \mathbf{V}_{i,\kappa} \end{bmatrix}^{\mathrm{T}} \quad i = 1, \ldots, f. \tag{18}$$

Failure to do so will result in the solution, in the hypothetical absence of dynamic measurement and modelling errors, converging to inaccurate member forces.

3.6. Mode tracing

The ordering of eigenpairs is arbitrary. Typically, eigenpairs are ordered according to the magnitudes of the eigenvalues. If this is done, then transitory eigenvalue coalescence with respect to some independent variable defines non-smooth eigenvalue functions of that variable.

However, while the eigenvalues exhibit considerable variation with respect to a parameter, the eigenvectors may experience quite small changes over particular ranges of parameter. Indeed, of relevance to the present context, Mead [12] notes that the mode shapes of fixed–jointed frames change little if at all with respect to loading. Previous work has exploited the steadfastness of eigenvectors in correlating modes, preceding and proceeding parameter perturbation. An assessment of how the mode shapes actually vary in such frames and how much confidence can be placed in them for the purpose of tracing modes is given by Bahra and Greening [45]. Ting et al. [28] have used the Modal Assurance Criterion [44] (MAC) to develop a Boolean operator, operating upon the nominated subset of eigenvectors of a current iterate and permuting them to concur in a physical sense with those of the preceding iterate. Kim and Kim [32] have utilised the MAC similarly in structural topology optimisation. Analogously, cross-mass-orthogonality (XOR) has been implemented to the same end with good effect by Gibson [29] and Eldred et al. [30]. Other methods include those that deal with perturbation expansions of the eigenproblem—Eldred et al. [27], but these are deemed computationally expensive and not as suitable for the present task of mode tracing in frame force identification.

Mode tracing here uses the root as a reference so that not only are the frequencies to be determined from the investigated physical frame, but also the mode shapes. It is desirable that the detail to which a mode shape has to be defined be minimal. Experience has shown that good consistency holds between mathematical eigenvectors and simulated mode shapes defined at a very limited number of freedoms; this is a subject for further investigation. Presently, the MAC suffices in tracing modes and is implemented as follows—note also that since the mass matrix is stationary with respect to the force parameters, XOR could also be used as an efficient mode tracer.

Let there be $1, \ldots, s$ modes in the root subset S, which are those modes observed in the physical structure, and let there be $1, \ldots, t$ modes in a subset T at the current iterate, k. Generally, t should be greater than s in order that T encompasses all of the modes in S. Further, let

$$\mathbf{MAC}_{ij}\left(\overline{\phi_i^k}, \phi_j^R\right) = \frac{\left|\left(\overline{\phi_i^k}\right)^{\mathrm{T}}\left(\phi_j^R\right)\right|^2}{\left\|\overline{\phi_i^k}\right\|_2^2\left\|\phi_j^R\right\|_2^2} \in [0, 1] \quad i = 1,\ 2,\ \ldots,\ s, \quad i = 1,\ 2,\ \ldots,\ t, \tag{19}$$

be the elements of the $\mathbf{MAC}$ matrix of dimension $s \times t$ connoting vector consistency between the ith eigenvector in S and the jth eigenvector in T. Over-bars stipulate that eigenvector bases be aligned to the root eigenvector bases prior to evaluation of consistency. For degenerate eigensystems, this alignment may be necessary and is the subject of the study by Bahra and Greening [40]; it is briefly discussed ahead. If the sets exist in t-column matrices, the permutation of modes in T may be expressed as

$$\left\{\overline{\phi_1^k}\ \cdots\ \overline{\phi_t^k}\right\} \to \left\{\overline{\phi_1^k}\ \cdots\ \overline{\phi_t^k}\right\}\mathbf{\Theta}^{\mathrm{T}} \tag{20}$$

and consistently

$$\left\{\lambda_1^k\ \cdots\ \lambda_t^k\right\} \to \left\{\lambda_1^k\ \cdots\ \lambda_t^k\right\}\mathbf{\Theta}^{\mathrm{T}}, \tag{21}$$

where the $s \times t$ Boolean operator $\mathbf{\Theta}$ is a binary matrix whose rows contain exclusive units corresponding to the row maxima of the matrix **MAC**. By Eqs. (20) and (21), with the assumption that there is no ambiguity in vector correlation, there are always s modes with physically consistent permutations in consideration.

If the Newton excursion is large, or if a region of loci veering—where there is coupling of eigenvectors—is encountered, the assurance with which modes can be traced may be inadequate. Bahra and Greening [40] propose a potential augmentation to assurance generally, where affine models of the freedoms of the eigenvectors at the kth iterate are made from knowledge of their derivatives and the amount by which the parameter has been perturbed by an iterative excursion; this allows a forward casting of the eigenvector freedoms, and a potential heightening of mode tracing assurance. This augmenting is recommended only for eigenvalue problems dependent upon a single parameter, since the eigenvectors may not be analytic functions of multiple parameters.

As stated, for axially loaded, cyclically symmetric space frames, there exists a two-fold permanent degeneracy of certain eigenvalues and the bases of the associated eigenvectors are arbitrary and non-unique. If mode tracing is to be performed on these degenerate modes, then the eigenvector bases need to be consistently aligned, and this is now briefly outlined. Note that what follows is essentially of relevance to once-redundant frames, since the degeneracy of eigenvalues in frames with multiple force distributions exists only in confined regions of the iteration space. For example, degeneracy exists in a multiply redundant frame when it is loaded in a single-force distribution. Permanent degeneracy is really only of concern in once-redundant space frames, where the single-force distribution can only alter the eigenvectors in a consistent manner.

Since mode tracing is performed with the modes at the root, that is to say those modes physically observed, as those to which the modes of the iteration are referenced, let the permanently degenerate mode eigenvectors at the root be considered as the datum set; all other bases can be aligned to these vector bases by defining the transformation

$$\overline{\mathbf{\Gamma}} = \left(\left(\begin{bmatrix}\phi_r & \phi_{r+1}\end{bmatrix}^k\right)^{\mathrm{T}}\begin{bmatrix}\phi_r & \phi_{r+1}\end{bmatrix}^k\right)^{-1}\left(\left(\begin{bmatrix}\phi_r & \phi_{r+1}\end{bmatrix}^k\right)^{\mathrm{T}}\begin{bmatrix}\phi_r & \phi_{r+1}\end{bmatrix}^R\right), \tag{22}$$

which serves to perform the least squares minimisation

$$\min\left(\mathrm{tr}\left(\left(\begin{bmatrix}\phi_r & \phi_{r+1}\end{bmatrix}^k\overline{\mathbf{\Gamma}} - \begin{bmatrix}\phi_r & \phi_{r+1}\end{bmatrix}^R\right)^{\mathrm{T}} \times\left(\begin{bmatrix}\phi_r & \phi_{r+1}\end{bmatrix}^k\overline{\mathbf{\Gamma}} - \begin{bmatrix}\phi_r & \phi_{r+1}\end{bmatrix}^R\right)\right)\right). \tag{23}$$

The aligned eigenvectors at iterate k are given as

$$\begin{bmatrix} \overline{\phi}_r & \overline{\phi}_{r+1} \end{bmatrix}^k = \begin{bmatrix} \phi_r & \phi_{r+1} \end{bmatrix}^k \overline{\boldsymbol{\Gamma}} \cdot \begin{bmatrix} w_1^{-1/2} & 0 \\ 0 & w_2^{-1/2} \end{bmatrix}, \tag{24}$$

where the vector of modal masses preserving mass-orthonormality is defined as

$$\mathbf{w} = \mathrm{diag}\left(\overline{\boldsymbol{\Gamma}}^{\mathrm{T}}\left(\begin{bmatrix} \phi_r & \phi_{r+1} \end{bmatrix}^k\right)^{\mathrm{T}} \mathbf{M}\left(\begin{bmatrix} \phi_r & \phi_{r+1} \end{bmatrix}^k\right)\overline{\boldsymbol{\Gamma}}\right). \tag{25}$$

Such vector alignment has been used in the reconciliation of mathematical and observed oscillatory modes in the presence of repeated frequencies by Lallement and Kosanek [46] and Pešek [47]. Presently, it is used in ensuring continuity of eigenvector bases for the purposes of mode tracing. Since it is not known which degenerate pair at the root should form the reference to which the degenerate modes at the kth iterate are aligned, all of the possible consistency matrices, equal in number to the number of degenerate modes in set T, need to be evaluated. The row units of the binary matrix $\boldsymbol{\Theta}$ in this instance are positioned in correspondence to the locations of the global row maxima of the *set* of consistency matrices.

For a more comprehensive account, the reader is referred to Bahra and Greening [40]. Other works of possible interest are those of D'Ambrogio and Fregolent [48], regarding consistency computation between an eigenvector and a subspace of degenerate mode eigenvectors, and Walther et al. [49], concerning consistency evaluation between mathematical and degenerate, disorientated and coupled physical modes.

3.7. Global convergence

Newton's method, although highly efficient in the neighbourhood of a root, is notorious for failing to converge in certain circumstances away from a root. The study of ensuring global convergence for Newton's method is well established. Such methods limit the lengths of wayward Newton excursions and seek to minimise some relevantly defined objective function. The literature on global methods of iterative convergence is readily available: pertinent texts include Ortega and Rheinboldt [50] and Dennis and Schnabel [51]. The topic is not here covered, the focus being on treating the particularities of Newton's method in the present context.

To confine iteration to the pre-buckled region, monitoring of the numerically designated fundamental eigenvalue may be utilised, since this goes to zero with the onset of buckling, marking the change from a strain mode to a zero frequency, rigid body mode.

4. Numerical simulations

Three numerical simulations demonstrating the concepts outlined in Section 3 are now given. General concepts, such as mode tracing and overdetermination, are demonstrated in the case of a frame with one distribution of axial force and hence force parameter. Frames of regular and irregular geometries are then used as illustrators of multidimensional force determination. The nominated geometries offer bases from which issues for more general space frames can be extrapolated. Effects of noise are included in the simulations to indicate the bounds upon errors that may be expected.

It is not the intention to provide high order, correlated models as illustrators. The intention is that the issues raised be made evident, so that if Newton's method is employed in space frame force identification, these are made apparent and therefore treatable. The models used in the numerical simulations suffice to do so. Note that there is an assumption of constant geometry with respect to frame loading. Of course, in practice, deviations in geometry would accompany the approach to buckling, and this second-order effect may need to be taken into account in the mathematical modelling.

For consistency, all distributions of axial force are normalised such that the maximum absolute member force is positive unity when tensile, so that the force parameters p, which are factors upon these force sets, are effectively the maximum absolute frame forces. All member discretisation for dynamic modelling is six elements. Modal designation is stated according to the ordering at the root and notwithstanding the ordering at any other loaded state, so that mode r is not necessarily the rth mode at a particular value of p. All member material properties are as follow: Young's modulus, $2.1 \times 10^{11}\,\mathrm{N\,m^{-2}}$; density, $7.85 \times 10^{3}\,\mathrm{kg\,m^{-3}}$.

Without being stated, mode tracing is in all places used as default. Lastly, the modes used in the force identification are in all places the initial, consecutive modes at the root, since it is felt that these would be the most readily measurable in a modal test.

4.1. Bi-tetrahedral frame

The bi-tetrahedral frame comprises two skeletal, regular tetrahedra having three mutual members; an internal spar joins the opposing apices to introduce static redundancy. The redundancy entails frame loading following member strain—a statically determinate frame would not be loaded but would deform geometrically. The frame can be seen in Fig. 4. The length of the external frame member is nominated as 0.5 m and the members are of circular section and diameter 0.01 m. The member centre-lines are concurrent at the joints so that the transfer of axial force alone is possible.

From a simple static arguments the frame member forces can be determined and these can be expressed as follows: internal member, 1; hemispherical members, $-(3\sin\alpha)^{-1}$; equatorial members, $2(9\sin\alpha)^{-1}$; $\alpha = \arccos\left(3^{-1/2}\right)$.

The variations of the frame eigenvalues up to 1×10^{6} with respect to the scalar parameter p are shown in Fig. 5 between the buckling limits at which the fundamental frequency—whatever mode that may be in that load region—vanishes. The curves are seen to be exceedingly nonlinear, with the eigenvalue coalescence and loci veering phenomena clearly visible. As acknowledged, Newton's method can be encumbered if the functions are non-smooth.

For the once-redundant, bi-tetrahedral frame, the eigenvector–force distribution relative orientation is at all loads conserved. In the mathematical model therefore, certain pairs of eigenvalues will be permanently degenerate in that they will not separate upon parameter perturbation. Indeed, many of the eigenvalue loci in Fig. 5 are two loci, one superimposing the other. The mode tracing of these types of mode needs special attention and it proves necessary to

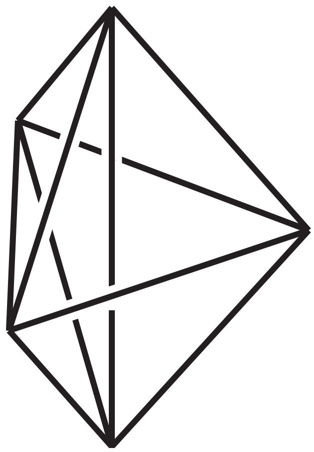

Fig. 4. Bi-tetrahedral frame geometry; complete *relative* fixity exists between the members at the joints.

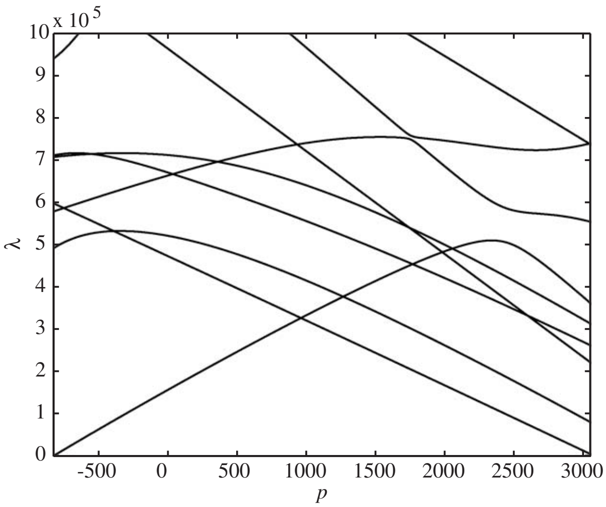

Fig. 5. Pre-buckled eigenvalue–force parameter loci for the bi-tetrahedral frame showing eigenvalue coalescences and various degrees of veering; p(N) is the maximum absolute frame force, positive when tensile.

align the arbitrary eigenvalue bases between iterates before employing consistency evaluation. The existence of multiple force distributions in the frame of subsequent examples means that the frame eigenvalues are generally distinct, and so it is only in the once-redundant case that this special treatment of vector bases is required.

Convergence will be very rapid if the eigenvalue function nominated for use in the iteration is near linear between the root and the point at which iteration commences. This is simply a result of the apt linear modelling of the indefinite integral in Eq. (10) in Newton's method. If such modes are utilised, then convergence even across the extreme buckling limits can be achieved practically after a single iteration. It is of course a matter of chance whether a linear eigenvalue function is nominated for the identification or not and is not a matter of choice, unless second derivatives and higher are inspected. This, however, may prove computationally expensive. Certainly eigenvalues with small first derivatives should be avoided as these will give instability to Newton's method. No general method can be suggested for the selection of eigenvalues to use in iteration, but given a set of m eigenvalues, with preferably $m > n$, it is likely that there will be a sufficient number of assistive eigenvalue functions to overcome antagonising ones—see Bahra and Greening [52].

The curvature of the coupled loci in Fig. 5 potentially give rise to root multiplicity, since for a given eigenvalue there can be two roots on a given, veered curve. This is important to know since convergence upon an erroneous root is misleading. The advantage of the present context of structural dynamics is that, within reason, there are many modes available as functions of the force parameter, and hence the system of equations (13) can be overdetermined, as is common practice, to alleviate the ambiguity of multiple roots. Fig. 6 shows convergence upon the negative buckling load, and as a harsh test, iteration is chosen to commence at the positive buckling load. For this task, all combinations of the three modes 3, 8 and 9 are used. For interest, these are represented by the three loci in Fig. 5 whose stationarity is in the vicinity of $p = -500$. Respectively, each of these modes used independently is seen to converge to its erroneous root since the true root is in each instance 'over hill' of the former. In combination, correct convergence is seen to result since mutuality between the modes must be met in convergence. One exception is the combination of modes 3 and 8: here, the Newton excursion at the erroneous root is near zero, and hence the iteration stagnates there. This is a rare occurrence and, nevertheless, overdetermination is a steadfast means for overcoming root ambiguity.

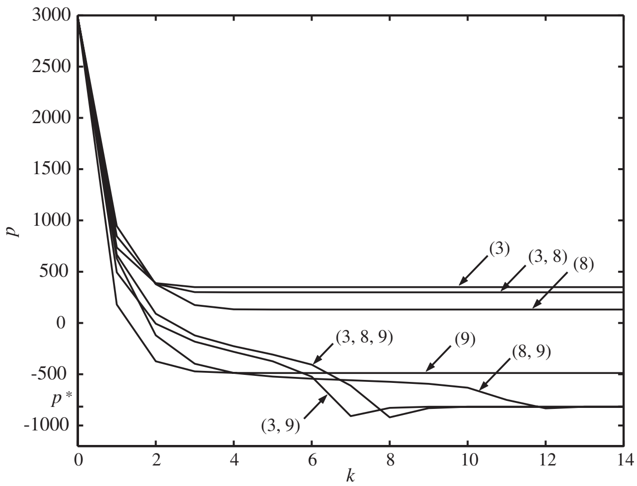

Fig. 6. Buckling-to-buckling, true and erroneous, convergences resulting from all combinations of three ambiguous eigenvalue loci (cf. those loci having their points of stationarity in the vicinity of $p = -500\,\text{N}$ in Fig. 5).

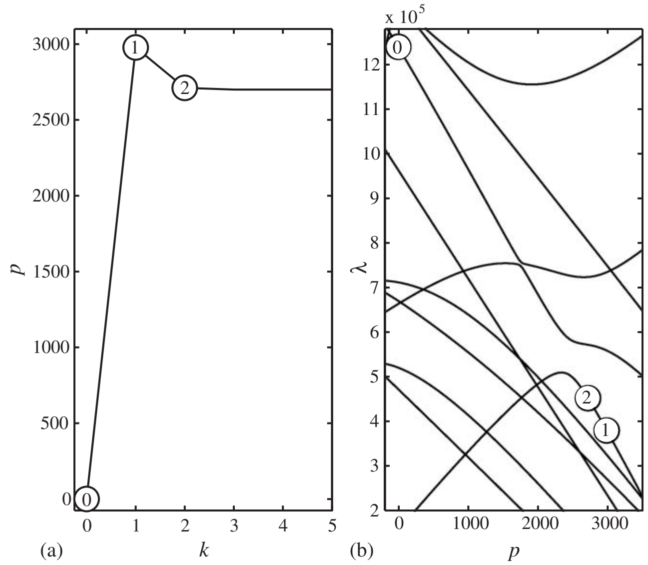

Fig. 7. Iteration past a veering event: (a) progression with respect to iteration number and (b) progression along eigenvalue loci with iteration transcending loci.

All of these curves were such that the iterates conformed to a single locus throughout the iteration. In loci veering however, it is possible for iterates to transcend curves when mode tracing is employed, as has been discussed. Fig. 7 shows how iteration transcends curves at the veering around $p = 2500$; the root is just passed the veering point at $p^R = 2700$ and iteration commences from $p^0 = 0$. Correct convergence ensues using mode 8, illustrating that the purpose of employing mode tracing is to conserve function smoothness for the benefit of the iteration; the definition of the functions between the initial parameter value and root is otherwise not of importance. Note that this is completely analogous to what happens in the limiting case of mode tracing passed an eigenvalue coalescence point.

4.2. Regular octahedral frame

The second frame to be analysed is octahedral. The opposing apices of the bi-tetrahedral frame make it possible to insert an internal redundancy; the octahedral frame has the potential to accommodate three internal members, introducing a degree of redundancy three. Again, the length of the external frame member is nominated as 0.5 m and the members are of circular section and diameter 0.01 m. The octahedral frame is depicted in Fig. 8.

From a simple compatibility analysis, it is found that about any single axis of the frame, there exist two classes of force distribution. These classes may be distinguished by the type of member that needs to be strained to produce them: external (class I) or internal (class II). However, since the frame possesses three axes of spatial periodicity and is identical from the aspect of each, the total number of distinct force distributions is six. These are given in the equilibrium matrix as

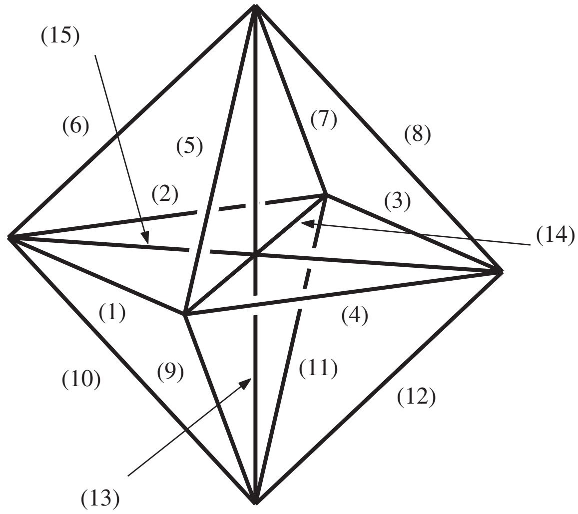

Fig. 8. Octahedral frame geometry; complete *relative* fixity exists between the members at the joints.

follows (member numbers correspond to those in Fig. 8):

$$\mathbf{L} = \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array} \left[\begin{bmatrix} f_4 & & f_4\\ f_4 & & f_4\\ f_4 & & f_4\\ f_4 & & f_4\\ f_2 & & f_2\\ f_2 & & f_2\\ f_2 & & f_2\\ f_2 & \cdots & f_2\\ f_2 & & f_2\\ f_2 & & f_2\\ f_2 & & f_2\\ f_2 & & f_2\\ f_3 & & f_3\\ f_1 & & f_1\\ f_1 & & f_1 \end{bmatrix}_{15\times 4} \begin{bmatrix} f_2 & & f_2\\ f_2 & & f_2\\ f_2 & & f_2\\ f_2 & & f_2\\ f_4 & & f_4\\ f_2 & & f_2\\ f_4 & & f_4\\ f_2 & \cdots & f_2\\ f_4 & & f_4\\ f_2 & & f_2\\ f_4 & & f_4\\ f_2 & & f_2\\ f_1 & & f_1\\ f_1 & & f_1\\ f_3 & & f_3 \end{bmatrix}_{15\times 4} \begin{bmatrix} f_2 & & f_2\\ f_2 & & f_2\\ f_2 & & f_2\\ f_2 & & f_2\\ f_2 & & f_2\\ f_4 & & f_4\\ f_2 & & f_2\\ f_4 & \cdots & f_4\\ f_2 & & f_2\\ f_4 & & f_4\\ f_2 & & f_2\\ f_4 & & f_4\\ f_1 & & f_1\\ f_3 & & f_3\\ f_1 & & f_1 \end{bmatrix}_{15\times 4} \begin{bmatrix} f_2\\ f_2\\ f_2\\ f_2\\ f_5\\ f_5\\ f_5\\ f_5\\ f_5\\ f_5\\ f_5\\ f_5\\ f_1\\ f_2\\ f_2 \end{bmatrix} \begin{bmatrix} f_5\\ f_5\\ f_5\\ f_5\\ f_2\\ f_5\\ f_2\\ f_5\\ f_2\\ f_5\\ f_2\\ f_5\\ f_2\\ f_2\\ f_1 \end{bmatrix} \begin{bmatrix} f_5\\ f_5\\ f_5\\ f_5\\ f_5\\ f_2\\ f_5\\ f_2\\ f_5\\ f_2\\ f_5\\ f_2\\ f_2\\ f_1\\ f_2 \end{bmatrix} \right],$$

where

$$f_1 = 1,\ f_2 = \beta,\ f_3 = -2\sqrt{2}\beta,\ f_4 = -\left(\sqrt{2}+3\right)\beta,\ f_5 = -\left(\sqrt{2}^{-1}+1\right)\beta,\ \beta = 2\left(\sqrt{2}+1\right)^{-1}.$$

The first three matrices comprise vectors that relate to force distributions resulting from the straining of an external frame member and therefore represent class I. They are four-fold multiple because of the symmetry of the regular octahedral frame—this can be appreciated by seeing that the external frame is formed of three orthogonal, square, plane frames and that straining of any one member of these is the same as straining any other. The last three, distinct vectors of the equilibrium matrix relate to straining of an internal frame member, and therefore represent class II distributions. Of course, the three distributions of the respective classes amongst themselves represent the same force distribution orientated according to the three different axes of spatial periodicity, so that variations of eigenvalues are three times identical. With p_1, p_2 and p_3 being defined as factors upon the class I distribution, and p_4, p_5 and p_6 those upon the class II distribution, Fig. 9 shows how the frame eigenvalues vary with respect to loading between the buckling limits.

It is found that the equilibrium matrix has rank three, and further that a reduced, full rank matrix can be established with the three distributions of class I alone, for example. The physical interpretation is that any possible state of equilibrium of the octahedral frame can be expressed as a linear combination of these three linearly independent force distributions. Therefore, let the force parameters governing the iteration space be factors upon the first three types of force distribution, p_1, p_2 and p_3.

The point made in Section 3.4 regarding root uniqueness is well demonstrated by seeking a root $\mathbf{p}^R = \begin{bmatrix} 1000 & 3000 & 5000 \end{bmatrix}^T$ and initiating iteration at a set $\mathbf{p}$ at which there exists a $\mathbf{p}^R_{prm}$ root: $\mathbf{p}^0 = \mathbf{p}^R_{prm} = \begin{bmatrix} 3000 & 5000 & 1000 \end{bmatrix}^T$. This coordinate simply represents the distribution of force in the frame as at the true root but at a different spatial orientation. Without mode tracing, the iteration would not make an excursion from this starting value since the eigenvalue hypersurface

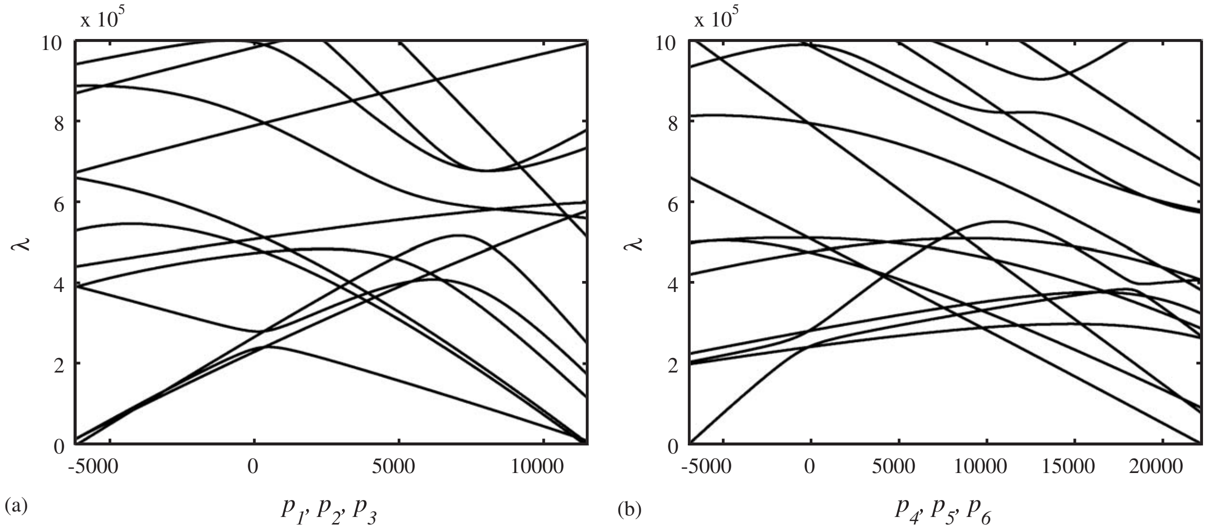

Fig. 9. Pre-buckled eigenvalue–force parameter loci for the octahedral frame showing various orders of eigenvalue coalescence and degrees of veering with fairly densely distributed loci: (a) distribution class I and (b) distribution class II; p_n(N) is the maximum absolute frame force in distribution n, positive when tensile.

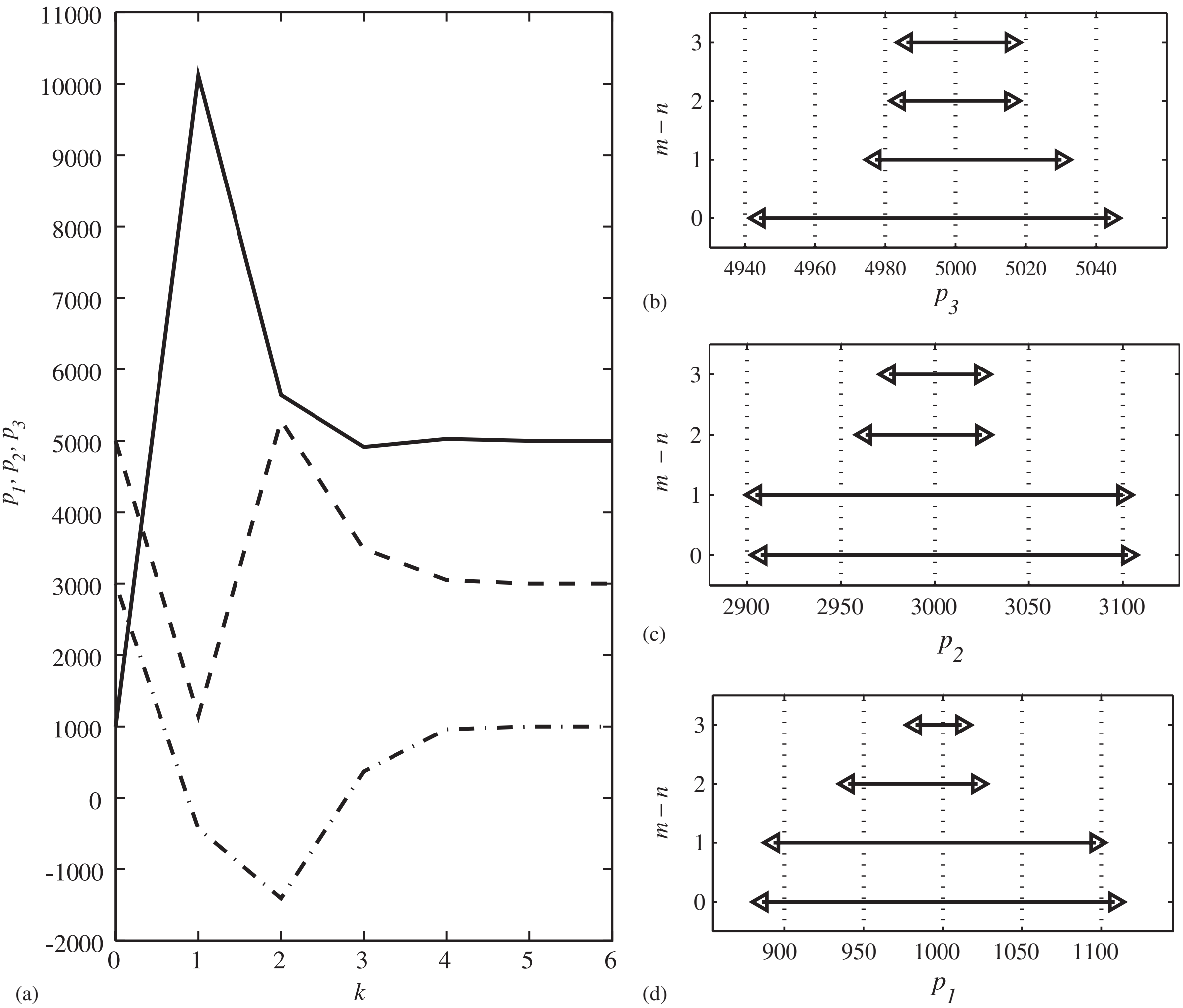

Fig. 10. Mode tracing overcoming root ambiguity in the force identification of the octahedral frame: (a) iteration progression (— · — · — p_1; – – – – p_2; ——— p_3); Monte Carlo noise simulations with varying degrees of overdeterminacy for (b) p_3, (c) p_2 and (d) p_1.

upon which the false root lies is a reflection through the three-dimensional focus of symmetry of the hypersurface upon which the true root lies, and therefore the eigenvalue constraint would automatically be satisfied. Fig. 10(a) shows that with mode tracing employed on the first three root modes, the correct eigenvalue curve is discerned so that the true root is discovered. The reason that the curve of the $\mathbf{p}^R_{prm}$ root is not traced with the curve of the true root $\mathbf{p}^R$, and that curve consistency is maintained, can be appreciated from seeing that the eigenvectors of these respective roots used in the mode tracing have a unique relation to their force distributions, which are further spatially orthogonal and therefore on a global level mode tracing does not pair them.

Noise is simulated by Monte Carlo runs in which the nominated random noise is at most $\pm 5\%$ of the root fundamental frequency, for any given frequency used in the identification. For each degree of overdeterminacy, one hundred runs were performed and the ranges of the results can be seen in Fig. 10. Expectedly, overdetermination helps to improve the precision of identification.

4.3. Irregular octahedral frame

A topologically similar frame is now considered. If the spatial locations of the five apices of the regular octahedral frame are described by the matrix

$$\mathbf{C}_0 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^{\mathrm{T}},$$

the geometry of the irregular octahedral frame is described by $\mathbf{C}_0 + \begin{bmatrix} \mathbf{I} & \mathbf{I} \end{bmatrix}^{\mathrm{T}}$. All other properties are conserved in the transformation from the regular frame. Owing to the more complex geometry, force distributions are determined from the static matrix analysis outlined in Section 3.5. The irregular octahedral frame is shown in Fig. 11 with member numbering consistent with that in Fig. 8.

Following conditioning of the equilibrium matrix $\mathbf{L}$ and the establishment of the reduced matrix $\hat{\mathbf{L}}$, it is found that the number of linearly independent force distributions—that is to say, the minimum number whose linear superposition is suffices in describing any possible equilibrium state—is conserved following the breaking of symmetry. This must hold since the topology is conserved. Physically, the member-joint relations remain identical and therefore the irregular frame cannot posses any further linearly independent force distributions in comparison to the regular frame from which it is developed. Although the present technique suffices to determine the number of linearly independent force distributions, it would be useful to posses a direct means for evaluating this number. The well-known Maxwell formula for computing the number of structural redundancies is not beneficial in that it does not hold universally; this remains a matter of further work.

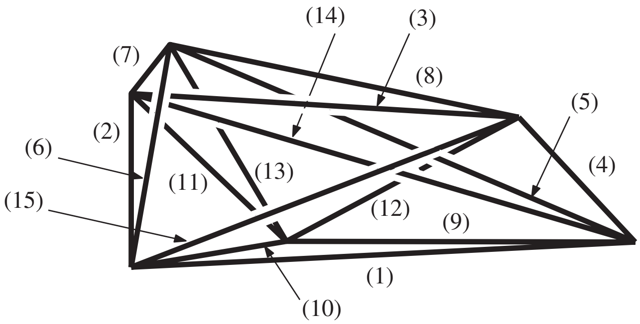

Fig. 11. Irregular octahedral frame geometry; complete *relative* fixity exists between the members at the joints.

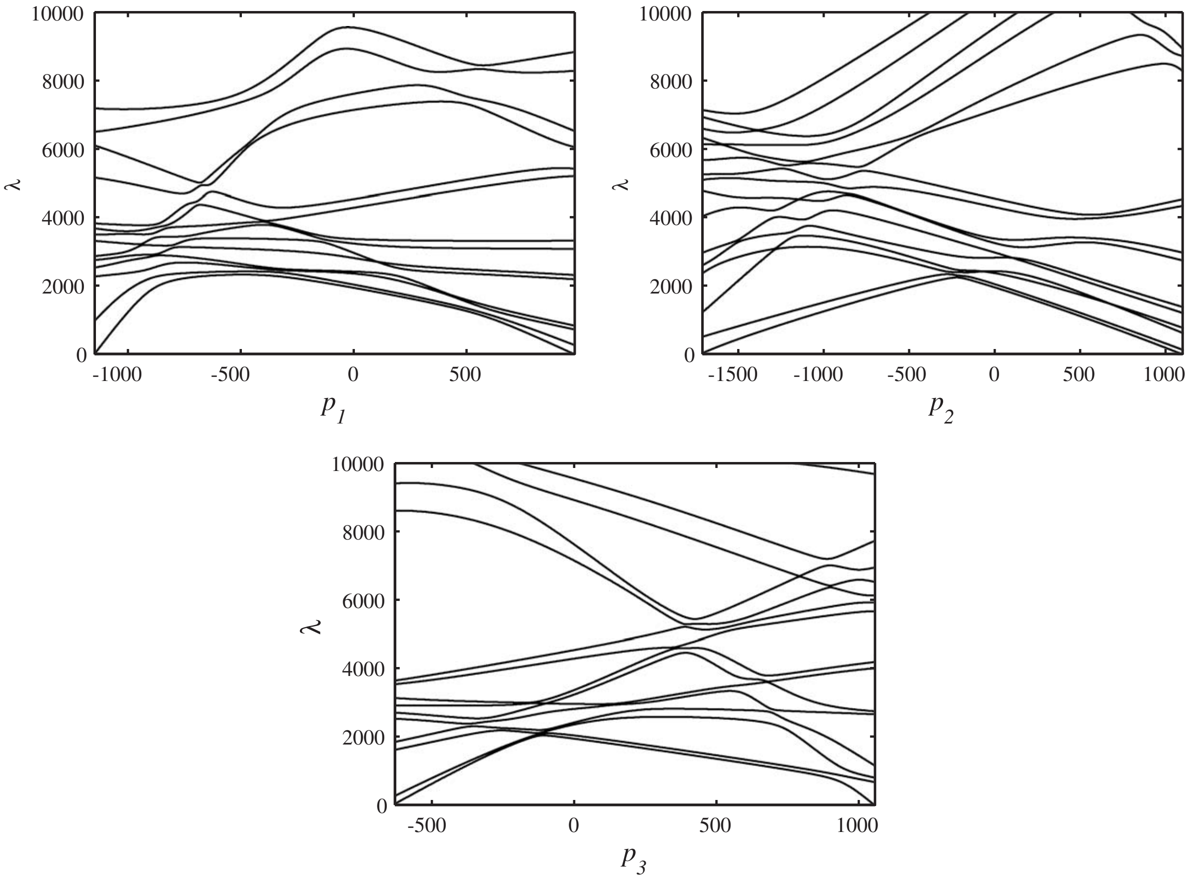

Fig. 12. Pre-buckled eigenvalue–force parameter loci for the irregular octahedral frame showing erratic behaviour (although not appreciable at this scale, all locus interactions are veerings with no coalescences); p_n(N) is the maximum absolute frame force in distribution n, positive when tensile.

The eigenvalue–force parameter loci for three particular distributions are given in Fig. 12. These distributions are those that arise from the straining of members 1, 2 and 3—refer to Fig. 11. Of immediate note is the increased erratic characteristic of the functions, potentially posing difficulty to the iteration. Upon closer inspection it is found that there is no coalescence in the loci, but only veerings. This is most probably ascribable to discretisation problems but as previously seen will not pose a problem if mode tracing is employed. Another point to note is that, owing to the increased slenderness of some of the members of the irregular frame, the frequencies are in general lower. Consequently, the irregular frame cannot withstand as much loading as its regular counterpart.

A target root of $\mathbf{p}^R = \begin{bmatrix} 200 & 200 & 200 \end{bmatrix}^T$ is nominated and convergence using the initial four root modes from an unloaded state is shown in Fig. 13(a). Four modes are used since a degree of overdeterminacy one is the minimum required for convergence to result at all. Again simulated noise is included in the manner as previously described and is found to have decreasing impact with respect to increasing overdetermination.

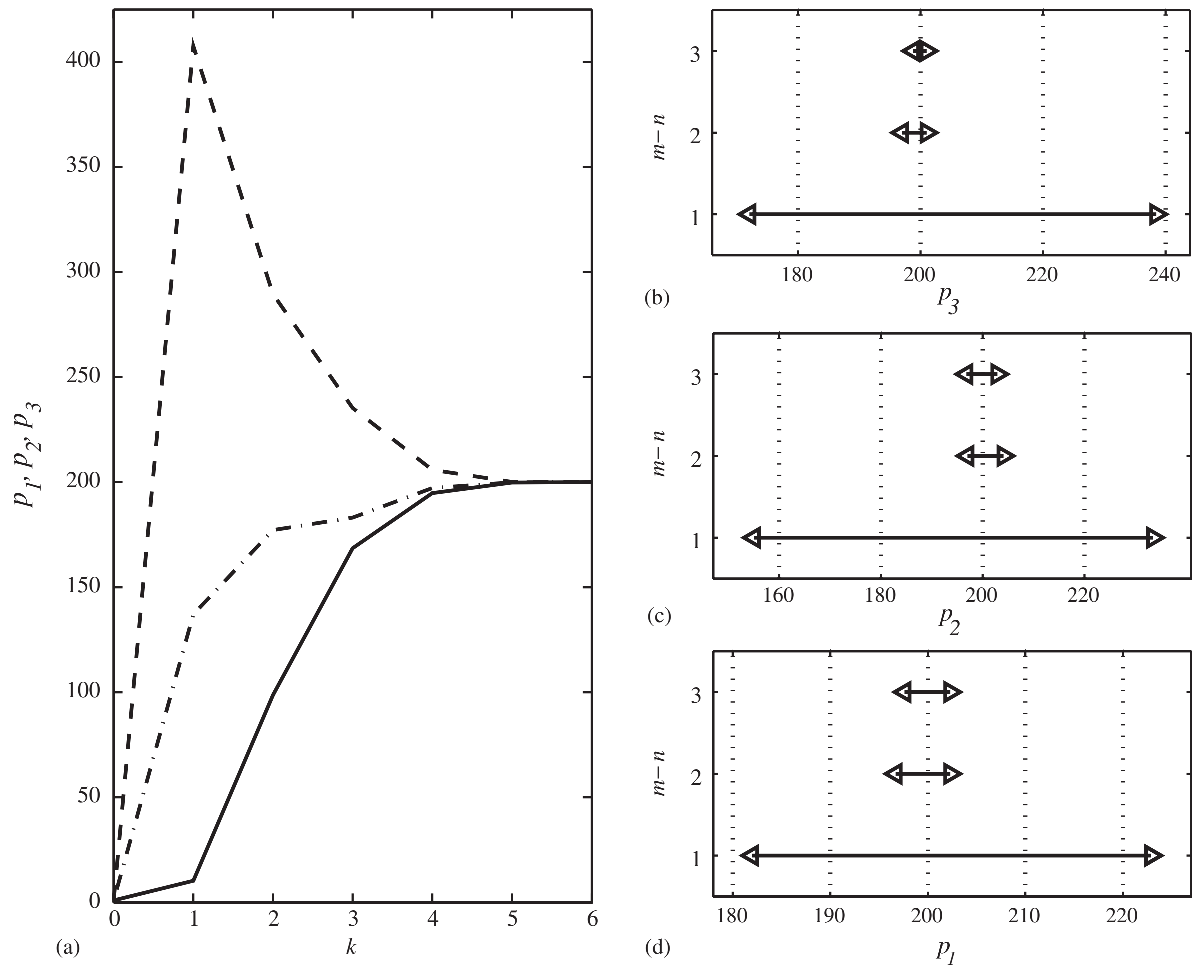

Fig. 13. Force identification of the irregular octahedral frame: (a) iteration progression (–·–·– p_1; – – – – p_2; —— p_3); Monte Carlo noise simulations with varying degrees of overdeterminacy for (b) p_3, (c) p_2 and (d) p_1

5. Conclusions

Following the outlining of several issues concerning the use of Newton's method in the axial force determination of redundant space frames, three illustrative numerical simulations are given. While it is not the intention to provide high order, correlated models as illustrators, it is intended that certain issues be made evident, so that if such a method is employed, these are anticipated and treated. The models used in the numerical simulations suffice to raise these issues. Indeed, the methodology of force determination here presented relies upon good model-to-structure correlation, and minimal noise in the modal data. It is hoped that both of these are achievable with the simple structural nature of space frames.

Since Newton's method extends readily and in a natural way to multiple dimensions, the one-dimensional problem has been used to illustrate the utility of overdetermination and mode tracing. The former is helpful in distinguishing between ambiguous roots and for reducing the error range due to noise; the latter helps to overcome the phenomena of eigenvalue coalescence and loci veering by defining smooth eigenvalue functions for the benefit of Newton's method. In fact, both eigenvalue coalescence and loci veering are overcome in the same manner, so that at veering iterates transcend loci. In all of the simulations, including one- and three-dimensional Newton's methods, absence of mode tracing and therefore the definition of non-smooth eigenvalue functions is known to have potentially detrimental consequences, such as divergence and, worse, convergence to erroneous roots. It is therefore found to be an imperative measure. Erroneous convergence can also occur with traced modes if the eigenvalue function curvature is such that root ambiguity is introduced. Overdetermination is here an aid to establishing root uniqueness, and so the two strategies should necessarily be used in conjunction. Mode tracing is seen to overcome a special problem of root ambiguity in the case of frames with multiple axes of spatial symmetry. Here, the unique root is isolated by virtue of the uniqueness of the eigenvector–force distribution relative orientation, and so the eigenvector-based mode tracing is afforded the discernment of the true solution.

Emphasis is placed upon minimising the number of force distributions needed to account for any state of frame equilibrium, and hence the dimensionality of Newton's method. The basis for doing so is the argument of equilibrium constraints. This progression from previous schemes of force identification is necessary in terms of easing the progression of iteration. Further, in cases of frames with multiple axes of spatial periodicity, this prevents a solution that, while discerning the correct forces, would imply incorrect member designations.

As is seen from the irregular octahedral frame, the slenderness of frame members governs the magnitudes of the frame frequencies. The low frequencies of slender frames may be of concern in terms of noise dependent errors in modal measurement. Further effects of excessive transverse, static deflection on vibration would need to be assessed.

Provided an eigenproblem accurately describing the dynamics of a space frame can be formulated, the method of force identification proposed in this paper provides an alternative to direct force measurement using strain gauges. The latter inconveniently requires that the forces be known at the instant of gauge application, which is not possible unless it is known that the frame is in a zero load state.

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