Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Sound and Vibration 291 (2006) 1029–1040
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Three-dimensional free vibration analysis of inhomogeneous thick orthotropic shells of revolution using differential quadrature

D. Redekop*

Department of Mechanical Engineering, University of Ottawa, Ottawa, Ont., Canada K1N 6N5

Received 9 February 2005; received in revised form 29 June 2005; accepted 7 July 2005
Available online 19 August 2005

Abstract

A procedure is developed to determine the natural frequencies of vibration of thick orthotropic shells of revolution consisting of a material having a radial variation of properties. Governing equations are developed using the linear orthotropic three-dimensional theory of elasticity, and a numerical solution is obtained using the differential quadrature method. The solution has geometric generality in that thick shells of revolution with arbitrary constant thickness and smoothly varying meridian can be considered. The method is validated through comparisons with previously published results, including results for a thick transversely isotropic spherical shell with radially varying material properties. Sample results are given for a thick inhomogeneous toroidal shell, and conclusions are drawn.

1. Introduction

Hollow bodies of revolution (thick shells) are used in a number of engineering applications including pressure vessels, piping, machinery, etc. The bodies consist mostly of cylindrical, spherical, or toroidal form. With the introduction of new applications employing new materials a capability to carry out analyses for inhomogeneous materials is desirable. The background theory

*Tel.: +1 613 562 5800x6290; fax: +1 613 562 5177.
E-mail address: dredekop@uottawa.ca.

0022-460X/$ - see front matter © 2005 Published by Elsevier Ltd.
doi:10.1016/j.jsv.2005.07.018

required is the three-dimensional theory of elasticity. A specific need is for the determination of accurate natural frequencies of vibration for thick shells of revolution made of inhomogeneous orthotropic materials.

Three-dimensional vibration analyses of thick shells of revolution have been reported in a number of studies. Among the geometries that have been considered are the cylindrical [1–3], spherical [4–10], and toroidal [11,12]. A few studies have sought to generate a single solution for general thick shells of revolution [13–15]. The earlier studies considered shells formed of isotropic homogeneous materials. Non-isotropic thick shells of revolution have been the subject of the more recent studies. They include work on thick cylindrical [2] and spherical [6–10] shells, and on general thick shells of revolution [14]. Vibration studies considering inhomogeneous materials, particularly radial functionally graded materials include those of Chen et al. [7], Chen and Ding [8], Chen [9] and Suzuki and Kosawada [14]. Additionally, there have been studies considering inhomogeneous materials that have dealt with stress or deformation analyses [16].

In the present work, the three-dimensional theory of elasticity is used to develop equations that predict the natural frequencies of vibration of orthotropic inhomogeneous thick shells of revolution. A general semi-analytical approach is adopted, in which solutions are sought for specified circumferential harmonic modes of vibration. The theory is specifically applicable to cylindrical, spherical, or toroidal shells of arbitrary constant thickness, having a radial variation in material properties. To obtain numerical results use is made of the differential quadrature method (DQM). The procedure is validated through comparisons with results cited in the literature. The validation examples involve thick isotropic cylindrical, spherical, and toroidal shells, and an inhomogeneous transversely isotropic thick spherical shell. Several different boundary conditions are covered. New results are then given for inhomogeneous thick toroidal shells. The paper ends with an appropriate set of conclusions.

2. Vibration theory

The position of a typical point P of the thick shell is defined by a radius vector $\mathbf{R} = \mathbf{R}(q_1, q_2, q_3)$, where the q_i are position variables. These variables are selected so that $q_1 \equiv \alpha$ and $q_2 \equiv \beta$ locate a point in the vertical radial plane (Fig. 1), while $q_3 \equiv \theta$ defines the angular position of that plane about the axis of symmetry of the body. The Lamé coefficients H_i are defined by

$$H_i = \sqrt{\mathbf{R}_{,i}.\mathbf{R}_{,i}}, \tag{1}$$

where the comma subscript indicates differentiation with respect to the position variable(s) q_i that follow. The H_i for shells of revolution are dependent only on two variables, α and β, and are readily determined. For a cylindrical coordinate system with $\alpha = r$, $\beta = z$ one has $H_1 = 1$, $H_2 = 1$, $H_3 = r$. For a spherical system with $\alpha = r$, $\beta = \phi$ one has $H_1 = 1$, $H_2 = r$, $H_3 = r\cos\phi$, where ϕ is the angle measured from the horizontal. For a circular toroidal system with $\alpha = r$, $\beta = \phi$ one has $H_1 = 1$, $H_2 = r$, $H_3 = R_o + r\cos\phi$, where R_o is the bend radius, r is measured from the bend center-line, and ϕ is measured from the positive horizontal.

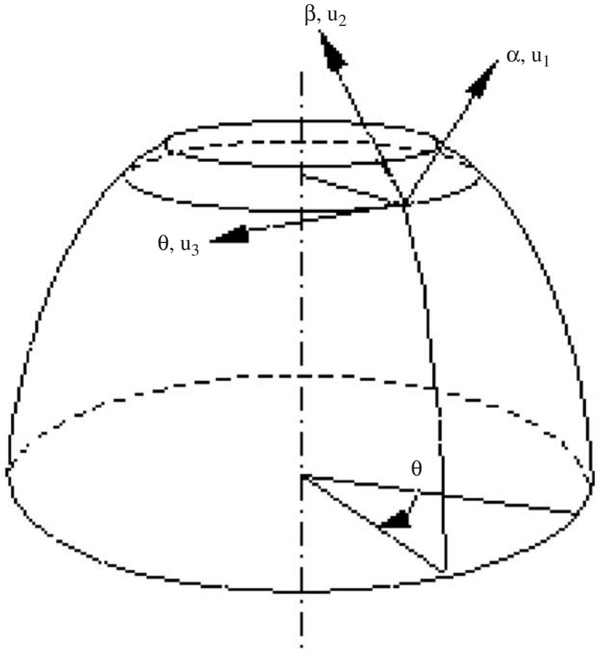

Fig. 1. Coordinates and displacements.

The equations of motion of the three-dimensional theory of elasticity are given in general curvilinear coordinates [17] as

$$
\begin{aligned}
&(H_1H_2H_3)^{-1}[(H_2H_3\sigma_1)_{,1} + (H_3H_1\sigma_{12})_{,2} + (H_1H_2\sigma_{13})_{,3}] + \sigma_{12}H_{1,2}/(H_1H_2) \\
&\quad - \sigma_2H_{2,1}/(H_1H_2) - \sigma_3H_{3,1}/(H_1H_3) - \rho\ddot{u}_1 = 0, \\
&(H_1H_2H_3)^{-1}[(H_3H_2\sigma_{12})_{,1} + (H_3H_1\sigma_2)_{,2} + (H_2H_1\sigma_{23})_{,3}] + \sigma_{12}H_{2,1}/(H_2H_1) \\
&\quad - \sigma_1H_{1,2}/(H_1H_2) - \sigma_3H_{3,2}/(H_3H_2) - \rho\ddot{u}_2 = 0, \\
&(H_1H_2H_3)^{-1}[(H_3H_2\sigma_{13})_{,1} + (H_3H_1\sigma_{23})_{,2} + (H_1H_2\sigma_3)_{,3}] + \sigma_{13}H_{3,1}/(H_3H_1) \\
&\quad + \sigma_{23}H_{3,2}/(H_3H_2) - \rho\ddot{u}_3 = 0,
\end{aligned}
\tag{2}
$$

where the σ_i, σ_{ij} are the normal and shear stresses, ρ is the mass density, and $\ddot{u}_1$, $\ddot{u}_2$, $\ddot{u}_3$ are the acceleration components. It can readily be demonstrated that these equations reduce to the equations of motion in cylindrical or spherical coordinates as given by Kachanov et al. [18], when appropriate choices are made for the Lamé coefficients. Account is made in Eq. (2) and subsequently of the fact that derivatives of the Lamé coefficients with respect to q_3 are zero for shells of revolution.

For an orthotropic material [19] the stress–strain relations are given as

$$
\begin{aligned}
\sigma_1 &= a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3; \quad \sigma_{12} = a_7\varepsilon_{12}, \\
\sigma_2 &= a_2\varepsilon_1 + a_4\varepsilon_2 + a_5\varepsilon_3; \quad \sigma_{13} = a_8\varepsilon_{13}, \\
\sigma_3 &= a_3\varepsilon_1 + a_5\varepsilon_2 + a_6\varepsilon_3; \quad \sigma_{23} = a_9\varepsilon_{23},
\end{aligned}
\tag{3}
$$

where the ε_i, ε_{ij} are the normal and shear strains, and the a_i, $i = 1, \ldots, 9$, are material properties. In the present study the a_i vary in the q_1 (radial) direction, i.e. $a_i = a_i(\alpha)$. Simplifications arise when there is material symmetry. For example, for an isotropic material the a_i are constants [19] given by

$$a_1 = a_4 = a_6 = \frac{E}{1+v}\frac{1-v}{1-2v}; \quad a_2 = a_3 = a_5 = \frac{E}{1+v}\frac{v}{1-2v},$$
$$a_7 = a_8 = a_9 = \frac{E}{2(1+v)}, \tag{4}$$

where E and v are the Young's modulus and the Poisson ratio.

The strain–displacement relations [17] in the linear theory are given by

$$\begin{aligned}
\varepsilon_1 &= u_{1,1}/H_1 + u_2 H_{1,2}/(H_1 H_2),\\
\varepsilon_2 &= u_{2,2}/H_2 + u_1 H_{2,1}/(H_1 H_2),\\
\varepsilon_3 &= u_{3,3}/H_3 + u_1 H_{3,1}/(H_1 H_3) + u_2 H_{3,2}/(H_2 H_3),\\
\varepsilon_{12} &= u_{2,1}/H_1 + u_{1,2}/H_2 - u_1 H_{1,2}/(H_1 H_2) - u_2 H_{2,1}/(H_1 H_2),\\
\varepsilon_{13} &= u_{3,1}/H_1 + u_{1,3}/H_3 - u_3 H_{3,1}/(H_1 H_3),\\
\varepsilon_{23} &= u_{3,2}/H_2 + u_{2,3}/H_3 - u_3 H_{3,2}/(H_2 H_3),
\end{aligned} \tag{5}$$

where the displacement components u_1, u_2, u_3 are, respectively, in the α-, β-, and θ-directions (Fig. 1). Substituting the strains (5) into the stresses (3) gives

$$\begin{aligned}
\sigma_1 &= a_1 B_1 u_{1,1} + (a_2 B_2 + a_3 B_3) u_1 + a_2 B_4 u_{2,2} + (a_1 B_5 + a_3 B_6) u_2 + a_3 B_7 u_{3,3},\\
\sigma_2 &= a_2 B_1 u_{1,1} + (a_4 B_2 + a_5 B_3) u_1 + a_4 B_4 u_{2,2} + (a_2 B_5 + a_5 B_6) u_2 + a_5 B_7 u_{3,3},\\
\sigma_3 &= a_3 B_1 u_{1,1} + (a_5 B_2 + a_6 B_3) u_1 + a_5 B_4 u_{2,2} + (a_3 B_5 + a_6 B_6) u_2 + a_6 B_7 u_{3,3},\\
\sigma_{12} &= a_7 B_4 u_{1,2} - a_7 B_5 u_1 + a_7 B_1 u_{2,1} - a_7 B_2 u_2,\\
\sigma_{13} &= a_8 B_7 u_{1,3} + a_8 B_1 u_{3,1} - a_8 B_3 u_3,\\
\sigma_{23} &= a_9 B_7 u_{2,3} + a_9 B_4 u_{3,2} - a_9 B_6 u_3,
\end{aligned} \tag{6}$$

where the B_i, $i = 1, \ldots, 7$, are known functions of the Lamé coefficients H_i and their derivatives, defined by Eq. (5).

Assuming cyclical vibrations the displacements for the typical circumferential harmonic mode are taken as

$$\begin{aligned}
u_1 &= u(\alpha, \beta) \cos m\theta \sin \omega t,\\
u_2 &= v(\alpha, \beta) \cos m\theta \sin \omega t,\\
u_3 &= w(\alpha, \beta) \sin m\theta \sin \omega t,
\end{aligned} \tag{7}$$

where m is the number of the circumferential harmonic, ω is the natural frequency, t is the time, and u, v, w are the displacement functions for the harmonic m.

Combining Eqs. (2) and (6), and substituting in the expressions (7), leads to three homogeneous differential equations for the three displacement functions and the frequency. In the development of the derivatives of the stresses of Eq. (6) with respect to $q_1 \equiv \alpha$, account is made of the variability of the material properties a_i in the radial direction. It is seen in particular from the form of

Eqs. (1) and (6) that the derivatives of $a_i, i = 1, 2, 3, 7, 8$, with respect to α are required. Developing the equations one obtains for the typical harmonic m a set of governing equations of the form

$$\begin{aligned} L_{11}u + L_{12}v + L_{13}w + \rho\omega^2 u &= 0, \\ L_{21}u + L_{22}v + L_{23}w + \rho\omega^2 v &= 0, \\ L_{31}u + L_{32}v + L_{33}w + \rho\omega^2 w &= 0, \end{aligned} \tag{8}$$

where the L_{ij} are differential operators. Eq. (8) is of the sixth order, two orders lower than that for a Love–Kirchhoff thin-shell theory. The operators are defined through

$$\begin{aligned} L_{11}u &= C_1 u_{,11} + C_2 u_{,22} + C_3 u_{,1} + C_4 u_{,2} + (-m^2 C_5 + C_6)u, \\ L_{12}v &= C_7 v_{,12} + C_8 v_{,1} + C_9 v_{,2} + C_{10} v, \\ L_{13}w &= mC_{11} w_{,1} + mC_{12} w, \\ L_{21}u &= C_{13} u_{,12} + C_{14} u_{,1} + C_{15} u_{,2} + C_{16} u, \\ L_{22}v &= C_{17} v_{,11} + C_{18} v_{,22} + C_{19} v_{,1} + C_{20} v_{,2} + (-m^2 C_{21} + C_{22})v, \\ L_{23}w &= mC_{23} w_{,2} + mC_{24} w, \\ L_{31}u &= -mC_{25} u_{,1} - mC_{26} u, \\ L_{32}v &= -mC_{27} v_{,2} - mC_{28} v, \\ L_{33}w &= C_{29} w_{,11} + C_{30} w_{,22} + C_{31} w_{,1} + C_{32} w_{,2} + (-m^2 C_{33} + C_{34})w, \end{aligned} \tag{9}$$

where the quantities C_i, $i = 1, \ldots, 34$, are lengthy but known functions of the Lamé coefficients H_i, and their derivatives with respect to α and β, and of the material properties a_i, and their derivatives with respect to α.

A solution is obtained herein for a thick shell whose meridional surfaces are stress-free. For these surfaces, defined by $\alpha =$ constant, the conditions to be satisfied are

$$\sigma_1 = \sigma_{12} = \sigma_{13} = 0. \tag{10}$$

End conditions for the body on radial surfaces defined by $\beta =$ constant, need not be enforced for a body complete in the meridional direction, such as a toroidal shell. Such boundary conditions need, however be enforced for a body not complete in the meridional direction, such as a cylindrical shell, or on a body whose meridian ends on the axis of symmetry. For these latter type bodies 'shear-free' boundary conditions are satisfied on the radial lines representing real or de-facto boundaries. The conditions to be satisfied on these lines are

$$u_2 = \sigma_{21} = \sigma_{23} = 0. \tag{11}$$

A two-dimensional eigenvalue problem is defined by Eqs. (8)–(11), for each choice of $m > 0$, in the variables α, β. It is noted that in the three-dimensional theory of elasticity the number of domain and boundary equations is equal, unlike the case of Love–Kirchhoff shell theory, where there are three domain and four boundary equations.

The $m = 0$ case, i.e. the axisymmetric harmonic, is a special simpler case, which can easily be extracted from the general case. For the axisymmetric case it is necessary to take $u_3 \equiv 0$, $\sigma_{13} \equiv 0$, $\sigma_{23} \equiv 0$, and to set all derivatives with respect to θ to zero. Furthermore, the domain equation (8), and each set of the boundary conditions (10)–(11), reduce from three to two.

3. Solution by differential quadrature

Application of the DQM allows for the conversion of the differential equations written for a particular harmonic m to a set of linear simultaneous algebraic equations [20]. A grid of sampling points is defined in the radial plane (for the $m = 0$ and $m > 0$ cases). The derivative of a function in a given direction is replaced by the weighted sum of the values of the function at specified sampling points in a line following the given direction. For a generic function $f(x)$ of a single variable, the series used to replace the rth derivative of the function at the sampling point x_i, is taken as

$$\left.\frac{\mathrm{d}^r f(x)}{\mathrm{d}x^r}\right|_{x_i} = \sum_{h=1}^{M} A_{ih}^{(r)} f(x_h), \tag{12}$$

where the $A_{ih}^{(r)}$ are the weighting coefficients of the rth-order derivative in the x-direction for the ith sampling point, $f(x_h)$ is the value of $f(x)$ at the sampling point x_h, and M is the number of sampling points in the x-direction. For a generic function of two variables $g(x, y)$ the series for the $(r+s)$th partial derivative at the sampling point x_i, y_j is taken as

$$\left.\frac{\partial^{(r+s)} g(x, y)}{\partial x^r \partial y^s}\right|_{x_i, y_j} = \sum_{h=1}^{M} A_{ih}^{(r)} \sum_{k=1}^{N} B_{jk}^{(s)} g(x_h, y_k), \tag{13}$$

where $B_{jk}^{(s)}$ and N describe the series for the y-direction, and $g(x_h, y_k)$ is the value of $g(x, y)$ at the sampling point x_h, y_k. The weighting coefficients are determined a priori with the help of an assumed grid and a set of trial functions. In the present study, when the meridian is incomplete, the well-known Chebyshev–Gauss–Lobatto spacing of sampling points is used, together with polynomial trial functions, for both directions. When the meridian is complete an equal spacing of points is used for the meridional direction, coupled with Fourier trial functions. For either scheme explicit formulas are available [20] for the weighting coefficients $A_{ih}^{(r)}$, $B_{jk}^{(s)}$.

Applying the quadrature rules (12)–(13) of the DQM to the differential equations for the domain (8) and enforcing these at the interior sampling points of the DQM grid leads to a set of linear simultaneous equations. Applying the quadrature rules to the boundary conditions (10)–(11) and enforcing these at the boundary sampling points leads to a second set of simultaneous linear equations. Combining the domain and boundary equations one obtains a single set of simultaneous linear equations which governs the problem. The matrix form of this set is

$$[K](U) = \lambda[M](U), \tag{14}$$

where (U) is the unknown vector of the displacement functions at the sampling points, λ is the eigenvalue, dependent on ω, and $[K]$, $[M]$ are the known 'stiffness' and 'mass' matrices. Standard matrix eigenvalue extraction routines may be used to solve Eq. (14).

The derivation of the lengthy governing equation according to the procedure described herein allows for an analysis of constant-thickness orthotropic inhomogeneous thick shells of revolution of arbitrary smooth meridian. The appropriate values of the Lamé coefficients H_i are inserted at the solution stage, i.e. on enforcing either the domain or boundary conditions at the DQM sampling points. Thus, the equations derived can readily be used for the common thick shells of revolution. Furthermore, the DQM approach is clearly more adept in handling changes in boundary conditions than the 'series approach', so that a single derivation can be used for several different types of boundary support.

4. Validation and results

The validity of the current procedure is demonstrated with the aid of five examples. The first three examples concern isotropic thick cylindrical, toroidal, and spherical shells. The cylindrical shells have boundary conditions which are 'shear-free'. The toroidal and spherical shells are completely free. The fourth example concerns a transversely isotropic thick spherical shell, with free boundary conditions. The fifth example concerns an inhomogeneous transversely isotropic thick spherical shell, also with free conditions. Unless otherwise indicated for all DQM solutions given herein a grid of 19×19 sampling points is used for cylindrical and spherical geometries, and a grid of 19×20 points for toroidal geometries. Results are given for a frequency parameter Ω defined as $\Omega = K\omega$, where K is a scalar constant defined in the following.

For isotropic materials the properties are taken as

$$v = 0.3; \quad E = 0.2e12\,\mathrm{Pa}; \quad \rho = 7800\,\mathrm{kg/m^2} \tag{15}$$

For the transversely isotropic material of the fourth validation problem the material properties in Eq. (3) are taken, in units of 10^{10} Pa, as

$$a_1 = a_2 = a_3 = 2; \quad a_4 = 20; \quad a_5 = 12, \quad a_6 = 20; \quad a_7 = a_8 = 1; \quad a_9 = 4. \tag{16}$$

For the inhomogeneous transversely isotropic material of the fifth validation problem the material properties are taken as

$$a_i = a_i^0 \xi^{\gamma}; \quad \rho = \rho_0 \xi^{\gamma}, \tag{17}$$

where $\xi = r/r_m$, $r_m = (r_i + r_o)/2$, r_i, r_o are the inside and outside radii of the sphere. The a_i^0 have the values given by the a_i of Eq. (16), γ is a specified constant, and ρ_0 is the mass density at the inside surface. The derivatives of $a_i, i = 1, 2, 3, 7, 8$, in the radial direction are given through the relation $a_{i,1} = a_i\gamma/r$. The material properties of the fifth validation example also apply to the inhomogeneous transversely isotropic thick toroidal shell discussed subsequently.

For the first validation example results are obtained for "shear-free" isotropic thick cylindrical shells of four different geometric cases. Results obtained previously from a Fourier–Bessel (FB) series solution are available for these shells [1]. The present procedure yields also results for plane strain modes which are not present in the previous solution.

Table 1 gives a comparison of results from the DQM with results of the previous solution. In the table h, L, R represent, respectively, the thickness, length, and mean radius of the cylinder. For all geometric cases the frequency parameter Ω is given for the first six modes, for the zeroth and third circumferential harmonics. Following the previous solution K is taken as $K = (h/\pi)\sqrt{\rho/G}$, where $G = E/[2(1 + v)]$. It is seen that the present DQM approach gives, for each of the natural frequencies cited in the previous work, results having relative differences less than 0.002%.

For the second validation example results are obtained for four cases of freely supported isotropic thick toroidal shells. The external cross-sectional radius r_o for each case is kept constant

Table 1
Comparison of Ω for isotropic thick cylindrical shell with Fourier Bessel (FB) method [1]

h/R	L/R	m	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.2	$\frac{1}{2}$	0	FB	0.22024	0.66791	1.22953	1.75438	2.24719	2.99800
			DQM	0.22024	0.66790	1.22953	1.75437	2.24719	2.99799
		3	FB	0.24332	0.44654	0.73649	1.10083	1.27496	1.74680
			DQM	0.24332	0.44654	0.73649	1.10083	1.27495	1.74680
	$\frac{1}{4}$	0	FB	0.58066	1.23753	1.67392	1.81630	2.62151	3.02871
			DQM	0.58065	1.23753	1.67391	1.81630	2.62150	3.02868
		3	FB	0.60084	0.82324	1.25731	1.30527	1.70117	1.83161
			DQM	0.60084	0.82324	1.25731	1.30526	1.70116	1.83160
0.5	$\frac{5}{4}$	0	FB	0.31092	0.67531	1.24088	1.76774	2.25634	3.00123
			DQM	0.31092	0.67531	1.24088	1.76773	2.25634	3.00122
		3	FB	0.36838	0.64417	0.97607	1.22449	1.54804	1.76134
			DQM	0.36838	0.64417	0.97607	1.22448	1.54803	1.76133
	$\frac{5}{8}$	0	FB	0.61465	1.24807	1.68683	1.82281	2.62998	3.03244
			DQM	0.61465	1.24807	1.68683	1.82281	2.62998	3.03243
		3	FB	0.69540	0.94557	1.31584	1.45837	1.83112	1.97190
			DQM	0.69539	0.94557	1.31582	1.45836	1.83111	1.97190

Table 2
Comparison of ω (Hz) for isotropic thick toroidal shell with FEM

R_o	r_i	Method	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
1.6	0.4	FEM	239.8	308.5	339.5	499.9	507.4	510.9
		DQM	239.8	308.5	339.1	499.7	506.9	510.8
		m	2	2	0	3	0	1
	0.8	FEM	150.9	227.0	258.8	279.1	291.6	299.7
		DQM	149.9	226.7	258.5	278.2	290.7	298.6
		m	0	2	2	1	1	2
2.4	0.4	FEM	135.8	159.1	234.0	311.0	325.1	348.2
		DQM	135.8	159.0	233.6	310.9	325.0	348.0
		m	2	2	0	3	1	0
	0.8	FEM	124.8	125.9	147.4	221.3	227.2	235.2
		DQM	124.0	125.7	147.1	220.3	226.2	234.2
		m	0	2	2	1	2	2

at 1.0, while the bend radius R_o is varied from 1.6 to 2.4, and the internal radius r_i is varied from 0.4 to 0.8. Results are also obtained using the finite element method (FEM). In the latter method the three-dimensional mesh used features a 20-node 60-degree-of-freedom element.

Table 2 gives a comparison of results obtained using the DQM with the FEM results. For each of the four geometric cases the natural frequency ω is given in Hz for the first six modes. The circumferential harmonic mode numbers as determined by the DQM are also given. It is seen that the present DQM gives, for each of the four cases, results having differences less than 1.0% of the FEM results.

For the third validation example results are obtained for four cases of freely supported isotropic thick spherical shells. The external radius r_o for each cases is taken as 1.0, while the internal radius r_i is varied from 0.3 to 0.9. Results obtained using an exact formula have been published previously for these spheres by Young and Dickinson [5].

Table 3 gives a comparison of results obtained using the DQM with the previous results [5]. For each of the four geometric cases the frequency parameter Ω is given for the six lowest eigenvalues. The factor K is taken as $K = r_o\sqrt{\rho/G}$. The type of the modes, either spherical (s), or toroidal (t), as quoted in Ref. [5] is also presented. All frequencies labelled 's' are found also in the $m = 0$ results. The presence in solutions for spherical shells of the zero harmonic modes in higher harmonics has been discussed previously by other authors [4,9]. It is seen that the present method gives, for each of the natural frequencies cited in Ref. [5], results that agree to four figures with those of the previous results.

For the fourth validation example results are obtained for a single geometric case of a transversely isotropic thick spherical shell. The shell radii r_i, r_o are, respectively, 1.0 and 2.0. The material properties are as defined by Eq. (16). Results obtained using the FEM [4], an analytical approach [8], and an exact formula [8] have been published previously. DQM results are determined for a total of four grid sizes, ranging from 7×7 to 19×19.

Table 4 gives a comparison of results obtained using the DQM with the previous results. For each of the various solutions the frequency parameter Ω is given for the first five, and eleventh modes. The factor K is taken as $K = r_i\sqrt{\rho/a_7}$. It is seen that the DQM solution converges rapidly. Furthermore, the DQM solution corresponding to the finest grid is closest to the exact solution of any of the approximate solutions, the maximum relative difference being less than 0.001%.

For the fifth validation example results are obtained for two geometric cases of transversely isotropic thick spherical shells having a radial variation of material properties. The cases are for thickness ratios $t^* = (r_o - r_i)/r_m$ of 0.2 and 1.2. Results are given for nine values of γ in Eq. (17)

Table 3
Comparison of Ω for isotropic thick spherical shell with exact solution [5]

r_i/r_o	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.3	Exact	2.391	2.494	3.674	3.809	3.862	4.498
	DQM	2.391	2.494	3.674	3.809	3.862	4.498
	Type	s	t	s	s	t	s
0.5	Exact	1.933	2.435	3.143	3.780	3.813	3.873
	DQM	1.933	2.435	3.143	3.780	3.813	3.873
	Type	s	t	s	s	t	s
0.7	Exact	1.523	2.256	2.292	3.147	3.246	3.620
	DQM	1.523	2.256	2.292	3.147	3.246	3.620
	Type	s	s	t	s	s	t
0.9	Exact	1.260	1.552	1.798	2.100	2.120	2.548
	DQM	1.260	1.552	1.798	2.100	2.120	2.548
	Type	s	s	s	t	s	s

Table 4
Comparison of Ω for transversely isotropic thick spherical shell with published results

Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_{11}
FEM [4]	1.783	2.502	3.120	3.394	3.707	5.152
Analytical [8]	1.78225	2.49629	3.10472	3.39334	3.67441	5.15141
Exact [8]	1.78224	2.49626	3.10448	3.39334	3.67332	5.15141
DQM 19 × 19	1.78224	2.49626	3.10448	3.39334	3.67331	5.15141
DQM 15 × 15	1.78224	2.49626	3.10446	3.39334	3.67330	5.15141
DQM 11 × 11	1.78218	2.49603	3.11385	3.39333	3.73397	5.15142
DQM 7 × 7	1.81384	2.67779	—	3.39314	4.11442	5.15209

Table 5
Comparison of Ω for inhomogeneous thick spherical shell with analytical solution

	$r_i = 0.9$, $r_o = 1.1$ ($t^* = 0.2$)				$r_i = 0.4$, $r_o = 1.6$ ($t^* = 1.2$)			
	$n = 2$		$n = 3$		$n = 2$		$n = 3$	
γ	[9]	DQM	[9]	DQM	[9]	DQM	[9]	DQM
−2.0	2.5394	2.5394	3.3657	3.3657	3.0274	3.0274	3.9377	3.9377
−1.5	2.5353	2.5353	3.3609	3.3609	2.8803	2.8803	3.7880	3.7879
−1.0	2.5312	2.5312	3.3559	3.3559	2.7440	2.7440	3.6467	3.6466
−0.5	2.5270	2.5270	3.3506	3.3506	2.6192	2.6192	3.5145	3.5144
0	2.5227	2.5227	3.3451	3.3451	2.5065	2.5064	3.3918	3.3917
0.5	2.5184	2.1584	3.3394	3.3394	2.4057	2.4056	3.2788	3.2787
1.0	2.5141	2.5141	3.3334	3.3334	2.3165	2.3164	3.1754	3.1753
1.5	2.5098	2.5098	3.3272	3.3272	2.2383	2.2382	3.0815	3.0813
2.0	2.5054	2.5054	3.3207	3.3207	2.1701	2.1700	2.9966	2.9963

ranging from −2.0 to 2.0. Results have been published previously for these γ values by Chen [9], who used an analytical approach.

Table 5 gives a comparison of results obtained using the DQM with the previous results. For each of the two geometric cases the lowest frequency parameter Ω is given for the $n = 2$ and $n = 3$ modes, where n is the term defining the spherical harmonic of the solution. These n values correspond to the circumferential harmonic m values of the current solution. The factor K is taken as $K = r_m\sqrt{\rho_0/a_7^0}$. It is seen that the DQM gives results that have maximum relative differences of about 0.01%.

New results are next given for a transversely isotropic thick toroidal shell having a radial variation of material properties. Four geometric cases are covered which are identical to the cases discussed in Table 2. The material properties are those defined by Eqs. (16)–(17), and the parameter γ is varied from −2 to +2. Results are given in Table 6 for the frequency parameter Ω for the lowest frequency for each of the circumferential harmonics $m = 0, 1, 2, 3$. The value of K is taken as $K = r_o\sqrt{\rho_0/a_7^0}$, where r_o is the outside cross-sectional radius.

Table 6
Results for Ω for inhomogeneous thick toroidal shell

	$R_o = 1.6$, $r_i = 0.4$, $r_o = 1$				$R_o = 1.6$, $r_i = 0.8$, $r_o = 1$			
γ	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 0$	$m = 1$	$m = 2$	$m = 3$
−2.0	1.2642	1.9600	0.9275	1.9833	0.6511	1.2080	0.9439	1.7217
−1.5	1.2450	1.9684	0.9375	1.9863	0.6512	1.2065	0.9447	1.7215
−1.0	1.2362	1.9719	0.9462	1.9868	0.6510	1.2048	0.9453	1.7210
−0.5	1.2170	1.9680	0.9535	1.9848	0.6505	1.2029	0.9459	1.7203
0	1.1941	1.9524	0.9596	1.9806	0.6499	1.2008	0.9464	1.7195
0.5	1.1690	1.9221	0.9644	1.9744	0.6498	1.1984	0.9469	1.7184
1.0	1.1428	1.8794	0.9681	1.9667	0.6478	1.1959	0.9473	1.7172
1.5	1.1140	1.8313	0.9709	1.9576	0.6464	1.1932	0.9475	1.7157
2.0	1.0902	1.7812	0.9729	1.9476	0.6448	1.1902	0.9478	1.7141
	$R_o = 2.4$, $r_i = 0.4$, $r_o = 1$				$R_o = 2.4$, $r_i = 0.8$, $r_o = 1$			
γ	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 0$	$m = 1$	$m = 2$	$m = 3$
−2.0	0.9482	1.3013	0.5228	1.2220	0.5441	0.9269	0.5311	1.2135
−1.5	0.9451	1.3018	0.5316	1.2349	0.5441	0.9272	0.5318	1.2138
−1.0	0.9386	1.3010	0.5395	1.2462	0.5438	0.9273	0.5323	1.2138
−0.5	0.9301	1.2988	0.5464	1.2557	0.5434	0.9271	0.5328	1.2135
0	0.9188	1.2952	0.5523	1.2635	0.5427	0.9266	0.5332	1.2130
0.5	0.9085	1.2903	0.5573	1.2696	0.5419	0.9260	0.5335	1.2121
1.0	0.8950	1.2838	0.5613	1.2744	0.5409	0.9251	0.5337	1.2109
1.5	0.8793	1.2758	0.5645	1.2778	0.5396	0.9240	0.5339	1.2094
2.0	0.8643	1.2664	0.5669	1.2803	0.5382	0.9226	0.5340	1.2077

The results of Table 6 indicate that the lowest frequency arises for the $m = 2$ harmonic for all geometric cases except the small thin shell ($R_o = 1.6$, $r_i = 1.0$, $r_o = 0.8$). For the thick shells the lowest frequency increases up to 8.4% when γ is increased from −2.0 to 2.0. For the thin shells the lowest frequency varies less than 1.0% for the same variation of γ. The extent of the variations is generally similar to that observed earlier for spherical shells [9].

5. Conclusions

The general procedure described herein gives results showing excellent agreement with previously published results obtained using analytical and numerical methods. The procedure, while semi-analytical, has an accuracy approaching that of series solutions. New results are given for inhomogeneous thick toroidal shells with radial variation of material properties, and the effect on the lowest frequency of the inhomogeneity is indicated. A number of important problems concerning thick shells of revolution remain to be studied, such as piezoelectric and thermoelastic behaviour, and different levels of inhomogeneity. The present procedure evidently is a very promising one for dealing with such problems.

References

[1] A.E. Armenakas, D.C. Gazis, G. Herrmann, *Free Vibrations of Circular Cylindrical Shells*, Pergamon, Oxford, 1969.

[2] F.H. Chou, J.D. Achenbach, Three-dimensional vibrations of orthotropic cylinders, *Journal of Engineering Mechanics* (1972) 813–821.

[3] K.M. Liew, L.A. Berman, T.Y. Ng, K.Y. Lam, Three-dimensional vibration of cylindrical shell panels—solution by continuum and discrete approaches, *Computational Mechanics* 26 (2000) 208–221.

[4] G.R. Buchanan, B.S. Rich, Effect of boundary conditions of free vibration of thick isotropic spherical shells, *Journal of Vibration Control* 8 (2002) 389–403.

[5] P.G. Young, S.M. Dickinson, Free vibration of a class of solids with cavities, *International Journal of Mechanical Sciences* 36 (1994) 1099–1107.

[6] B.P. Gautham, N. Ganesan, Free vibration analysis of orthotropic thick shells of revolution using discrete layer theory, *Journal of Sound and Vibration* 171 (1994) 549–556.

[7] W.Q. Chen, Z.B. Bian, H.J. Ding, On deformations of a spherically isotropic hollow sphere of a functionally graded material, *Materials Science Forum* 423–425 (2003) 743–748.

[8] W.Q. Chen, H.J. Ding, Free vibration of multi-layered spherically isotropic hollow spheres, *International Journal of Mechanical Sciences* 43 (2001) 667–680.

[9] W.Q. Chen, Effect of radial inhomogeneity on natural frequencies of an anisotropic hollow sphere, *Journal of Sound and Vibration* 226 (4) (1999) 787–794.

[10] H.J. Ding, W.Q. Chen, Natural frequencies of an elastic spherically isotropic hollow sphere submerged in a compressible fluid medium, *Journal of Sound and Vibration* 192 (1) (1996) 173–198.

[11] W. Jiang, D. Redekop, Polar axisymmetric vibration of a hollow toroid using the DQM, *Journal of Sound and Vibration* 251 (2002) 761–765.

[12] Y.J. Liu, G.R. Buchanan, Three-dimensional vibration analysis of a thick-walled circular torus, *Proceedings of the Seventh National Congress on Computing Mechanics*, Alburquerque, NM, 2002.

[13] A.W. Leissa, J.H. Kang, Three-dimensional vibration analysis of thick shells of revolution, *Journal of Engineering Mechanics* 125 (1999) 1365–1371.

[14] D. Suzuki, T. Kosawada, Free vibrations of inhomogeneous thick shells of revolution having meridionally varying curvature, *ASME PVP-178* (1989) 33–38.

[15] D. Redekop, Free vibration of hollow bodies of revolution, *Journal of Sound and Vibration* 273 (2004) 415–520.

[16] W.Q. Chen, Y. Lu, F.R. Ye, J.B. Cai, 3D electroelastic fields in a functionally graded piezoceramic hollow sphere under mechanical and electric loadings, *Archive of Applied Mechanics* 72 (2002) 39–51.

[17] A.I. Lur'e, *Three-Dimensional Problems of the Theory of Elasticity*, Wiley, New York, NY, 1964.

[18] M. Kachanov, B. Shafiro, I. Tsukrov, *Handbook of Elasticity Solutions*, Kluwer Academic Publishers, London, 2003.

[19] S. Nemat-Nasser, M. Hori, *Micromechanics: Overall Properties of Heterogeneous Materials*, North-Holland, New York, NY, 1993.

[20] C. Shu, *Differential Quadrature and its Application in Engineering*, Springer, Berlin, 2000.