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Recursive filter estimation for feedforward noise cancellation with acoustic coupling

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Abstract

The basic principle of feedforward noise cancellation for broadband active noise reduction is based on the availability of a reliable measurement of the noise source. Reliability is compromised in case the noise measurement is acoustically coupled to the actual active noise cancellation (ANC), as stability and performance of the feedforward compensation becomes ambiguous. This paper presents a framework to recursively estimate a feedforward filter in the presence of acoustic coupling, addressing both stability and performance of the active feedforward noise cancellation algorithm. The framework is based on fractional model representations in which a feedforward filter is parameterized by coprime factorization. Conditions on the parameterization of the coprime factorization formulated by the existence of a stable perturbation enables stability in the presence of acoustic coupling. In addition, the paper shows how the stable perturbation can be estimated on-line via a recursive least-squares estimation of a generalized FIR filter to improve the performance of the feedforward filter for ANC.

1. Introduction

In application of active noise cancellation (ANC) a feedforward filter has been widely used for broadband noise cancellation [1–4]. In case the noise measurement in feedforward compensation

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is not influenced by the feedforward control signal, feedforward ANC provides an effective resource to create a controlled emission for active sound attenuation. Algorithms based on recursive (filtered) least mean squares (LMS) minimization [5] can be quite effective for the estimation and self-tuning of feedforward-based sound cancellation [6]. To facilitate an output-error-based affine optimization of the feedforward compensation, a linearly parameterized finite impulse response (FIR) filter [2] is widely used for the recursive estimation and adaptation.

Unfortunately, in many ANC systems a strong acoustic coupling may be present, indicating that the noise measurement in feedforward compensation is influenced by the ANC signal. From a control point of view, the ANC system will no longer be a pure feedforward ANC system, because a positive feedback induced by acoustic coupling tends to destabilize the ANC system [7]. Therefore, modifications to the control algorithm have to be made to stabilize the feedforward-based ANC system as indicated for example in Refs. [3,8]. Techniques to deal with acoustic coupling can be hardware based: dual-microphone reference sensing, motional feedback loudspeakers and directional microphones and loudspeakers or control based: feedback neutralization filter, filter-u LMS method and the use of distributed parameter models. In addition, a control synthesis method such as H_∞ optimal control can also be used to design feedforward filter, as the acoustic coupling can automatically be incorporated during the design process [9].

In this paper, a new approach is adopted for the estimation of a feedforward filter in an ANC system, where the feedforward filter is estimated using coprime factors [10–12]. The coprime factor approach allows for a parameterization of all stabilizing feedforward filters using knowledge of the acoustic coupling in the ANC system. For the parameterization of the feedforward filter only a initial (low order) feedforward filter is needed that is known to be stable in the presence of the acoustic coupling. Subsequently, the performance of the stabilizing feedforward filter is optimized by minimizing the error signal in the ANC system via the estimation of a stable perturbation on the coprime factors. The stable perturbation on the coprime factors is also known as a dual-Youla parameterization [13–15].

The estimation of the stable perturbation is posed as an output-error system identification problem, in which the dual-Youla transfer function is parameterized by a generalized FIR filter [16]. Using the generalized FIR filter structure, an output error recursive least-squares (RLS) estimation is applied to facilitate the on-line self-tuning of the feedforward compensation. Since the self-tuning of the feedforward filter is done within the coprime factor framework using a dual-Youla parameterization, stability in the presence of acoustic coupling can be maintained while optimizing the performance of the feedforward filter for noise cancellation. The theoretical contributions of this paper are illustrated by the actual modeling and implementation of this algorithm on an active sound silencer for an air-ventilation system. It is shown that a stabilizing feedforward filter can be estimated in the presence of a strong acoustic coupling and the ANC system demonstrates the effect of noise cancellation over a broad frequency range from 30 till 400 Hz.

2. Analysis of feedforward compensation

In order to analyze the design of the feedforward filter F, consider the schematic representation of a linear airduct which is depicted in Fig. 1. Sound waves from an external noise source are

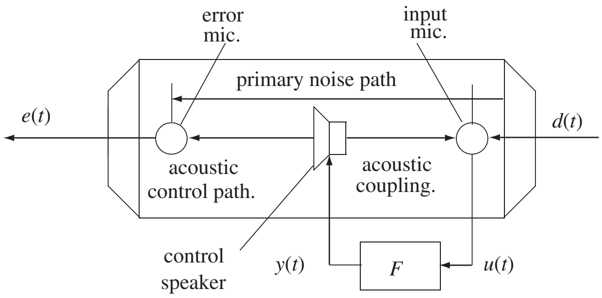

Fig. 1. Schematics of feedforward-based ANC system.

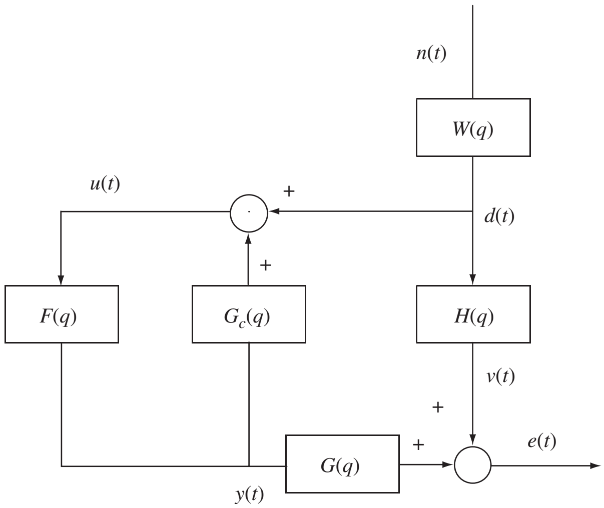

Fig. 2. Block diagram of ANC system with feedforward and acoustic coupling.

predominantly traveling from right to left and can be measured by the pick-up microphone at the inlet and the error microphone at the outlet.

In the ANC system described in Fig. 1, the control signal $y(t)$ is generated by processing $u(t)$ with a feedforward filter F to cancel the noise acoustically in the airduct. Because the control speaker in the airduct will generate plane waves propagating both upstream and downstream, the control signal $y(t)$ to the control speaker not only cancels noise downstream, but also radiates upstream to the input microphone and interferes with the reference signal $u(t)$. The coupling of the acoustic waves from the control speaker to the input microphone is denoted by acoustic coupling.

The block diagram that models the dynamical relationships between the signals in the ANC is given in Fig. 2. Following this block diagram, dynamical relationship between signals in the ANC system are characterized by discrete time transfer functions, with $qd(t) = d(t+1)$ indicating a unit step time delay, and q is a shift operator. For notational convenience, the shift operator q will be dropped in most of the remaining part of the paper.

The spectrum of noise disturbance $d(t)$ at the input microphone is characterized by filtered white-noise signal $n(t)$, where $W(q)$ is a (unknown) stable and stable invertible noise filter [17]. The dynamic relationship between the input $d(t)$ and the error microphone signals $e(t)$ is characterized by the primary noise path $H(q)$, whereas the acoustic control path $G(q)$ characterizes the relationship between control speaker signal $y(t)$ and error microphone signal $e(t)$. Finally, $G_c(q)$ is used to indicate the acoustic coupling from the control speaker signal $y(t)$ back to the input microphone signal $d(t)$.

The error microphone signal $e(t)$ can be described by

$$\begin{aligned} e(t) &= \left[H(q) + \frac{G(q)F(q)}{1 - G_c(q)F(q)}\right] d(t) \\ &= H(q)d(t) + \frac{F(q)}{1 - G_c(q)F(q)} \cdot G(q)d(t) \end{aligned} \tag{1}$$

and definition of the signals

$$v(t) := H(q)d(t), \quad r(t) := G(q)d(t) \tag{2}$$

leads to

$$\begin{aligned} e(t) &= v(t) + \frac{F(q)}{1 - G_c(q)F(q)} r(t) \\ &= v(t) + L(q)r(t) \quad L(q) := \frac{F(q)}{1 - G_c(q)F(q)}. \end{aligned} \tag{3}$$

From Eq. (3) it can be seen that the acoustic coupling $G_c(q)$ creates a positive feedback loop with the feedforward filter $F(q)$. The presence of the acoustic coupling $G_c(q)$ might lead to an undesirable or unstable feedforward compensation if $G_c(q)$ is not taken into account in the design of the feedforward filter $F(q)$ for ANC [7]. To address the issues of stability it can be noted that certain signals can be used for estimation purposes of the dynamics of the various transfer functions in the ANC system. In case the signals $v(t)$ can be measured and the signal $r(t)$ can be created by filtering the measured signal $d(t)$ through a filter that models the dynamics of the acoustic control path $G(q)$, the estimation of the feedforward filter $F(q)$ can be considered as a closed-loop identification problem, where the error $e(t, \theta)$

$$e(t, \theta) = v(t) + L(q, \theta)r(t)$$

is minimized according to

$$\hat{\theta} = \min_{\theta} \|e(t, \theta)\|_2. \tag{4}$$

In case the design of the feedforward filter is an off-line estimation, the signal $v(t)$ can be measured by performing an experiment using the external noise as excitation signal and measuring the error microphone signal as $v(t)$. However, for an on-line feedforward filter adaptation, the signal $v(t)$ is not available during the operation of ANC because it is intended to be cancelled by the control path noise. Therefore, we need to estimate the signal $v(t)$ by

$$v(t) = e(t) - \hat{G}(q)y(t) \tag{5}$$

and use it as reference signal for feedforward filter.

Minimization of $\|e(t,\theta)\|_2$ in Eq. (4) is an output error-based identification problem [17]. In this paper, the presence of acoustic coupling G_c during the estimation of $F(q)$ is taken into account by using a model $\hat{G}_c$ of the acoustic coupling. Using a model $\hat{G}_c$, the estimation and computation of F can be done in several ways.

The first possibility is an indirect identification method, where the closed-loop transfer function $\hat{L} = L(q,\hat{\theta})$ is estimated. Subsequently, the feedforward filter $\hat{F} = F(q,\hat{\theta})$ is computed via

$$\hat{F} = \frac{\hat{L}}{1 + \hat{L}\hat{G}_c}. \tag{6}$$

The model $\hat{F}$ can be computed only in the case that the inverse of $(1 + \hat{L}\hat{G}_c)$ is well-defined. Provided the computed $\hat{F}$ in Eq. (6) is stable, stability of the estimate $\hat{L}$ implies stability of the feedback connection $\mathscr{T}(\hat{F},\hat{G}_c)$ of the feedforward filter $\hat{F}$ and the model of the acoustic coupling $\hat{G}_c$. Stability of $\hat{L}$ can be enforced during output error optimization, but stability of $\hat{F}$ cannot be guaranteed with this indirect estimation method.

A second possibility is to use a tailor-made parameterization of the closed-loop transfer function [18,19]. In that case $L(q,\theta)$ is parameterized via

$$L(q,\theta) := \frac{F(q,\theta)}{1 - \hat{G}_c(q)F(q,\theta)}$$

and the minimization in Eq. (4) would require a nonlinear optimization over a intricate restricted model structure. Although, gradient expression for the nonlinear optimization are available [19], such an optimization will be hard to implement in a real-time adaptive filter estimation of the feedforward filter.

Closely related to the use of a tailor-made parameterization is the third possibility exploited in this paper, which is an estimation based on a so-called dual-Youla parameterization. This will be discussed in Section 3 and opens a possibility to guarantee the internal stability of $\mathscr{T}(\hat{F},\hat{G}_c)$ by constructing a feedforward filter $\hat{F}$ via estimation of a stable dual-Youla transfer function.

3. Dual-Youla parameterization

3.1. Structure of feedforward filter via coprime factorization

Using the theory of fractional representations, a feedforward filter $F(q)$ can be expressed by $F(q) = N(q)D(q)^{-1}$, where $N(q)$ and $D(q)$ are two stable mappings. Referring to Refs. [20,21], the following definitions are used in this paper.

Definition 1. Let $N(q), D(q) \in \mathscr{RH}_\infty$ (where $\mathscr{RH}_\infty$ indicates the set of all rational stable transfer functions), the pair $(N(q), D(q))$ is called a right coprime factorization (rcf), if there exists $X_r, Y_r \in \mathscr{RH}_\infty$, such that $X_rN + Y_rD = I$. Similarly, $\tilde{N}(q), \tilde{D}(q) \in \mathscr{RH}_\infty$ are left coprime factorization (lcf), if there exists $X_l, Y_l \in \mathscr{RH}_\infty$, such that $X_l\tilde{N} + Y_l\tilde{D} = I$.

Definition 2. Let $N(q), D(q)$ be a rcf, then the pair $(N(q), D(q))$ is a rcf of a filter $F(q)$ if $\det\{D(q)\} \neq 0$ and $F(q) = N(q)D(q)^{-1}$. Similarly, a lcf of $F(q)$ has the form $F(q) = \tilde{D}(q)^{-1}\tilde{N}(q)$, where $\tilde{D}(q)$ and $\tilde{N}(q)$ are left-coprime over $\mathscr{RH}_\infty$.

Using Definitions 1 and 2, a characterization of the set of feedforward filters $F(q) = N(q)D(q)^{-1} = \tilde{D}(q)^{-1}\tilde{N}(q)$ that yields an internally stable feedback connection $\mathcal{T}(F(q), \hat{G}_c(q))$ of the feedforward filter $F(q)$ and the model for the acoustic coupling $\hat{G}_c(q)$ can be expressed via a well-known dual-Youla parameterization [10,11,13–15] and is given in the following.

Lemma 1. *Let (N_x, D_x) be a rcf of an auxiliary feedforward filter $F_x = N_x D_x^{-1}$ over $\mathcal{RH}_\infty$, and (N_c, D_c) be a rcf of the model $\hat{G}_c$ of the acoustic coupling G_c with $\hat{G}_c = N_c D_c^{-1}$, such that $\mathcal{T}(F_x, \hat{G}_c) \in \mathcal{RH}_\infty$, then a feedforward filter F with a rcf(N, D) satisfies $\mathcal{T}(F, \hat{G}_c) \in \mathcal{RH}_\infty$ if and only if there exists $R_0 \in \mathcal{RH}_\infty$such that*

$$\begin{aligned} N &= N_x + D_c R_0, \\ D &= D_x + N_c R_0. \end{aligned} \tag{7}$$

The proof of this lemma is given in Appendix A. From Eq. (7) it is obtained that R_0 can vary over all possible transfer functions in $\mathcal{RH}_\infty$ such that $\det\{D_x + D_c R_0\} \neq 0$, which characterizes a set of filters F that are internally stabilized by $\hat{G}_c$. For the interpretation of the result in Lemma 1, consider the following prior information to estimate the optimal feedforward filter F.

Firstly, assume the availability of an initial (not optimal) feedforward controller F_x that is used only to create a stable feedback connection $\mathcal{T}(F_x, \hat{G}_c)$ in the presence of the acoustic coupling G_c. Secondly, if a model $\hat{G}_c = N_c D_c^{-1}$ for the acoustic coupling G_c is available, then a set of feedforward filters can be parameterized that is known to be stabilized by the model $\hat{G}_c$ of the acoustic coupling. With this prior information, the optimal feedforward filter F to minimize $\|e(t,\theta)\|_2$ in Eq. (4) can be constructed by means of the nominal filter F_x plus a possible perturbation R_0 given by Eq. (7). Because R_0 is the only unknown parameter, the estimation of a stable model $\hat{R}$ of R_0 will yield an estimate $(\hat{N}, \hat{D})$ of a rcf of the feedforward filter $\hat{F} = \hat{N}\hat{D}^{-1}$ described by

$$\begin{aligned} \hat{N} &:= N_x + D_c \hat{R}, \\ \hat{D} &:= D_x + N_c \hat{R}. \end{aligned} \tag{8}$$

If the estimate $\hat{R}$ is stable, then the feedforward filter $\hat{F} = \hat{N}\hat{D}^{-1}$ computed in Eq. (8) is guaranteed to create a stable feedback connection with the model $\hat{G}_c$ of the acoustic coupling. In this way, instabilities of the feedforward ANC can be avoided in the presence of acoustic coupling.

3.2. Estimation of dual-Youla transfer function

From Eq. (7), it is seen that the set of feedforward filters in Fig. 2 is parameterized by the rcf's (N_x, D_x), (N_c, D_c) and the stable transfer function R_0. The representation of the feedback connection $\mathcal{T}(F, G_c)$ in terms of the dual-Youla parameterization has been depicted in Fig. 3 with the knowledge of G_c represented by the model $\hat{G}_c$.

From Fig. 2 it can be seen that the acoustic control path G follows the feedforward filter F. In order to ensure the accuracy of the LMS algorithm, an identical filter G is presented in the reference signal path to filter the noise disturbance $d(t)$ for LMS adaptation, which is shown in Fig. 3. In practical ANC applications, G is unknown and must be estimated by a filter $\hat{G}$.

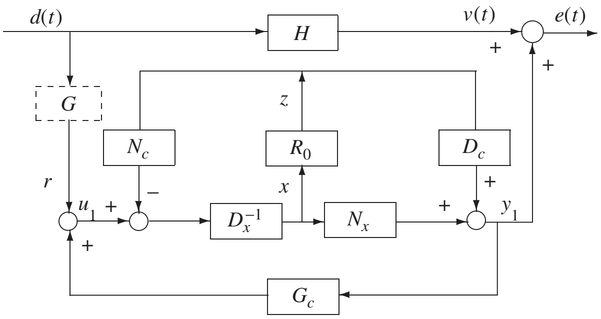

Fig. 3. Block diagram of dual-Youla representation of the feedback connection $\mathcal{T}(G_c, F)$.

Therefore, the reference signal $r(t)$ in Fig. 3 is given by

$$r(t) = \hat{G}(q)d(t). \tag{9}$$

The output signal $y_1(t)$ is defined as

$$y_1(t) := -v(t), \tag{10}$$

where $v(t)$ can be obtained by performing an experiment to measure the error microphone signal or by Eq. (5). The input signal $u_1(t)$ is defined as

$$u_1(t) := r(t) + \hat{G}_c(q)y_1(t), \tag{11}$$

where $\hat{G}_c$ is the model of the acoustic feedback path G_c. The use of this filtered (closed-loop) input signal is needed to address stability of the feedforward compensation in the presence of acoustic coupling. The use of the filtered input signal $u_1(t)$ is in addition to the filtering in Eq. (10) used in filtered LMS estimation of feedforward filters. With the definition of these signals, the open-loop estimate problem of the dual-Youla transfer function R_0 can be formalized as follows.

Lemma 2. *Let (N_x, D_x) be a rcf of an auxiliary feedforward filter $F_x = N_x D_x^{-1}$ and (N_c, D_c) be a rcf of the model $\hat{G}_c = N_c D_c^{-1}$ of the acoustic coupling G_c, such that $\mathcal{T}(F_x, \hat{G}_c) \in \mathcal{RH}_\infty$. Then the intermediate signals $x(t)$ and $z(t)$ are related by*

$$z(t) = R_0(q)x(t), \tag{12}$$

where the intermediate input signal x is defined by the filter operation

$$x := (D_x - \hat{G}_c N_x)^{-1} [-\hat{G}_c \;\; I] \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} \tag{13}$$

and dual-Youla signal z is defined by the filter operation

$$z := (D_c - F_x N_c)^{-1} [I \;\; -F_x] \begin{bmatrix} y_1 \\ u_1 \end{bmatrix}, \tag{14}$$

where the signals r, y_1 and u_1 are defined, respectively, in Eqs. (9)–(11).

The proof of this lemma is given in Appendix B and from Eq. (12) it can be seen that the estimation of a stable model $\hat{R} = R(q, \hat{\theta})$ can be obtained via a standard output-error (OE) minimization by

$$\hat{\theta} = \min_{\theta} \|z(t) - R(q, \theta)x(t)\|_2 \tag{15}$$

Although Eq. (12) and the proof of Lemma 2 indicate a noise-free open-loop identification problem, the OE minimization in Eq. (15) is robust in the presence of possible measurement noise $e_n(t)$ on the error microphone signal $e(t)$. Under the viable assumption that additional measurement noise $e_n(t)$ on $e(t)$ is uncorrelated with the signal $d(t)$ from the input microphone, it is easy to verify that the reference $r(t)$ in Eq. (9) in uncorrelated with $e_n(t)$. With

$$u_1(t) - \hat{G}_c(q)y_1(t) = r(t) = \hat{G}(q)d(t)$$

it can be seen that the intermediate input signal $x(t)$ in Eq. (13) is uncorrelated with $e_n(t)$, making Eq. (15) a standard open-loop identification problem even in the presence of additional measurement noise $e_n(t)$ on the error microphone signal $e(t)$.

3.3. Robustness against modeling errors

The result in Lemma 1 states the following important result for ANC in the presence of acoustic coupling. Provided the estimate $\hat{R}$ is stable, the feedforward filter $\hat{F} = \hat{N}\hat{D}^{-1}$ in the presence of the acoustic coupling modeled by $\hat{G}_c$, is guaranteed to create a stabilizing feedforward-based ANC system. Obviously, this results only holds if $\hat{G}_c$ is an accurate model of the actual acoustic coupling G_c in the ANC system. In case the model $\hat{G}_c$ is (only) an approximation of the actual acoustic coupling G_c, robustness results with respect to modeling errors can be easily formulated using the coprime factor framework.

Let (N_c, D_c) be a rcf of the model $\hat{G}_c$. Subsequently, consider (N_x, D_x) as a rcf of an initial feedforward filter F_x that is known to be stabilizing in the presence of the acoustic coupling, i.e. the feedback connection of G_c and F_x is stable. The knowledge of the initial feedforward filter F_x and the the model $\hat{G}_c$ can be used to characterize the uncertain rcf $(\bar{N}_c, \bar{D}_c)$ of the actual acoustic coupling G_c as

$$\begin{aligned} \bar{N}_c &:= N_c + D_x\Delta, \\ \bar{D}_c &:= D_c + N_x\Delta, \end{aligned} \tag{16}$$

where the uncertainty is characterized by a bounded but unknown $\Delta \in \mathcal{RH}_\infty$. Using the standard small gain theorem, the following lemma concerning the robust stability with respect to the model uncertainty Δ can be derived.

Lemma 3. *Let the uncertain acoustic coupling G_c be given by $G_c = \bar{N}_c\bar{D}_c^{-1}$ where the rcf $(\bar{N}_c, \bar{D}_c)$ is given in Eq.* (16). *Consider a feedforward filter*

$$\hat{F} = (N_x + D_c\hat{R})(D_x + N_c\hat{R})^{-1}$$

similar as in Eq. (8). *With $\Delta, \hat{R} \in \mathscr{RH}_\infty$ the feedback connection of the feedforward filter $\hat{F}$ and the actual acoustic coupling G_c is well posed and internally stable for all $\|\Delta\hat{R}\|_\infty < 1$.*

The proof of this lemma is given in Appendix C. In the presence of a (coprime factor) modeling uncertainty Δ, the condition of estimating a stable $\hat{R}$ is further restricted to the condition that $\|\hat{R}\Delta\|_\infty < 1$. As expected, the robust stability condition limits the size of the stable $\hat{R}$ in case of large modeling errors.

3.4. Summary of feedforward estimation

Since the intermediate signal $x(t)$ and the dual-Youla signal $z(t)$ can be created by Eqs. (13) and (14), the estimation of the dual-Youla transfer function R_0 from Eq. (12) is an open-loop identification problem that can be computed by standard system identification techniques [17]. For more information about the dual-Youla parameterization, please refer to Refs. [10–12] for more details. As a result, the estimation of the feedforward filter $F(q)$ using the dual-Youla parameterization can be summarized by the following steps.

(1) A model $\hat{G}$ of the acoustic control path G is needed for filtering purpose to create the reference signal $r(t)$. The model $\hat{G}$ can be estimated via a standard open-loop identification by performing an experiment using the control speaker signal $y(t)$ as excitation signal and the error microphone signal $e(t)$ as output signal. Such a filtering is commonly used in filtered LMS algorithms to avoid bias of the estimate of the feedforward filter [5].
(2) A model $\hat{G}_c$ of the acoustic coupling G_c is needed to design an initial nominal filter $F_x = N_x D_x^{-1}$ to stabilize the acoustic feedback loop. The model $\hat{G}_c$ along with the initially stabilizing F_x is used to parameterize the feedforward filter F according to Lemma 1. The model $\hat{G}_c$ can be estimated via a standard open-loop identification by performing an experiment using the control speaker signal $y(t)$ as excitation signal and the input microphone signal $d(t)$ as output signal.
(3) With the models $\hat{G}$, $\hat{G}_c$ and the initial feedforward filter F_x, the reference signal r, input signal u_1 and output signal y_1 can be created. With these signals, the optimal feedforward filter $\hat{F}$ can be estimated by minimizing $\|e(t,\theta)\|_2$ in Eq. (4) using the dual-Youla parameterization of the filter $F(q,\theta)$.

Although both the acoustic coupling G_c and the acoustic control path G require an additional modeling effort for the implementation of the ANC, the use of the models $\hat{G}$ and $\hat{G}_c$ is beneficial for the ANC system. Since in most ANC systems both the acoustic coupling G_c and the acoustic control path G are fixed, adaptation of the feedforward filter F is not required to adjust for varying acoustics. Furthermore, using both models $\hat{G}$ and $\hat{G}_c$ for filtering purposes can be seen as a generalization of the filtering used in filtered LMS estimation techniques. In comparison, the FXLMS method in Ref. [8] is not able to incorporate the acoustic coupling explicitly, which is especially of importance when the acoustic coupling cannot be neglected. Similarly, the FULMS method as listed in Ref. [8] does incorporate the acoustic coupling, but no (formal) guarantee on the stability of the inherent feedback connection between the IIR filter and the acoustic coupling can be given.

4. Perturbation estimation via generalized FIR filter

By adopting the theory of dual-Youla parameterization, the design of optimal feedforward controller F is transformed to the estimation of the perturbation R_0 from Eq. (12) using a standard open-loop OE identification technique [17]. It should be stated that the estimation of the perturbation R_0 depends on the initial feedforward filter F_x. In the unlikely event where the initial feedforward filter F_x equals the optimal filter $\hat{F}$, R_0 is equal to 0 and does not need to be updated. Since the initial feedforward filter F_x is chosen only to stabilize the acoustic system in the presence of acoustic coupling, further estimation of the perturbation R_0 will improve the feedforward filter to minimize the signal of the error microphone and the performance of the ANC system.

In general, the OE minimization of Eq. (15) is a nonlinear optimization but reduces to a convex optimization problem in case $R(q,\theta)$ is parameterized linearly in the parameter θ. Linearity in the parameter θ is also favorable for on-line recursive estimation of the filter. A linear parameterization of $R(q,\theta)$ can be obtained by using a FIR filter

$$R(q,\theta) = D + \sum_{k=0}^{N} \theta_k q^{-(k+1)}, \tag{17}$$

where D is a (possible) feedthrough term. Unfortunately, many parameters θ_k may be required to approximate the perturbation R_0 for an optimal feedforward filter $\hat{F}$. Especially for a complex ANC with many lightly damped resonance modes, the use of FIR models is detrimental for the performance of the system. To improve these aspects, generalized FIR filters can be used that are based on orthonormal basis function expansions (ORTFIR) [22].

To improve the approximation properties of the perturbation R_0, the linear combination of tapped delay functions in the FIR filter of Eq. (17) are generalized to

$$R(q,\theta) = D + \sum_{k=0}^{N} \theta_k V_k(q), \quad \theta = [D \;\; \theta_0 \;\; \cdots \;\; \theta_N], \tag{18}$$

where $V_k(q)$ are generalized (orthonormal) basis functions [22] and can be computed by

$$V_k(q) = (qI - A)^{-1} B P^k(q) = V_0(q) P^k(q), \tag{19}$$

where $P(q)$ is an all pass function which is built based on the possible prior knowledge of the dynamics of the perturbation R_0. For details on the construction of the functions $V_k(q)$ one is referred to Refs. [22,16]. A block diagram of the generalized FIR filter $R(q)$ in Eq. (18) is depicted in Fig. 4 and it can be seen that it exhibits the same tapped delay line structure found in a conventional FIR filter, where the difference lies in the use of more general basis functions $V_k(q)$.

An important property and advantage of the generalized FIR filter is that knowledge of the (desired) dynamical behavior can be incorporated in the basis function $V_k(q)$. If a more elaborate choice for the basis function $V_k(q)$ is incorporated, then Eq. (18) can exhibit better approximation properties for a much smaller number of parameters than used in a conventional FIR filter. Consequently, the accuracy of the model $R(q,\theta)$ of the stable dual-Youla transfer function R_0 can substantially increase over a standard FIR parameterization, when the same number of parameters is estimated.

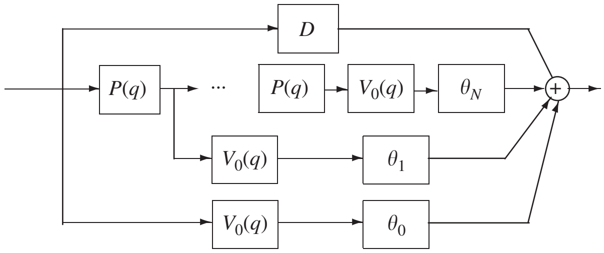

Fig. 4. Basic structure of generalized FIR filter.

To facilitate the use of the generalized FIR filter, the basis functions $V_k(q)$ in Eq. (19) have to be selected. A low-order model for the basis functions will suffice, as the generalized FIR model will be expanded on the basis of $V_k(q)$ to improve the accuracy of the feedforward compensator. For the initialization of the parameterization of the generalized FIR model, an initial low-order IIR model $\bar{R}(q) = R(q, \bar{\theta})$ of the perturbation $R_0(q)$ can be estimated using an OE-minimization

$$\bar{\theta} = \min_{\theta} \|z(t) - R(q, \theta)x(t)\|_2 \tag{20}$$

with the intermediate signal x and the dual-Youla signal z available from Eqs. (13) and (14). The initial low-order IIR model $\bar{R}(q)$ can be used to generate the basis functions $V_k(q)$ of the generalized FIR filter. An input balanced state–space realization of the low-order model $\bar{R}(q)$ is used to construct the basis function $V_k(q)$ in Eq. (19). With the basis function $V_k(q)$ in place, the linear parameterization of $R(q, \theta)$ in Eq. (18) is obtained.

Since the parameterization of $R(q, \theta)$ is based on the generalized FIR model, the intermediate signal $x(t)$ is filtered by the tapped delay line of basis functions

$$\bar{x}_k(t) = V_k(q)x(t), \quad k = 0, \ldots, N \tag{21}$$

creating filtered intermediate signals $\bar{x}_k(t)$. With the generalized FIR filter expansion given in Eq. (18), the relation between the signal $z(t)$ in Eq. (14) and $\bar{x}_k(t)$ in Eq. (21) can be rewritten in a linear regression form

$$z(t) = \phi^{\mathrm{T}}(t)\theta, \quad \theta = [D \quad \theta_0 \quad \cdots \quad \theta_N]^{\mathrm{T}}, \tag{22}$$

where $\phi^{\mathrm{T}}(t) = [x(t) \;\; \bar{x}_0^{\mathrm{T}}(t) \;\; \cdots \;\; \bar{x}_N^{\mathrm{T}}(t)]$ is the available input data vector and θ is the parameter vector of $R(q, \theta)$ in Eq. (18) to be estimated. Therefore, the parameters θ can be estimated by RLS algorithm with variable forgetting factor [23].

5. Application of feedforward ANC

5.1. Modeling of ANC system dynamics

The ACTA silencer depicted in Fig. 5 and located at the System Identification and Control Laboratory at UCSD was used for the experimental verification of the proposed feedforward noise cancellation. The system is a wall-isolated open-ended airduct with a diameter of 0.4 m that

Fig. 5. ACTA airduct silencer located in the System Identification and Control Laboratory at UCSD.

holds an inner tube with a diameter of 0.2 m diameter. The inner tube houses the control speaker and the input and error microphones. The total length of the airduct is 1.2 m, the input microphone is located at the inlet and the error microphone is located at the outlet of the airduct. The control speaker is also located at the outlet, approximately 0.15 m inside the airduct. With these dimensions, the cutoff frequency of this airduct can be calculated to be approximately $f = 999$ Hz [24]. Experimental data and real-time digital control is implemented at a sampling frequency of 2.56 kHz.

With the given mechanical and geometrical properties of the ANC system in Fig. 5, the acoustic control path G and the acoustic coupling G_c both are fixed. For initialization and calibration of ANC algorithm, the models of the acoustic control path G and the acoustic coupling G_c can be identified off-line. Estimation of a model $\hat{G}$ can be done by performing an experiment using the control speaker signal $y(t)$ as excitation signal and the error microphone signal $e(t)$ as output signal. The same experiment can also be used to measure the input microphone signal $d(t)$ as an additional output to estimate the model of the acoustic coupling G_c. Because these models $\hat{G}$ and $\hat{G}_c$ will be used to design nominal feedforward filter F_x and feedforward filter $\hat{F}$, the order of these models should be controlled. In order to estimate a low-order feedforward filter $\hat{F}$, a 20th-order ARX model $\hat{G}$ was estimated for filtering purpose and a 17th-order ARX model of $\hat{G}_c$ was estimated for feedforward filter design purposes. The identification results of $\hat{G}$ and $\hat{G}_c$ can be found in Figs. 6 and 7, respectively.

5.2. Estimation of basis function for dual-Youla transfer function

A low-order model $\hat{R}$ is estimated to compute the basis functions $V_k(q)$ for the parameterization and estimation of the dual-Youla transfer function R_0. On the basis of the model $\hat{G}_c$ a simple 2nd-order nominal feedforward filter F_x is pre-computed that internally stabilize the positive feedback loop connection $\mathscr{T}(F_x, \hat{G}_c)$. The initial feedforward controller is given by the discrete time transfer function

$$F_x(q) = \frac{-1.577q + 1.611}{q^2 - 1.99q + 0.9913} \tag{23}$$

and is lightly damped 2nd-order system with one-step time delay and a resonance mode at approximately 100 rad/s.

Given the experimental data and the prior information, consisting of the model $\hat{G}$ of the acoustic control path G and the model $\hat{G}_c$ of acoustic coupling G_c, the filtered signals $r(t)$, $y_1(t)$

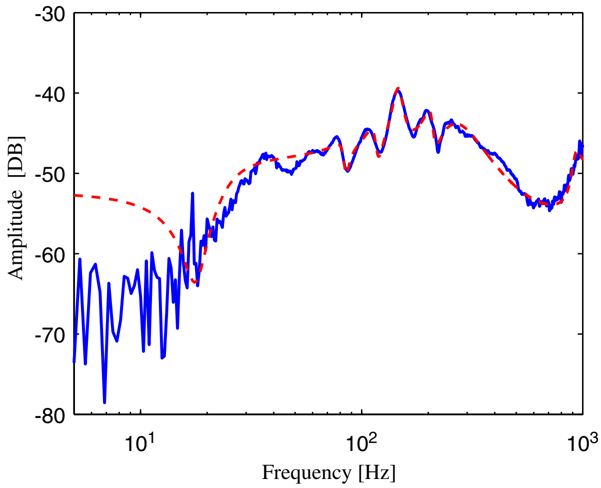

Fig. 6. Amplitude Bode plot of spectral estimate of acoustic control path G (solid) and 20th-order parameteric model $\hat{G}$ (dashed).

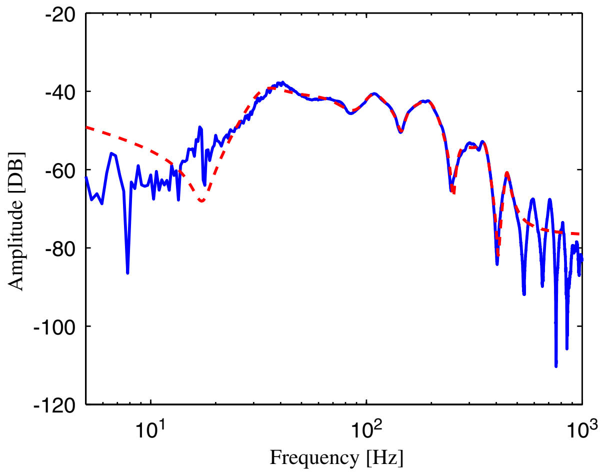

Fig. 7. Amplitude Bode plot of spectral estimate of acoustic coupling G_c (solid) and 17th-order parameteric model $\hat{G}_c$ (dashed).

and $u_1(t)$ can be created, respectively, via Eqs. (9)–(11). With the computation of a normalized right coprime factorization (N_x, D_x) and (N_c, D_c) of the initial filter F_x and the model $\hat{G}_c$, the signals x and z can be created using Eqs. (13) and (14). With the intermediate input signal x and the dual-Youla signal z a spectral estimate of the dual-Youla transfer function R_0 can be computed. The spectral estimate is plotted in Fig. 8 and compared with a simple 6th-order OE model estimate $\bar{R}$. The 6th-order OE model estimate $\bar{R}$ is found by Eq. (20) using standard open-loop identification technique [17] and will be used to generate the basis functions $V_k(q)$ in Eq. (19) for the generalized FIR parameterization of $R(q, \theta)$ in Eq. (18).

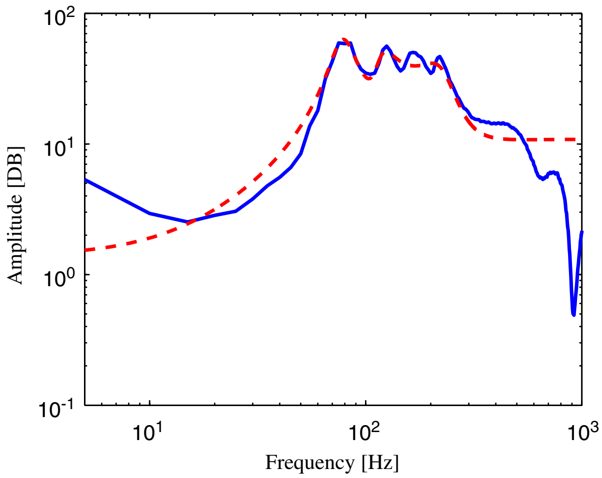

Fig. 8. Amplitude Bode plot of spectral estimate of dual-Youla transfer function R_0 (solid) and 6th-order parameteric model $\bar{R}$ (dashed) used for basis function generation.

From Fig. 8, it can be observed that the simple 6th-order model $\bar{R}$ does not capture the spectral estimate of the dual-Youla transfer function R_0 very well. However, the model $\bar{R}$ will be used only to create the basis functions $V_k(q)$ in Eq. (19) for the orthonormal FIR expansion of the dual-Youla transfer function. The generalized FIR parameterization of $R(q, \theta)$ in Eq. (18) will then allow for a recursive estimation and self-tuning of the feedforward filter.

5.3. Application of feedforward ANC

The prior information reflected in the models $\hat{G}$ of the acoustic control path, $\hat{G}_c$ of the acoustic coupling, the initial filter $F_x(q)$ and the basis functions $V_k(q)$ generated by the low-order model $\bar{R}(q)$ all serve as an initialization for the recursive estimation of the feedforward controller in the presence of acoustic coupling.

For the recursive least-squares estimation of generalized FIR filter in Eq. (18) only $N = 3$ parameters θ_i, $i = 0, \ldots, 3$ were estimated. Since no feedthrough term was expected in the feedforward filter, the feedthrough term D in Eq. (22) was set to $D = 0$. With a 6th-order basis functions $V_k(q), k = 0, \ldots, 3$ generated by the low-order model $\bar{R}$, each parameter $\theta_i \in R^{1\times 6}$. As a result a generalized FIR filter $R(q, \theta)$ of order 24 is estimated by minimizing the error signal $e(t)$ using a recursive least-squares estimation. The recursive estimation is implemented on a Pentium II based personal computer system using a 12 AD/DA Quanser card a sampling time of 2.56 kHz.

From Figs. 6 and 7 it can be observed that the acoustic coupling G_c in the ANC system is relatively large compared to the acoustic control path G. As a result, a straightforward implementation of a filtered LMS algorithm for the computation of a 24th-order FIR filter leads to an unstable feedforward ANC system, where harmonic oscillation are observed due to destabilizing effects of the acoustic feedback path.

The performance of the feedforward compensator $\hat{F}$ that is estimated recursively using a dual-Youla parameterization with generalized FIR filters is confirmed by the estimate of the spectral content of the microphone error signal $e(t)$ plotted in Fig. 9. The spectral content of the error microphone signal has been reduced significantly by the feedforward compensator $\hat{F}$ which is estimated by the recursive least-squares dual-Youla parameterization in the frequency range from 40 till 400 Hz.

A final confirmation of the performance of the ANC has been depicted in Fig. 10. The significant reduction of the error microphone signal observed in the time-domain traces and the norm of the signal displayed on the right part of Fig. 10 indicates the effectiveness of the feedforward filter $\hat{F}$ estimated via recursive least-squares dual-Youla parameterization for feedforward sound compensation.

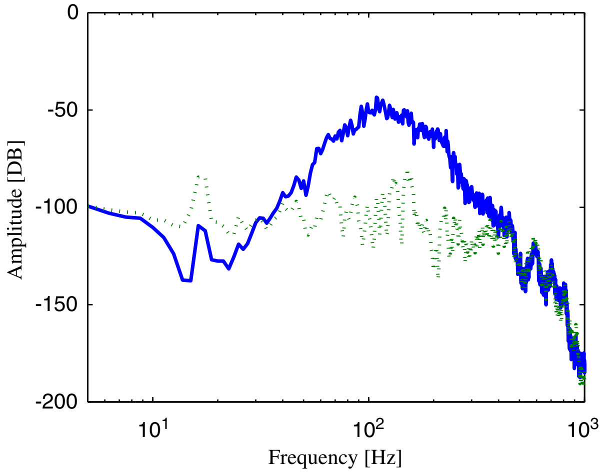

Fig. 9. Spectral estimate of error microphone signal $e(t)$ without ANC (solid) and with ANC (dashed) using feedforward filter $\hat{F}$ estimated via recursive least-squares dual-Youla parameterization.

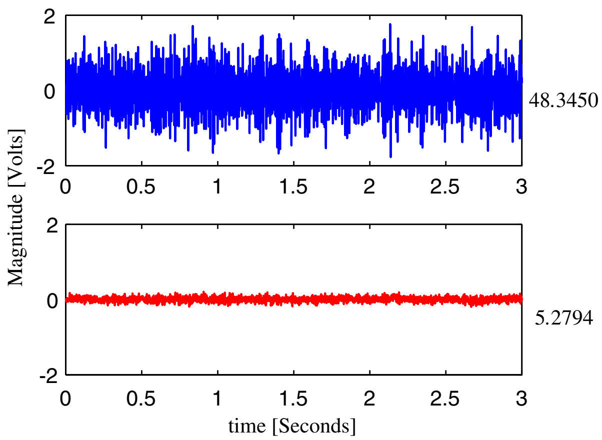

Fig. 10. Time trace of reduction of error microphone signal $e(t)$ without ANC (top) and with ANC turned on at $t = 0$ (bottom) using feedforward filter $\hat{F}$ estimated via recursive least-squares dual-Youla parameterization.

6. Conclusions

In this paper, a new methodology has been proposed for the active feedforward noise control using a dual-Youla parameterization with recursive least-squares (RLS) estimation. In this new approach the dual-Youla parameterization is used to incorporate the acoustic coupling to avoid instabilities of the feedforward ANC. Moreover, generalized FIR filters based on orthonormal basis function expansions can be used to efficiently parameterize and estimate the dual-Youla transfer function. The linear parameterization obtained by generalized FIR filters also facilitates the online RLS implementation using a variable forgetting factor.

The algorithm presented in this paper combines the dual-Youla parameterization to guarantee stability in the presence of acoustic coupling with the generalized FIR filter for online recursive least-squares (RLS) implementation. The algorithm does require prior information that include models of acoustic control path and the acoustic coupling, but this information is a generalization of the filtering used in filtered LMS estimation of feedforward filters.

The practical results of the algorithm are illustrated by an implementation on a commercial silencer for an air-ventilation system. Using relative simple models 6th-order models for initialization of the recursive estimation along with relatively accurate models for the acoustic control and acoustic coupling path, excellent noise cancellation properties were obtained at a broad low-frequency spectrum.

Appendix A. Proof of Lemma 1

Let $(\tilde{D}_x, \tilde{N}_x)$ be a lcf of the filter $F_x = \tilde{D}_x^{-1}\tilde{N}_x$, and $(\tilde{D}_c, \tilde{N}_c)$ be a lcf of the model of acoustic coupling $\hat{G}_c = \tilde{D}_c^{-1}\tilde{N}_c$, which satisfy the Bezout identifies

$$\tilde{D}_c D_x - \tilde{N}_c N_x = I \quad \tilde{D}_x D_c - \tilde{N}_x N_c = I. \tag{A.1}$$

Moreover, it also satisfies

$$\begin{bmatrix} \tilde{D}_c & -\tilde{N}_c \\ -\tilde{N}_x & \tilde{D}_x \end{bmatrix} \begin{bmatrix} D_x & N_c \\ N_x & D_c \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{A.2}$$

Then

$$\begin{aligned} \tilde{D}_c D - \tilde{N}_c N &= \tilde{D}_c(D_x + N_c R_0) - \tilde{N}_c(N_x + D_c R_0) \\ &= (\tilde{D}_c D_x - \tilde{N}_c N_x) + (\tilde{D}_c N_c - \tilde{N}_c D_c) R_0 \\ &= I. \end{aligned} \tag{A.3}$$

By using the fact that $\tilde{D}_c D - \tilde{N}_c N = \Lambda$, where $\Lambda, \Lambda^{-1} \in \mathcal{RH}_\infty$ for internal stability of the closed-loop system, thus, $F = ND^{-1}$ satisfies $\mathcal{T}(F, \hat{G}_c) \in \mathcal{RH}_\infty$.

Conversely, suppose F has a rcf with $F = ND^{-1}$ which satisfy $\mathcal{T}(F, \hat{G}_c) \in \mathcal{RH}_\infty$, $Z := \tilde{D}_c D - \tilde{N}_c N$ is invertible over $\mathcal{RH}_\infty$. Computing R_0 with the equation

$$NZ^{-1} = N_x + D_c R_0, \tag{A.4}$$

one obtains

$$R_0 = D_c^{-1}(NZ^{-1} - N_x). \tag{A.5}$$

With the Bezout identity

$$\begin{aligned} D_x + N_c R_0 &= D_x + N_c D_c^{-1}(NZ^{-1} - N_x) \\ &= \tilde{D}_c^{-1}\tilde{D}_c D_x + \tilde{D}_c^{-1}\tilde{N}_c(NZ^{-1} - N_x) \\ &= \tilde{D}_c^{-1}(\tilde{D}_c D_x - \tilde{N}_c N_x + \tilde{N}_c N Z^{-1}) \\ &= \tilde{D}_c^{-1}(Z + \tilde{N}_c N)Z^{-1} \\ &= \tilde{D}_c^{-1}\tilde{D}_c D Z^{-1} \\ &= DZ^{-1} \end{aligned} \tag{A.6}$$

and the computation of F follows:

$$F = ND^{-1} = NZ^{-1}ZD^{-1} = (N_x + D_c R_0)(D_x + N_c R_0)^{-1}. \tag{A.7}$$

From (A.4) and (A.6), it can be observed that $N_c R_0 \in \mathcal{RH}_\infty$ and $D_c R_0 \in \mathcal{RH}_\infty$, then

$$R_0 = (\tilde{D}_x D_c - \tilde{N}_x N_c)R_0 = \tilde{D}_x \underline{D_c R_0} - \tilde{N}_x \underline{N_c R_0} \in \mathcal{RH}_\infty. \qquad \square \tag{A.8}$$

Appendix B. Proof of Lemma 2

From Fig. 3, the relationship between intermediate signal x and reference signal r can be easily obtained

$$\begin{aligned} x &= (D_x - \hat{G}_c N_x)^{-1} r = (D_x - \hat{G}_c N_x)^{-1}(u_1 - \hat{G}_c y_1) \\ &= (D_x - \hat{G}_c N_x)^{-1}[-\hat{G}_c \quad I]\begin{bmatrix} y_1 \\ u_1 \end{bmatrix}. \end{aligned} \tag{B.1}$$

Pre-multiplying $\hat{G}_c$ to the first equation of Eq. (7) and then subtracting second equation of Eq. (7) yields $D - \hat{G}_c N = D_x - \hat{G}_c N_x$ and one obtains

$$(I - \hat{G}_c F)^{-1} = D(D_x - \hat{G}_c N_x)^{-1}. \tag{B.2}$$

The data coming from the plant F operating with positive feedback $\hat{G}_c$ under closed-loop can be described as follows:

$$\begin{aligned} \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} &= \begin{bmatrix} F(I - \hat{G}_c F)^{-1} \\ (I - \hat{G}_c F)^{-1} \end{bmatrix} r \\ &= \begin{bmatrix} N(D_x - \hat{G}_c N_x)^{-1} \\ D(D_x - \hat{G}_c N_x)^{-1} \end{bmatrix} r \end{aligned}$$

$$= \begin{bmatrix} N_x(D_x - \hat{G}_c N_x)^{-1} + D_c R_0 (D_x - \hat{G}_c N_x)^{-1} \\ D_x(D_x - \hat{G}_c N_x)^{-1} + N_c R_0 (D_x - \hat{G}_c N_x)^{-1} \end{bmatrix} r$$

$$= \begin{bmatrix} N_x \\ D_x \end{bmatrix} x + \begin{bmatrix} D_c \\ N_c \end{bmatrix} R_0 x. \tag{B.3}$$

Computing $y_1 - F_x u_1$ with $F_x = N_x D_x^{-1}$, yields

$$y_1 - F_x u_1 = (D_c - F_x N_c) R_0 x. \tag{B.4}$$

Define $z := (D_c - F_x N_c)^{-1}(y_1 - F_x u_1)$ which is Eq. (14), then Eq. (B.4) reduces to Eq. (12). □

Appendix C. Proof of Lemma 3

Define $\bar{x} = [x_1, x_2]^{\mathrm{T}}$ and $\bar{z} = [z_1, z_2]^{\mathrm{T}}$ where $z_1 = \hat{R} x_1$ and $z_2 = \Delta x_2$. With the rcf $(\hat{N}, \hat{D})$ of the feedforward filter $\hat{F}$ given by Eq. (8) and the rcf $(\bar{N}_c, \bar{D}_c)$ of the acoustic coupling G_c given by Eq. (16), $\bar{x}$ can be written as a function of $\bar{z}$ and $\bar{z}$ as a function of $\bar{x}$ via

$$\bar{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bar{z} \tag{C.1}$$

and

$$\bar{z} = \begin{bmatrix} \hat{R} & 0 \\ 0 & \Delta \end{bmatrix} \bar{x}. \tag{C.2}$$

From Eqs. (C.1) and (C.2), the closed-loop connection between x_2 and z_2 can be written as

$$z_2 = \Delta x_2,$$
$$x_2 = z_1 = \hat{R} x_1 = \hat{R} z_2.$$

With $\hat{R} \in \mathcal{RH}_\infty$ and by use of the small gain theory, the closed-loop system is well posed and internal stable for all $\Delta \in \mathcal{RH}_\infty$ if and only if

$$\|\Delta \hat{R}\|_\infty < 1. \tag{C.3}$$

Similarly, the closed-loop connection between x_1 and z_1 can also be obtained with the same procedure, and the same result (C.3) is obtained. □

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