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Classical solutions of forced vibration of rectangular plate driven by displacement boundary conditions

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Abstract

Forced vibration of a rectangular plate excited by displacement boundary conditions is investigated. The system of boundary conditions is decomposed into two fundamental types of sub-systems, which involves only one side or one corner. The necessary conditions for the solutions to be of classical sense are discussed. The transformation is designed to convert the problems to those of forced vibration by body forces with homogeneous boundary conditions. The closed-form solutions are derived for these two types of problems, using Fourier series. Additional conditions are proposed to the boundary functions so that the differentiated series converges uniformly. And consequently, the solutions are verified to satisfy the differential equations in a classical sense with continuous derivatives.

1. Introduction

The forced vibration due to impact, traveling or transient loadings has important applications in structural analysis, and is often used as examples in the textbooks [1–6]. Besides the vibration excited by the body force, the forced vibration driven by time-dependent boundary conditions has attracted many researchers. Closed-form solutions are of particular interest. For example, the axial vibration of a rod is studied in Refs. [7–9]. The beam bending problems can be found in

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Refs. [10–17]. The membrane deflection problems and plate bending problems are reported in Refs. [18,19] and [20–22], respectively.

Several approaches have been developed for solving the forced vibration with time-dependent boundary conditions. The time-dependent boundary value problems can be treated by using Laplace transform, see Ref. [10] for example. If the functions involved are of simple form, the closed-form transformation is possible. It is however not always easy to invert the solution of the transformed system in a closed form. On the other hand, a general transform is proposed in Mindlin and Goodman [11] to convert a class of vibration systems with time-dependent boundary conditions to a forced-vibration system by body force with homogeneous boundary conditions. Then the classical approach of separation of variables is applied to obtain a series solution in terms of superposition of eigenfunctions. Example of Bernoulli–Euler beam is illustrated there. The Mindlin–Goodman method has been extended to various types of applications, e.g., rod [7,9], beam [11,15–17], membrane [19], and plate [21,22]. Included are also the non-uniform beam [15,16] and plate [22]. Another approach is studied by Williams [23], which introduces an auxiliary quasi-static problem, and transforms the system to one in the same form of that resulting from Mindlin–Goodman method. Applications of Williams method can be found, e.g., in Refs. [8,9] for rod, [12] for beam, [18] for membrane and [20] for plate. Improvement of Mindlin–Goodman method and Williams method is discussed in Refs. [9,24]. Besides these methods, a special transformation is investigated in Refs. [13,14]. The transform is designed to let the homogeneous system remain homogeneous, with homogeneous boundary conditions and non-homogeneous initial conditions. With only special examples shown however, the application of the method to a general case might still be a challenge.

It is commented in Ref. [9] that the displacement by Williams method may converge faster than Mindlin–Goodman method. The displacement obtained in series form is expected to be differentiated term by term (twice in time, two or four times in space) to satisfy the partial differential equations. It is however a challenging task to assure the needed convergence of the differentiated series and the solution to be of classical sense with continuous derivatives. Although the expression for stress (first derivative) or even the shear force of the beam (third derivative) [10] is discussed in the literature sometimes, the solution is seldom rigorously verified in the classical sense. As a matter of fact, to assure the continuous derivatives by differentiating the series, additional requirements need to be satisfied by the transformation. An improvement from Mindlin–Goodman method is recently reported in Ref. [25] to construct the closed-form solutions for rod and beam that satisfy the differential equations in the classical sense with continuous derivatives. Certain homogeneous boundary conditions are imposed in Ref. [25] on the transform function and the transformed body force. A similar requirement on the body force term after transformation is studied in Ref. [9]. In addition, certain consistency conditions need to be imposed on the boundary functions, so that the differential equations are satisfied at $t = 0$, at the boundary.

This paper is to develop an approach, as an extension of Ref. [25], for solving forced vibration of a type of a rectangular plate driven by the displacement boundary conditions. The system of general boundary conditions is decomposed into two fundamental types of sub-systems, a side type involving only one side and a corner type involving only one corner and the two neighboring sides. For each of the sub-systems, a transform is designed to convert the system to a forced vibration excited by body forces with homogeneous boundary conditions. Then the separation of variables is applied and the Fourier series with a Duhamel's integral form the solution. To guarantee that the solutions satisfy the equations in a classical sense, the data must satisfy certain conditions (necessary), consistently at time $= 0$ at the boundary, and compatible at the corners. With additional conditions on the data (sufficient), the constructed solutions are verified in the classical sense with continuous derivatives, by using theories of Fourier series, particularly the properties of Fourier coefficients.

In what follows, Section 2 describes the decomposition of displacement boundary conditions into a side type and a corner type. Sections 3 and 4 develop the closed-form solutions for these two types of systems respectively and verify the solutions in the classical sense. Section 5 gives an example of side-type system, followed by the concluding remarks in Section 6.

Through the text, the discussions are for the solutions in the classical sense, which has continuous second temporal derivatives and fourth spatial derivatives that appear in the differential equations on a closed rectangular domain. The following traditional notations are used for derivatives with respect to time and space variables, for example,

$$\begin{aligned}&\ddot{u}(x,t)=\partial^2 u/\partial t^2, \quad u_{t^4}(x,t)=\partial^4 u/\partial t^4,\ldots,\\&u''(x,t)=\partial^2 u/\partial x^2, \quad u^{(4)}(x,t)=u_{x^4}(x,t)=\partial^4 u/\partial x^4,\ldots.\end{aligned}$$

2. Forced vibration of a rectangular plate by displacement boundary conditions

Consider a general case of forced vibration driven by displacement boundary conditions

$$\begin{aligned}&\rho h\ddot{w}+D\nabla^4 w(x,y,t)=0,\\&w(0,y,t)=h_1(y,t), \quad w_{x^2}(0,y,t)=0,\\&w(a,y,t)=h_2(y,t), \quad w_{x^2}(a,y,t)=0,\\&w(x,0,t)=g_1(x,t), \quad w_{y^2}(x,0,t)=0,\\&w(x,b,t)=g_2(x,t), \quad w_{y^2}(x,b,t)=0,\\&w(x,y,0)=0,\\&\dot{w}(x,y,0)=0.\end{aligned} \tag{1}$$

By superposition, the cases of non-zero initial conditions or body forces can be solved separately with homogeneous boundary conditions using Fourier series.

Obviously, functions g_α and h_α ($\alpha = 1, 2$) should be compatible at the corners

$$\begin{aligned}&g_1(0,t)=h_1(0,t)=\lambda_1(t),\\&g_1(a,t)=h_2(0,t)=\lambda_2(t),\\&g_2(a,t)=h_2(b,t)=\lambda_3(t),\\&g_2(0,t)=h_1(b,t)=\lambda_4(t).\end{aligned}$$

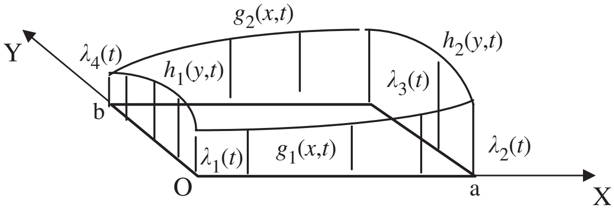

Fig. 1. Displacement at the boundary of a plate.

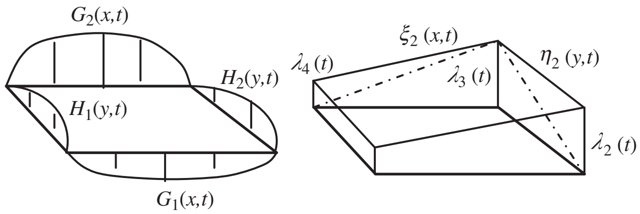

Fig. 2. Decomposition of displacement boundary conditions: Left—Type A; Right—Type B.

The distribution of the boundary conditions is depicted in Fig. 1. Decomposition is essential now for this complex system. Take one side: $x \in [0, a]$ and $y = b$ as example. If we define

$$\begin{aligned}&\xi_2(x,t) = \lambda_4(t)\xi_{21}(x) + \lambda_3(t)\xi_{22}(x),\\&\xi_{21}(0) = 1, \quad \xi_{21}(a) = 0, \quad \xi_{22}(0) = 0, \quad \xi_{22}(a) = 1,\\&g_2(x,t) = G_2(x,t) + \xi_2(x,t).\end{aligned}$$

Then function $G_2(x, t)$ will vanish at both ends of this side. $\lambda_4(t)\xi_{21}(x)$ and $\lambda_3(t)\xi_{22}(x)$ contribute to the corners $(0, b)$ and (a, b), respectively. Thus the system of displacement boundary conditions can be decomposed into two types of sub-systems, the side-type (A) and the corner-type (B). Each of them can be further decomposed into four sub-systems, involving one side or one corner only, depicted in Fig. 2.

The following development devotes to the approaches solving these two types of problems.

3. Solution of sub-system with Type A boundary conditions

Consider the sub-system with the side-type boundary conditions, for $y = b$, $x \in [0, a]$

$$\begin{aligned}&\rho h\ddot{w} + D\nabla^4 w(x,y,t) = 0,\\&w(0,y,t) = 0, \quad w_{x^2}(0,y,t) = 0,\\&w(a,y,t) = 0, \quad w_{x^2}(a,y,t) = 0,\\&w(x,0,t) = 0, \quad w_{y^2}(x,0,t) = 0,\\&w(x,b,t) = g(x,t), \quad w_{y^2}(x,b,t) = 0,\\&w(x,y,0) = 0,\\&\dot{w}(x,y,0) = 0.\end{aligned} \tag{2}$$

The cases studied in Refs. [20–22] have the Type A boundary conditions. Continuity of w, a solution in the classical sense, at $y = b$ requires that $\ddot{g}$ is continuous and the following consistency conditions at $t = 0$ are satisfied

$$g(x,0) = \dot{g}(x,0) = \ddot{g}(x,0) = 0. \tag{3}$$

The first two equations are due to the zero initial conditions defined in Eq. (2). The third one is due to the differential equation at $t = 0$ and $\nabla^4 w(x,b,0) = 0$, which is from the differentiation of $w(x,y,0) = 0$ at $y = b$. Furthermore, the continuity of w at the corner points requires that g_{x^4} is continuous and the following compatibility conditions at (0, b) and (a, b) are satisfied:

$$g(0,t) = g(a,t) = 0, \quad g''(0,t) = g''(a,t) = 0. \tag{4}$$

These are due to the second and third equations of Eq. (2). The t- and y-derivatives of $w(0,y,t) = 0$ lead to $\ddot{w}(0,y,t) = w_{y^2}(0,y,t) = w_{y^4}(0,y,t) = 0$. Meanwhile, $w_{x^2}(0,y,t) = 0$ gives $w_{x^2y^2}(0,y,t) = 0$. Then $w_{x^4}(0,y,t) = 0$ results from the differential equation at $(0,y,t)$. In the same way, we have $w_{x^4}(a,y,t) = 0$. Hence, at $y = b$,

$$g^{(4)}(0,t) = g^{(4)}(a,t) = 0. \tag{5}$$

Note that Eqs. (3)–(5) are the necessary conditions for system (2) to have a solution in the classical sense. This is an extension form [25] for the rod and beam problems. This type of requirement is however not fully discussed in the publications.

With Eq. (4), function $g(x, t)$ can be viewed as periodic and have Fourier series over the basis

$$\{\sin \alpha_m x, \ m = 1,2,\ldots\}, \quad \alpha_m = m\pi/a = O(m). \tag{6}$$

This suggests an x expansion for w along with g

$$\begin{aligned} g(x,t) &= \sum_m g_m(t) \sin \alpha_m x, \quad g_m(t) = \frac{2}{a}\int_0^a g(x,t) \sin \alpha_m x \, \mathrm{d}x, \\ w(x,y,t) &= \sum_m w_m(y,t) \sin \alpha_m x. \end{aligned}$$

For each m, it results in a reduced system of equations in y and t,

$$\begin{aligned} &\rho h \ddot{w}_m + D(w_{m,y^4} - 2\alpha_m^2 w_{m,y^2} + \alpha_m^4 w_m) = 0, \\ &w_m(0,t) = 0, \quad w_{m,y^2}(0,t) = 0, \\ &w_m(b,t) = g_m(t), \quad w_{m,y^2}(b,t) = 0, \\ &w_m(y,0) = 0, \\ &\dot{w}_m(y,0) = 0. \end{aligned}$$

A transform is introduced

$$\begin{aligned} &w_m(y,t) = p_m(y,t) + v_m(y,t), \\ &q_m(y,t) = -(\rho h \ddot{p}_m + D(p_{m,y^4} - 2\alpha_m^2 p_{m,y^2} + \alpha_m^4 p_m)). \end{aligned} \tag{7}$$

Let p_m satisfy the following conditions:

$$\begin{aligned} &p_m(0,t) = 0, \quad p''_m(0,t) = 0, \\ &p_m(b,t) = g_m(t), \quad p''_m(b,t) = 0. \end{aligned} \tag{8}$$

This yields a system of v_m,

$$\begin{aligned} &\rho h \ddot{v}_m + D(v_{m,y^4} - 2\alpha_m^2 v_{m,y^2} + \alpha_m^4 v_m) = q_m, \\ &v_m(0,t) = 0, \quad v_{m,y^2}(0,t) = 0, \\ &v_m(b,t) = 0, \quad v_{m,y^2}(b,t) = 0, \\ &v_m(y,0) = -p_m(y,0), \\ &\dot{v}_m(y,0) = -\dot{p}_m(y,0). \end{aligned} \tag{9}$$

A Fourier series solution v_m can be tried, though not the only way, over the basis

$$\{\sin \beta_n y;\ \ n = 1, 2, \ldots\}, \quad \beta_n = n\pi/b = O(n), \tag{10}$$

$$\begin{aligned} q_m(y,t) &= \sum_n q_{mn}(t) \sin \beta_n y, \\ \varphi_m(y) &= \sum_n \varphi_{mn} \sin \beta_n y, \\ \psi_m(y) &= \sum_n \psi_{mn} \sin \beta_n y, \\ v_m(y,t) &= \sum_n v_{mn}(t) \sin \beta_n y. \end{aligned} \tag{11}$$

Using this approach, additional conditions are proposed

$$q_m(0,t) = 0, \quad q_m(b,t) = 0. \tag{12}$$

This is required by the end values of the series. Similar requirement is discussed in Ref. [9]. For the six conditions in Eqs. (8) and (12), a six-term polynomial (5th-degree in y) is suggested, as an extension of the approach developed in Ref. [25] for the rod and beam problems

$$p_m(x,t) = C_0(t) + C_1(t)y + C_2(t)y^2 + C_3(t)y^3 + C_4(t)y^4 + C_5(t)y^5.$$

The answer is straightforward,

$$p_m(y,t) = \frac{y}{b}\, g_m(t) + \frac{1}{D}\, F_m(t)\zeta(y,b), \tag{13}$$

$$q_m(y,t) = 2\alpha_m^2 F_m(t)\zeta''(y,b) - (\alpha_m^4 F_m(t) + D^{-1}\rho h \ddot{F}_m(t))\zeta(y,b), \tag{14}$$

where

$$F_m(t) = D\alpha_m^4 g_m(t) + \rho h \ddot{g}_m(t), \tag{15}$$

$$\begin{aligned}
&\zeta(y,b) = \frac{b^4}{360}\left(-7\,\frac{y}{b} + 10\left(\frac{y}{b}\right)^3 - 3\left(\frac{y}{b}\right)^5\right),\\
&\zeta(0,b) = \zeta(b,b) = \zeta''(0,b) = \zeta''(b,b) = 0,\\
&\zeta^{(4)}(y,b) = -y/b.
\end{aligned} \tag{16}$$

Straight application of Mindlin–Goodman method or Williams method will yield

$$p_m(y,t) = g_m(t)y/b, \tag{13$'$}$$

$$q_m(y,t) = -(\rho h \ddot{g}_m(t) + D\alpha_m^4 g_m(t))y/b, \tag{14$'$}$$

satisfying Eq. (8), but not Eq. (12) at $y = b$. The Fourier series of q_m in Eq. (11) takes zero value at the end point $y = b$. If the derivatives are taken from differentiation of the series, the solution is not expected to satisfy the differential equation (9) at $y = b$. This discontinuity will at least affect a small zone near the boundary.

From Eqs. (13), (15) and (3), the initial conditions are $p_m(y,0) = 0$ and $\dot{p}_m(y,0) = D^{-1}\rho h \ddot{g}(0)$ $\zeta(y,b)$. Using Eq. (11), system (9) is now reduced to a set of separated second-order ODE subsystems

$$\begin{aligned}
&\rho h \ddot{v}_{mn} + D(\alpha_n^4 + 2\alpha_m^2\beta_n^2 + \beta_n^4)v_{mn} = q_{mn},\\
&v_{mn}(0) = 0,\\
&\dot{v}_{mn}(0) = -\dot{p}_{mn}(0).
\end{aligned}$$

The solution is in the standard form with a Duhamel's integral

$$\begin{aligned}
&v_{mn}(t) = -\frac{\dot{p}_{mn}(0)}{\omega_{mn}}\sin\,\omega_{mn}t + \frac{1}{\rho h\omega_{mn}}\int_0^t q_{mn}(\tau)\,\sin\,\omega_{mn}(t-\tau)\,\mathrm{d}\tau,\\
&\omega_{mn} = \sqrt{D/\rho h}(\alpha_m^2 + \beta_n^2) = O(m^2 + n^2).
\end{aligned} \tag{17}$$

With Eqs. (13)–(16), the Fourier coefficients are

$$\begin{aligned}
&\dot{p}_{mn}(0) = D^{-1}\rho h \ddot{g}_m(0)\zeta_n,\\
&q_{mn}(t) = 2\alpha_m^2 F_m(t)\zeta_{1n} - (\alpha_m^4 F_m(t) + D^{-1}\rho h \ddot{F}_m(t))\zeta_n,
\end{aligned} \tag{18}$$

where

$$\zeta_n = (-1)^n\,\frac{2}{b\beta_n^5} \quad \text{and} \quad \zeta_{1n} = (-1)^{n-1}\,\frac{2}{b\beta_n^3} \tag{19}$$

are the Fourier coefficients of $\zeta(y)$ and $\zeta_1(y) = \zeta''(y) = b^2/6(y/b - (y/b)^3)$, respectively. Finally, the solution is formed

$$w(x,y,t) = \sum_m (p_m(y,t) + \sum_n v_{mn}(t)\,\sin\,\beta_n y)\,\sin\,\alpha_m x. \tag{20}$$

The next step is to verify the convergence conditions for the series so that the equation is satisfied in the classical sense. A formal differentiation yields

$$\begin{aligned}&\rho h\ddot{w}(x,y,t)+D\nabla^4 w(x,y,t)\\&\quad=\sum_m(\rho h\ddot{p}_m(y,t)+D(\alpha_m^4 p_m-2\alpha_m^2 p_m''+(p_m)_{y^4})\sin\alpha_m x\\&\qquad+\sum_m\left(\sum_n(\rho h\ddot{v}_{mn}(t)+D(\alpha_m^2+\beta_n^2)^2 v_{mn}(t))\sin\beta_n y\right)\sin\alpha_m x.\end{aligned}\tag{21}$$

The first summation of Eq. (21) has a dominant series, in view of Eqs. (13)–(16),

$$\begin{aligned}S_1&=\sum_m(|\ddot{p}_m(y,t)|+|\alpha_m^4 p_m|+|\alpha_m^2 p_m''|+|p_m^{(4)}|)\\&\leqslant C\sum_m(|\ddot{g}_m|+|\alpha_m^4\ddot{g}_m|+|(g_{t^4})_m|+\alpha_m^4(|g_m|+|\alpha_m^4 g_m|+|\ddot{g}_m|)+\alpha_m^2(|\alpha_m^4 g_m|+|\ddot{g}_m|)).\end{aligned}$$

Assume now, in addition to Eqs. (4) and (5) that

$$\partial^{10}g/\partial x^9\partial t,\quad \partial^9 g/\partial x^7\partial t^2,\quad \partial^8 g/\partial x^5\partial t^3,\quad \partial^7 g/\partial x^3\partial t^4 \text{ are continuous,}\tag{22}$$

$$g^{(6)}(0,t)=g^{(6)}(a,t)=g^{(8)}(0,t)=g^{(8)}(a,t)=0.\tag{23}$$

According to the theory of Fourier series [26], in particular, the properties of Fourier coefficients

$$S_1\leqslant C\sum_m(|\ddot{g}_m|+|(\ddot{g}_{x^4})_m|+|(g_{t^4})_m|+|(g_{x^4})_m|+|(g_{x^8})_m|+|(g_{x^6})_m|+|\ddot{g}_m''|)<\infty.$$

The terms involved above are all the Fourier coefficients of the corresponding derivative functions of g. The estimate of Fourier coefficients, used above to bound S_1, is proved for periodic functions. The proposed conditions (4, 5, 22, 23) assure that the corresponding functions can be treated as periodic functions. See Refs. [27–30] for more discussions on Fourier series.

For the second summation of Eq. (21), with $v_{mn}(t)$ of Eq. (17),

$$\ddot{v}_{mn}(t)=\omega_{mn}\dot{p}_{mn}(0)\sin\omega_{mn}t+q_{mn}(t)/\rho h-\omega_{mn}\int_0^t q_{mn}(\tau)\sin\omega_{mn}(t-\tau)\,\mathrm{d}\tau/\rho h.$$

Several items contain the factor $\alpha_m=O(m)$, via Eqs. (14) and (15). The highest power of factor α_m in these items is 10. It can be reduced to 8, after integration by parts with respect to t, for F_m contained in q_{mn},

$$-\omega_{mn}\int_0^t F_m(\tau)\sin\omega_{mn}(t-\tau)\,\mathrm{d}\tau=F_m(t)-F_m(0)\cos\omega_{mn}(t)-\int_0^t\dot{F}_m(\tau)\cos\omega_{mn}(t-\tau)\,\mathrm{d}\tau.$$

Note that $F_m(0) = 0$, by Eqs. (15) and (3). Hence, a dominant series for the second summation in Eq. (21) is found

$$\begin{aligned} S_2 &= \sum_m \sum_n (|\omega_{mn}\dot{p}_m(0)| + |q_{mn}(t)| + (|\alpha_m^2\zeta_{1n}| + |\alpha_m^4\zeta_n|)(|F_m(t)| + \int (|\dot{F}_m(\tau)|\,\mathrm{d}\tau) \\ &\quad + \omega_{mn}\zeta_n \int |\ddot{F}_m|\,\mathrm{d}\tau \\ &\leqslant C\sum_m \sum_n (|\omega_{mn}\ddot{g}_m(0)\zeta_n| + (|\alpha_m^2\zeta_{1n}| + |\alpha_m^4\zeta_n|)(\alpha_m^4|g_m| + |\ddot{g}_m| + \alpha_m^4|\dot{g}_m| + |\dddot{g}_m|) \\ &\quad + |\omega_{mn}\zeta_n|(\alpha_m^4|\ddot{g}_m| + |(\partial^4 g/\partial t^4)_m|)). \end{aligned}$$

By Eq. (19) and $\omega_{mn} = O(m^2 + n^2)$, the summation over n for all the terms are uniformly convergent. Then using the property of Fourier series with $\alpha_m = O(m)$,

$$\begin{aligned} S_2 &\leqslant C\sum_m (|\alpha_m^2\ddot{g}_m(0)| + \alpha_m^4(\alpha_m^4(|g_m| + |\dot{g}_m|) + |\ddot{g}_m| + |\dddot{g}_m|) \\ &\quad + \alpha_m^2(\alpha_m^4|\ddot{g}_m| + |(\partial^4 g/\partial t^4)_m|)) \\ &\leqslant C\sum_m (|(\ddot{g}_{x^2})_m(0)| + (|(g_{x^8})_m| + |(\dot{g}_{x^8})_m| + |(\ddot{g}_{x^4})_m| + |(\dddot{g}_{x^4})_m|) \\ &\quad + (|(\ddot{g}_{x^6})_m| + |(\partial^6 g/\partial t^4 \partial x^2)_m|)) \\ &< \infty. \end{aligned}$$

Thus the series are uniformly convergent under the given assumptions, the derivatives are valid, and the solution is verified in the classical sense, summarized in the following.

Conclusion 1—solution of Type A problem (2), in case (23) is satisfied. The consistency conditions (3) and the compatibility conditions (4) and (5) with continuous g_{x^4} and $\ddot{g}$ are the necessary conditions for Eq. (2) to have a classical solution. If in addition, assumptions (22) and (23) hold, then a closed-form solution of Eq. (2), in the classical sense, can be constructed by using Eq. (20) with p_m defined in Eq. (13) with Eqs. (15) and (16), and v_m defined in Eq. (17) with Eqs. (18) and (19).

It is worth noting that if Eqs. (13′) and (14′) are used from a straight application of Mindlin–Goodman method or Williams method, the Fourier coefficients of y/b are of the order of $O(n^{-1})$ only. The uniform convergence of derivative series will be hard.

Note that assumption (23) seems to be more than what could be naturally expected. However, the following decomposition is possible:

$$\begin{aligned} &g(x,t) = \tilde{g}(x,t) + \xi_1(x,t) + \xi_2(x,t), \\ &\xi_1^{(2i)}(0,t) = 0, \quad \xi_1^{(2i)}(a,t) = g^{(2i)}(a,t), \\ &\xi_2^{(2i)}(0,t) = g^{(2i)}(0,t), \quad \xi_2^{(2i)}(a,t) = 0, \\ &i = 0,1,2,3,4. \end{aligned}$$

The functions $\xi_1(x,t)$ and $\xi_2(x,t)$ satisfying the above 10 conditions can be found, e.g., in a form of ninth-degree polynomials in x with the coefficients as functions of t. For example,

$$\xi_1(x,t) = C_1(t)\frac{x}{a} + C_3(t)\left(\frac{x}{a}\right)^3 + C_5(t)\left(\frac{x}{a}\right)^5 + C_7(t)\left(\frac{x}{a}\right)^7 + C_9(t)\left(\frac{x}{a}\right)^9. \tag{24}$$

The five conditions at $x = 0$ are satisfied automatically. The other five conditions, at $x = a$, form a linear system with an upper triangular matrix. The solution is unique. Then $\tilde{g}(x, t)$ satisfies conditions (4), (5) and (23). System (2) with function $\tilde{g}(x, t)$ in the place of $g(x, t)$ can be solved as summarized in Conclusion 1.

Then, consider system (2) with the polynomial form $\xi_1(x, t)$ of Eq. (24) in the place of $g(x, t)$. In fact, Eqs. (4) and (5) are satisfied by $\xi_1(x, t)$ defined in this way,

$$\xi_1^{(2i)}(a, t) = g^{(2i)}(a, t) = 0, \quad i = 0, 1, 2. \tag{25}$$

The consistency condition (3) implies

$$C_j(0) = \dot{C}_j(0) = \ddot{C}_j(0) = 0, \quad j = 1, 3, 5, 7, 9. \tag{26}$$

A transform can be devised

$$\begin{aligned} w(x, y, t) &= p(x, y, t) + v(x, y, t), \\ q(x, y, t) &= -(\rho h \ddot{p} + D \nabla^4 p). \end{aligned} \tag{27}$$

Let p and q satisfy the following boundary conditions, as an extension of the approach of Ref. [25] to the two-dimensional problem:

$$\begin{aligned} &p(0, y, t) = 0, \quad p_{x^2}(0, y, t) = 0, \\ &p(a, y, t) = 0, \quad p_{x^2}(a, y, t) = 0, \\ &p(x, 0, t) = 0, \quad p_{y^2}(x, 0, t) = 0, \\ &p(x, b, t) = \xi_1(x, t), \quad p_{y^2}(x, b, t) = 0, \\ &q(0, y, t) = 0, \quad q(a, y, t) = 0, \quad q(x, 0, t) = 0, \quad q(x, b, t) = 0. \end{aligned} \tag{28}$$

A polynomial form is chosen,

$$\begin{aligned} p(x, y, t) &= D_{11} \frac{xy}{ab} + D_{13} \frac{x}{a} \left(\frac{y}{b}\right)^3 + D_{33} \left(\frac{xy}{ab}\right)^3 + D_{31} \left(\frac{x}{a}\right)^3 \frac{y}{b} + \cdots \\ &= \sum_{i,j=1}^{k} D_{(2i-1)(2j-1)} \left(\frac{x}{a}\right)^{2i-1} \left(\frac{y}{b}\right)^{2j-1}. \end{aligned} \tag{29}$$

The even order terms are purposely left out so that the conditions at $x = 0$ and $y = 0$ are satisfied automatically. Thus there are six equations left for p and q. At the boundary $x = a$, $p(a, y, t)$ is a polynomial in y. If p is up to the degree of $2k - 1$ in y, there are k terms in y. It is the same situation for x components at $y = b$. So there are a total of $6k$ equations. On the other hand, the total number of parameters D_{ij} for use is $\sum_{i=1}^{k}(2i - 1) = k^2$. A good match is $k = 6$. That means an *11*th-degree (in x and y) polynomial for p. These 36 equations form a linear system with constant coefficients. The right-hand side is composed of $g^{(2i)}(a, t)$, which form the coefficients of $\xi_1(x, t)$, and their time derivatives. A solution should be available. Thus, the auxiliary system of v

satisfies all the homogeneous boundary conditions:

$$\begin{aligned}
&\rho h\ddot{v} + D\nabla^4 v(x,y,t) = q(x,y,t),\\
&v(0,y,t) = 0, \quad v_{x^2}(0,y,t) = 0,\\
&v(a,y,t) = 0, \quad v_{x^2}(a,y,t) = 0,\\
&v(x,0,t) = 0, \quad v_{y^2}(x,0,t) = 0,\\
&v(x,b,t) = 0, \quad v_{y2}(x,b,t) = 0,\\
&v(x,y,0) = -p(x,y,0),\\
&\dot{v}(x,y,0) = -\dot{p}(x,y,0).
\end{aligned} \tag{30}$$

Note that ξ and p are in the polynomial form. All the space derivatives of p and q are continuous. $\ddot{p}$ is involved in q. $\ddot{C}_i(t)$ are involved in p. Condition (26) assures $p(x,y,0) = 0$. System (30) is a forced vibration by body force with homogeneous boundary conditions. The solution is standard, using a double Fourier series in the form of tensor product, and a Duhamel's integral

$$v(x,y,t) = \sum v_{mn}(t) \sin \alpha_m x \sin \beta_n y, \tag{31}$$

$$v_{mn} = -\frac{\dot{p}_{mn}(0)}{\omega_{mn}} \sin \omega_{mn} t + \frac{1}{\rho h \omega_{mn}} \int_0^t q_{mn}(\tau) \sin \omega_{mn}(t-\tau)\, \mathrm{d}\tau. \tag{32}$$

The next step is to examine the convergence of the differentiated series of v, for example,

$$\begin{aligned}
\ddot{v}_{mn} &= \omega_{mn}\dot{p}(0)_{mn} \sin \omega_{mn} t - \frac{\omega_{mn}}{\rho h}\int_0^t q_{mn}(\tau) \sin \omega_{mn}(t-\tau)\,\mathrm{d}\tau + \frac{q_{mn}(t)}{\rho h}\\
&= \omega_{mn}\dot{p}_{mn}(0) \sin \omega_{mn} t + 2\,\frac{q_{mn}(t)}{\rho h} - \frac{q_{mn}(0)}{\rho h} \cos \omega_{mn}(t) - \frac{1}{\rho h}\int_0^t \dot{q}_{mn}(\tau) \cos \omega_{mn}(t-\tau)\,\mathrm{d}\tau\\
&\leqslant C(|\omega_{mn}\dot{p}_{mn}(0)| + |q_{mn}(t)| + |q_{mn}(0)| + |\dot{q}_{mn}(t)|).
\end{aligned}$$

Note that the integration by parts for $\dot{q}_{mn}(\tau)$, in the last step, needs continuous $C_j^{(5)}(t)$. On the other hand, from Eqs. (28) and (26), $\dot{p}(x,b,0) = \dot{\xi}_1(x,0) = 0$. Hence, $\dot{p}(x,y,0)$ vanishes on the boundary. According to the property of Fourier coefficients,

$$|\omega_{mn}\dot{p}_{mn}(0)| \leqslant C(m^2+n^2)|\dot{p}_{mn}(0)| \leqslant C(|\dot{p}_{x^2}(0)_{mn}| + |\dot{p}_{y^2}(0)_{mn}|).$$

Eq. (28) assures $\sum_{mn}|\ddot{v}_{mn}(t)| \leqslant \infty$. Therefore, the differentiated series converge uniformly, the derivatives are valid, and then the solution of Eq. (30) is verified in the classical sense. The above discussion is summarized in the following.

Conclusion 2—solution of Type A problem (2), in case (24) is satisfied. For Eq. (2), with ξ_1 of Eq. (24) in the place of g, to have a classical solution, it is necessary that $\ddot{C}_i$ are continuous, that the consistency conditions (26) and the compatibility conditions (25) are satisfied. If in addition, $C_i^{(5)}(t)$ are continuous, then a closed-form solution, in the classical sense, can be constructed by using Eq. (27) with p_m defined in Eq. (29), and v_m defined in Eqs. (31) and (32).

4. Solution of sub-system with Type B boundary conditions

A corner involves two edges. Consider a system for corner (a, b):

$$
\begin{aligned}
&\rho h\ddot{w} + D\nabla^4 w(x,y,t) = 0,\\
&w(0,y,t) = 0, \quad w_{x^2}(0,y,t) = 0,\\
&w(a,y,t) = \eta(y,t), \quad w_{x^2}(a,y,t) = 0,\\
&w(x,0,t) = 0, \quad w_{y^2}(x,0,t) = 0,\\
&w(x,b,t) = \xi(x,t), \quad w_{y^2}(x,b,t) = 0,\\
&w(x,y,0) = 0,\\
&\dot{w}(x,y,0) = 0.
\end{aligned} \tag{33}
$$

By continuity, it is necessary that ξ_{x^4}, η_{y^4}, $\dddot{\xi}$, and $\ddot{\eta}$ are continuous, and that the following conditions are satisfied at the boundary, by the same reasoning as for Eqs. (4) and (5):

$$
\begin{aligned}
&\xi(a,t) = \eta(b,t) = \lambda(t) \in C^2(\Omega_t),\\
&\rho h\ddot{\lambda}(t) + D(\xi_{x^4}(a,t) + \eta_{y^4}(b,t)) = 0,\\
&\xi(0,t) = 0, \quad \eta(0,t) = 0,\\
&\xi''(0,t) = \xi''(a,t) = 0,\\
&\eta''(0,t) = \eta''(b,t) = 0,\\
&\xi^{(4)}(0,t) = 0,\\
&\eta^{(4)}(0,t) = 0
\end{aligned} \tag{34}
$$

and at $t = 0$, by the same reasoning as for Eq. (3),

$$
\begin{aligned}
&\xi(x,0) = 0, \quad \eta(y,0) = 0, \quad \lambda(0) = 0,\\
&\dot{\xi}(x,0) = 0, \quad \dot{\eta}(y,0) = 0, \quad \dot{\lambda}(0) = 0,\\
&\ddot{\xi}(x,0) = 0, \quad \ddot{\eta}(y,0) = 0, \quad \ddot{\lambda}(0) = 0.
\end{aligned} \tag{35}
$$

Consider a further decomposition

$$
\begin{aligned}
&\xi(x,t) = \tilde{\xi}(x,t) + \hat{\xi}_1(x,t) + \hat{\xi}_2(x,t),\\
&\hat{\xi}_1^{(2i)}(0,t) = 0, \quad \hat{\xi}_1^{(2i)}(a,t) = \xi^{(2i)}(a,t),\\
&\hat{\xi}_2^{(2i)}(0,t) = \xi^{(2i)}(0,t), \quad \hat{\xi}_2^{(2i)}(a,t) = 0,\\
&i = 0,1,2,3,4,
\end{aligned}
$$

$$
\begin{aligned}
&\eta(x,t) = \tilde{\eta}(x,t) + \hat{\eta}_1(x,t) + \hat{\eta}_2(x,t),\\
&\hat{\eta}_1^{(2i)}(0,t) = 0, \quad \hat{\eta}_1^{(2i)}(b,t) = \eta^{(2i)}(b,t),\\
&\hat{\eta}_2^{(2i)}(0,t) = \eta^{(2i)}(0,t), \quad \hat{\eta}_2^{(2i)}(b,t) = 0,\\
&i = 0,1,2,3,4.
\end{aligned}
$$

This is a similar situation discussed in Section 3, for a general case where Eq. (23) may not be satisfied. $\hat{\hat{\xi}}_1, \hat{\hat{\xi}}_2, \hat{\eta}_1$ and $\hat{\eta}_2$ can be solved by the ninth-degree polynomials. Then $\tilde{\tilde{\xi}}$ and $\tilde{\eta}$ form two Type A problems with the boundary conditions (4), (5) and (23) satisfied, which can be solved as summarized in Conclusion 1. Note that $\hat{\hat{\xi}}_2^{(2j)}(0,t) = \hat{\hat{\xi}}_2^{(2j)}(a,t) = 0$ and $\hat{\eta}_2^{(2j)}(0,t) = \hat{\eta}_2^{(2j)}(b,t) = 0$, $j = 0, 1, 2$. They can be further separated and form two Type A problems with the ninth-degree polynomial form, and can be solved as summarized in Conclusion 2. These cases have been discussed in Section 3.

The only case needs further investigation is system (33) with Eqs. (34) and (35), with the leftover $\hat{\hat{\xi}}_1$ and $\hat{\eta}_1$ which are in the following form of the ninth-degree polynomials and replace ξ and η, still denoted by ξ and η in the following discussion:

$$\xi(x,t) = C_1(t)\frac{x}{a} + C_3(t)\left(\frac{x}{a}\right)^3 + C_5(t)\left(\frac{x}{a}\right)^5 + C_7(t)\left(\frac{x}{a}\right)^7 + C_9(t)\left(\frac{x}{a}\right)^9,$$
$$\eta(y,t) = D_1(t)\frac{y}{b} + D_3(t)\left(\frac{y}{b}\right)^3 + D_5(t)\left(\frac{y}{b}\right)^5 + D_7(t)\left(\frac{y}{b}\right)^7 + D_9(t)\left(\frac{y}{b}\right)^9. \tag{36}$$

Without the even order terms, the homogeneous boundary conditions (34) at $x = 0$ and $y = 0$ are satisfied automatically. The rest of Eq. (34) are

$$\begin{aligned} &\xi(a,t) = \eta(b,t) = \lambda(t), \\ &\xi''(a,t) = \eta''(b,t) = 0, \\ &\rho h\ddot{\lambda}(t) + D(\xi^{(4)}(a,t) + \eta^{(4)}(b,t)) = 0. \end{aligned} \tag{37}$$

Again, a transform is used

$$\begin{aligned} &w(x,y,t) = p(x,y,t) + v(x,y,t), \\ &q(x,y,t) = -(\rho h\ddot{p} + D\nabla^4 p). \end{aligned} \tag{38}$$

Let p and q satisfy the boundary conditions

$$\begin{aligned} &p(0,y,t) = 0, \quad p(a,y,t) = \eta(y,t), \quad p_{x^2}(0,y,t) = 0, \quad p_{x^2}(a,y,t) = 0, \\ &p(x,0,t) = 0, \quad p(x,b,t) = \xi(x,t), \quad p_{y^2}(x,0,t) = 0, \quad p_{y^2}(x,b,t) = 0, \\ &q(0,y,t) = 0, \quad q(a,y,t) = 0, \\ &q(x,0,t) = 0, \quad q(x,b,t) = 0. \end{aligned} \tag{39}$$

Since the symmetric pattern in x and y is observed, p is constructed in a symmetric format

$$\begin{aligned} p(x,y,t) = {} & \xi(x,t)\frac{y}{b} + \eta(y,t)\frac{x}{a} + \frac{\rho h}{D}(\ddot{\xi}(x,t)\zeta(y,b) + \ddot{\eta}(y,t)\zeta(x,a)) \\ & - \lambda(t)\frac{xy}{ab} - \frac{\rho h}{D}\ddot{\lambda}(t)\left(\frac{x}{a}\zeta(y,b) + \frac{y}{b}\zeta(x,a)\right) + r(x,y,t), \end{aligned} \tag{40}$$

where function ζ is defined in Eq. (16) and $r(x,y,t)$ is to be determined. Examine the conditions at $x = a$. Using Eq. (37) and the properties of function ζ of Eq. (16), we have

$$\begin{aligned} p(a,y,t) &= \xi(a,t)\frac{y}{b} + \eta(y,t) + \frac{\rho h}{D}\ddot{\xi}(a,t)\zeta(y,b) - \lambda(t)\frac{y}{b} - \frac{\rho h}{D}\ddot{\lambda}(t)\zeta(y,b) + r(a,y,t) \\ &= \eta(y,t) + r(a,y,t), \end{aligned}$$

$$\begin{aligned}p_{x^2}(a,y,t) &= \xi''(a,t)\frac{y}{b}+\frac{\rho h}{D}(\ddot{\xi}''(a,t)\zeta(y,b)+\ddot{\eta}(y,t)\zeta''(a,a))-\frac{\rho h}{D}\ddot{\lambda}(t)\zeta''(a,a)\frac{y}{b}+r_{x^2}(a,y,t)\\ &= r_{x^2}(a,y,t).\end{aligned}$$

Using Eq. (39), we obtain $r(a,y,t)=r_{x^2}(a,y,t)=0$. Consequently, $r_{t^2}(a,y,t)=0$, $r_{y^4}(a,y,t)=0$, $r_{x^2y^2}(a,y,t)=0$, and $\nabla^4 r(a,y,t)=r_{x^4}(a,y,t)$. Furthermore, we have from Eq. (40)

$$\begin{aligned}p_{x^4}(a,y,t) &= \xi^{(4)}(a,t)\frac{y}{b}+\frac{\rho h}{D}\left(\ddot{\xi}^{(4)}(a,t)\zeta(y,b)+\ddot{\eta}(y,t)\zeta^{(4)}(a,a)-\ddot{\lambda}(t)\zeta^{(4)}(a,a)\frac{y}{b}\right)+r_{x^4}(a,y,t)\\ &= \xi^{(4)}(a,t)\frac{y}{b}+\frac{\rho h}{D}\left(\ddot{\xi}^{(4)}(a,t)\zeta(y,b)-\ddot{\eta}(y,t)+\ddot{\lambda}(t)\frac{y}{b}\right)+r_{x^4}(a,y,t),\end{aligned}$$

$$\begin{aligned}p_{y^4}(a,y,t) &= \eta^{(4)}(y,t)+\frac{\rho h}{D}(\ddot{\xi}(a,t)\zeta^{(4)}(y,b)+\ddot{\eta}^{(4)}(y,t)\zeta(a,a)-\ddot{\lambda}(t)\zeta^{(4)}(y,b))+r_{y^4}(a,y,t)\\ &= \eta^{(4)}(y,t)+r_{y^4}(a,y,t),\end{aligned}$$

$$p_{x^2y^2}(a,y,t)=r_{x^2y^2}(a,y,t),$$

$$\nabla^4 p(a,y,t)=\xi^{(4)}(a,t)\frac{y}{b}+\frac{\rho h}{D}\left(\ddot{\xi}^{(4)}(a,t)\zeta(y,b)-\ddot{\eta}(y,t)+\ddot{\lambda}(t)\frac{y}{b}\right)+\eta^{(4)}(y,t)+\nabla^4 r(a,y,t),$$

$$\begin{aligned}\ddot{p}(a,y,t) &= \ddot{\lambda}(t)\frac{y}{b}+\ddot{\eta}(y,t)+\frac{\rho h}{D}\lambda^{(4)}(t)\zeta(y,b)-\ddot{\lambda}(t)\frac{y}{b}-\frac{\rho h}{D}\lambda^{(4)}(t)\zeta(y,b)+\ddot{r}(a,y,t)\\ &= \ddot{\eta}(y,t)+\ddot{r}(a,y,t).\end{aligned}$$

Thus

$$\begin{aligned}q(a,y,t) &= -\rho h\ddot{p}(a,y,t)-D\nabla^4 p(a,y,t)\\ &= -\rho h\ddot{r}(a,y,t)-D\nabla^4 r(a,y,t)-D(\xi^{(4)}(x,t)\frac{y}{b}+\eta^{(4)}(y,t))\\ &\quad -\rho h\left(\ddot{\xi}^{(4)}(a,t)\zeta(y,b)+\ddot{\lambda}(t)\frac{y}{b}\right).\end{aligned}$$

Similarly, we obtain the conditions at $y=b$. The requirements on r, to let p and q satisfy the conditions in Eq. (39), are summarized below:

$$r(0,y,t)=r_{x^2}(0,y,t)=r_{x^4}(0,y,t)=r(a,y,t)=r_{x^2}(a,y,t)=0,$$

$$r(x,0,t)=r_{y^2}(x,0,t)=r_{y^4}(x,0,t)=r(x,b,t)=r_{y^2}(x,b,t)=0,$$

$$Dr_{x^4}(a,y,t)+D(\xi^{(4)}(a,t)y/b+\eta^{(4)}(y,t))+\rho h(\ddot{\xi}^{(4)}(a,t)\zeta(y,b)+\ddot{\lambda}(t)y/b)=0,$$

$$Dr_{y^4}(x,b,t)+D(\xi^{(4)}(x,t)+\eta^{(4)}(b,t)x/a)+\rho h(\ddot{\eta}^{(4)}(b,t)\zeta(x,a)+\ddot{\lambda}(t)x/a)=0.$$

Similar to the case of using Eq. (29) to solve Eq. (28), we assume:

$$r(x,y,t)=\sum_{i,j=1}^{k}C_{(2i-1)(2j-1)}\left(\frac{x}{a}\right)^{2i-1}\left(\frac{y}{b}\right)^{2j-1}. \tag{41}$$

A solution for C_{ij}'s with $k=6$ is available, in the form of a linear combination of ξ,η,λ and their derivatives.

This results in a system for v, which has the same form of Eq. (30), with body force and homogeneous boundary conditions. The next step is to check the initial conditions formed by p, with the help of Eq. (35)

$$\begin{aligned} p(x,y,0) &= \xi(x,0)\frac{y}{b} + \eta(y,0)\frac{x}{a} + \frac{\rho h}{D}(\ddot{\xi}(x,0)\zeta(y,b) + \ddot{\eta}(y,0)\zeta(x,a)) \\ &\quad - \lambda(0)\frac{xy}{ab} - \frac{\rho h}{D}\ddot{\lambda}(0)\left(\frac{x}{a}\zeta(y,b) + \frac{y}{b}\zeta(x,a)\right) + r(x,y,0) \\ &= \frac{\rho h}{D}(\ddot{\xi}(x,0)\zeta(y,b) + \ddot{\eta}(y,0)\zeta(x,a)) - \frac{\rho h}{D}\ddot{\lambda}(0)\left(\frac{x}{a}\zeta(y,b) + \frac{y}{b}\zeta(x,a)\right) + r(x,y,0) \\ &= r(x,y,0). \end{aligned}$$

The equations for determining $r(x,y,t)$ include functions ξ, η, λ, and their second t-derivatives. Their initial values at $t = 0$ are all zeroes, due to Eq. (35). Hence these equations at $t = 0$ form a homogeneous system. A solution $C_{ij}(0) = 0$ is ready. Then $r(x,y,0) = 0$ and $p(x,y,0) = 0$. The solution can be constructed now, in the same form of Eqs. (31) and (32).

The same process in deriving Conclusion 2 leads to the conclusion that the solution satisfies the differential equation in the classical sense. The highest t-derivatives involved in p are of second-order. Therefore, q contains fourth t-derivatives. The discussion is summarized below.

Conclusion 3—solution of Type B problem (33). For Eq. (33), with ξ and η defined in Eq. (36), to have a classical solution, it is necessary that $\ddot{C}_i$ and $\ddot{D}_i$ are continuous, that the compatibility conditions (34) and the consistency conditions (35) are satisfied. If in addition, $C_i^{(5)}(t)$ and $D_i^{(5)}(t)$ are continuous, then a closed-form solution of Eq. (33), in the classical sense, can be constructed by using Eq. (38) with p defined in Eq. (40) with Eq. (41), and v defined in the form of Eqs. (31) and (32).

5. Example—forced vibration of a rectangular plate by displacement specified along one side

For a case based on the discussion in Section 3, the parameters and the function $g(x,t)$ are defined below:

$$\begin{aligned} &E = 200.0\ (\text{kn/mm}^2), \\ &\rho = 8.0e - 6\ (\text{kg/mm}^3), \\ &v = 0, \\ &a = 500.0\ (\text{mm}), \\ &b = 500.0\ (\text{mm}), \\ &h = 4.0\ (\text{mm}), \\ &g(x,t) = w_0 \sin \alpha_1 x g(t)\ (\text{mm}), \\ &g(t) = \sin \omega_k t - 0.5 \sin 2\omega_k t, \end{aligned}$$

where $\omega_k \neq \omega_{1n}$ for any n, to avoid resonance. As discussed in Conclusion 1, with one term only, the following solution is obtained, with β_n, ω_{1n}, ζ_n and ζ_{1n} defined in Eqs. (10), (17)

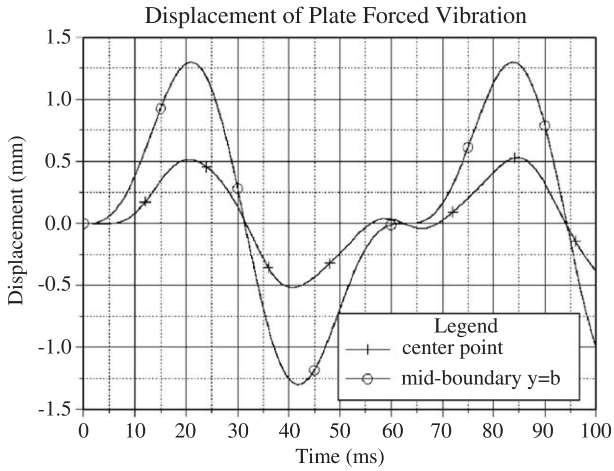

Fig. 3. Displacement time history of the plate.

and (19),

$$w = w_0 \sin \alpha_1 x \left(\frac{y}{b} g(t) + \frac{1}{D} \zeta(y, b) F(t) + \sum v_n(t) \sin \beta_n y \right)$$

$$v_n(t) = -\frac{\rho h \zeta_n}{D\omega_{1n}} \ddot{g}(0) \sin \omega_{1n} t + \frac{1}{\rho h \omega_{1n}} (Q_{1n} S_{1n}(t) + Q_{2n} S_{2n}(t)),$$

$$F(t) = D\alpha_1^4 g(t) + \rho h \ddot{g}(t),$$

$$S_{1n}(t) = \frac{\omega_{1n} \sin \omega_k t - \omega_k \sin \omega_{1n} t}{\omega_{1n}^2 - \omega_k^2},$$

$$S_{2n}(t) = \frac{\omega_{1n} \sin 2\omega_k t - 2\omega_k \sin \omega_{1n} t}{\omega_{1n}^2 - 4\omega_k^2},$$

$$Q_{1n}(t) = D(\alpha_1^4 - D^{-1} \rho h \omega_k^2)(2\alpha_1^2 \zeta_{1n} - (\alpha_1^4 - D^{-1} \rho h \omega_k^2) \zeta_n),$$

$$Q_{2n}(t) = -D(\alpha_1^4 - 4D^{-1} \rho h \omega_k^2)(2\alpha_1^2 \zeta_{1n} - (\alpha_1^4 - 4D^{-1} \rho h \omega_k^2) \zeta_n)/2.$$

It is clear that $v_n(t) = O(n^{-7})$. The derivative series $\sum v_n''(t) \sin \beta_n y$ and $\sum (\beta_n)^4 v_n(t) \sin \beta_n y$ should converge uniformly, and w satisfies the equation in the classical sense.

For $w_0 = 1.0$ and $w_k = 0.1$, the time history of displacement at the center point $(a/2, b/2)$ and the specified function at mid-side $g(a/2, t)$ are shown in Fig. 3.

6. Conclusions

The forced vibrations of a rectangular plate excited by displacement boundary conditions were discussed. The general boundary conditions were decomposed into two fundamental types, a side type involving only one side and a corner type involving only one corner and the two neighboring sides. For each of the problems, a transform was designed to convert the system to a forced vibration excited by body forces with homogeneous boundary conditions. Then the Fourier series with a Duhamel's integral was employed to construct the closed-form solutions. The necessary and sufficient conditions for the solutions to satisfy the partial differential equations in a classical sense were discussed. The key development was that the transform satisfied certain conditions so that the derivative series converged uniformly.

References

[1] S. Timoshenko, *Vibration Problems In Engineering*, second ed., D. Van Nostrand Co. Inc., New York, 1937.
[2] R. Courant, D. Hilbert, *Methods of Mathematical Physics (two volumes)*, Interscience publishers, Inc., New York, 1953.
[3] A.N. Tychonov, A.A. Samarski, *Partial Differential Equations Of Mathematical Physics*, Vol. 1, Holden-Day, Inc., San Francisco, CA, 1964.
[4] I.G. Petrovsky, *Lectures On Partial Differential Equations*, first English ed., Interscience publishers, Inc., New York, 1954.
[5] L. Meirovitch, *Analytical Methods In Vibrations*, The MacMillan Company, New York, 1967.
[6] S. Timoshenko, D.H. Young, W. Weaver Jr., *Vibration Problems In Engineering*, fourth ed., Wiley, New York, 1974.
[7] G. Herrmann, Forced motion of elastic rods, *Journal of Applied Mechanics* 21 (1954) 221–224.
[8] H. Reismann, On the forced motion of elastic solids, *Applied Scientific Research* 18 (1967) 156–165.
[9] G.L. Anderson, C.R. Thomas, A forced vibration problem involving time derivatives in the boundary conditions, *Journal of Sound and Vibration* 14 (1971) 193–214.
[10] G.A. Nothmann, Vibration of a cantilever beam with prescribed end motion, *Journal of Applied Mechanics* 16 (1948) 327–334.
[11] R.D. Mindlin, L.E. Goodman, Beam vibrations with time-dependent boundary conditions, *Journal of Applied Mechanics* 17 (1950) 377–380.
[12] J.G. Berry, P.M. Naghdi, On the vibration of elastic bodies having time-dependent boundary conditions, *Quarterly Applied Mathematics* 14 (1956) 43–50.
[13] C.R. Edstrom, The vibrating beam with nonhomogeneous boundary conditions, *Journal of Applied Mechanics* 48 (1981) 669–670.
[14] D.A. Grant, Beam vibrations with time-dependent boundary conditions, *Journal of Sound and Vibration* 89 (1983) 519–522.
[15] S.Y. Lee, S.M. Lin, Dynamic analysis of nonuniform beams with time-dependent elastic boundary conditions, *Journal of Applied Mechanics* 63 (1996) 474–478.
[16] S.Y. Lee, S.M. Lin, Non-uniform Timoshenko beams with time-dependent elastic boundary conditions, *Journal of Sound and Vibration* 217 (1998) 223–238.
[17] S.M. Lin, S.Y. Lee, The forced vibration and boundary control of pretwisted Timoshenko beams with general time dependent elastic boundary conditions, *Journal of Sound and Vibration* 254 (2002) 69–90.
[18] N.Y. Ölçer, General solution to the equation of the vibrating membrane, *Journal of Sound and Vibration* 6 (1967) 365–374.
[19] K. Sato, Vibrations of non-homogeneous membranes with time-dependent boundary conditions, *Journal of Sound and Vibration* 74 (1980) 273–278.
[20] H. Reismann, Forced motion of elastic plates, *Journal of Applied Mechanics* 35 (1968) 510–515.
[21] J. Venkataramana, M. Maiti, R.K. Srinivasan, Vibration of rectangular plates with time-dependent boundary conditions, *Journal of Sound and Vibration* 62 (1979) 327–337.
[22] S.M. Lin, The closed-form solution for the forced vibration of non-uniform plates with distributed time-dependent boundary conditions, *Journal of Sound and Vibration* 232 (2000) 493–509.

[23] D. Williams, Displacement of a linear elastic system under a given transient load. *Br. R.A.E. Report S.M.E. C/7219/DW/19*, 1946.
[24] S. Chandra, Some comments on the Mindlin–Goodman method of solving vibrations problems with time dependent boundary conditions, *Journal of Sound and Vibration* 67 (1979) 275–277.
[25] S.R. Wu, Classical solutions of forced vibration of rod and beam driven by displacement boundary conditions, *Journal of Sound and Vibration* 279 (2005) 481–486.
[26] G.P. Tolstov, *Fourier Series*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1962.
[27] A. Zygmund, Trigonometric Series, Vol. 2, second ed., The University Press, Cambridge, UK, 1959.
[28] N.K. Bary, *A Treatise on Trigonometric Series*, Pergamon Press Ltd., Oxford, England, 1964.
[29] V.I. Smirnov, *A Course of Higher Mathematics*, Vol. 2, Addison-Wesley Publishing, Reading, MA, 1964.
[30] E.W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, Vol. 2, second ed., The University Press, Cambridge, UK, 1926.